题目 (本题 10 分): 已知函数 $f(x)$ 在 $[0,1]$ 上连续, $f(x) \geq 0$, 且 $\int_0^1 f(x)\,\mathrm{d}x = 1$. 证明

$$\Big(\int_0^1 f(x)\cos x\,\mathrm{d}x\Big)^2 + \Big(\int_0^1 f(x)\sin x\,\mathrm{d}x\Big)^2 \leq 1.$$

证明 1 (得 10 分): 由 Cauchy-Schwarz 不等式,

$$\begin{aligned}
&\Big(\int_0^1 f(x)\cos x\,\mathrm{d}x\Big)^2 + \Big(\int_0^1 f(x)\sin x\,\mathrm{d}x\Big)^2 \\
&= \Big(\int_0^1 \sqrt{f(x)}\sqrt{f(x)}\cos x\,\mathrm{d}x\Big)^2 + \Big(\int_0^1 \sqrt{f(x)}\sqrt{f(x)}\sin x\,\mathrm{d}x\Big)^2 \\
&\leq \int_0^1 f(x)\,\mathrm{d}x \int_0^1 f(x)\cos^2 x\,\mathrm{d}x + \int_0^1 f(x)\,\mathrm{d}x \int_0^1 f(x)\sin^2 x\,\mathrm{d}x \\
&= \Big(\int_0^1 f(x)\,\mathrm{d}x\Big)^2 = 1.
\end{aligned}$$

证明 2 (得 3 分): 由 Cauchy-Schwarz 不等式,

$$\begin{aligned}
&\Big(\int_0^1 f(x)\cos x\,\mathrm{d}x\Big)^2 + \Big(\int_0^1 f(x)\sin x\,\mathrm{d}x\Big)^2 \\
&\leq \int_0^1 f(x)^2\,\mathrm{d}x \int_0^1 \cos^2 x\,\mathrm{d}x + \int_0^1 f(x)^2\,\mathrm{d}x \int_0^1 \sin^2 x\,\mathrm{d}x \\
&= \int_0^1 f(x)^2\,\mathrm{d}x.
\end{aligned}$$

证明 3 (得 0 分): 由 Cauchy-Schwarz 不等式,

$$\begin{aligned}
&\Big(\int_0^1 f(x)\cos x\,\mathrm{d}x\Big)^2 + \Big(\int_0^1 f(x)\sin x\,\mathrm{d}x\Big)^2 \\
&\leq \int_0^1 f(x)^2\,\mathrm{d}x \int_0^1 \cos^2 x\,\mathrm{d}x + \int_0^1 f(x)^2\,\mathrm{d}x \int_0^1 \sin^2 x\,\mathrm{d}x \\
&= \int_0^1 f(x)^2\,\mathrm{d}x \leq 1.
\end{aligned}$$