Relation between the solitons of Yang–Mills–Higgs fields in 2+1 dimensional Minkowski space–time and anti-de Sitter space–time

Zixiang Zhou[a)]
Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

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The Yang–Mills–Higgs–Bogomolny equations in both 2+1 dimensional Minkowski space–time and 2+1 dimensional anti-de Sitter space–time are known to be integrable and their soliton solutions have already been obtained. In this article, we show that there is a natural relation between the Lax pairs and soliton solutions in these two space–times when the curvature changes from 0 to −1. The changes of the asymptotic behaviors of the solitons are also discussed. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1398585]

I. INTRODUCTION

The Yang–Mills–Higgs–Bogomolny equations in both 2+1 dimensional Minkowski space–time and 2+1 dimensional anti-de Sitter space–time are known to be integrable.[1–4] There are several ways to solve them explicitly. The Darboux transformation method is one of them, which gives an easy way to obtain explicit soliton solutions.[5,6] Since the Lax pairs in both Minkowski and anti-de Sitter cases are known, the Darboux transformations can be constructed separately in these two cases.

The standard anti-de Sitter space–time has curvature −1. Naturally we can consider the anti-de Sitter space–time with constant curvature $-1/\rho^2(\rho>0)$. When $\rho\to+\infty$, the space–time tends to the Minkowski space–time. In this article, we consider the following problem: When ρ changes from 1 to $+\infty$, do the solitons in the anti-de Sitter space–time change to solitons in the Minkowski space–time?

In Sec. II, the Yang–Mills–Higgs–Bogomolny equations and their Lax pairs for general ρ are considered. When $\rho=1$ and $\rho\to+\infty$, they become the known equations and their Lax pairs for the Minkowski and anti-de Sitter cases. In Sec. III, the Darboux transformation is discussed. Using the Darboux transformation, we construct solitons in the SU(2) case in Sec. IV and give some examples. When ρ changes from 1 to $+\infty$, the shape of the solitons changes a lot. However, when the coordinates of the space–time depend on ρ suitably, the position of the solitons keeps in a finite region and the solitons in part of the anti-de Sitter space–time change to the solitons in the Minkowski space–time.

II. YANG–MILLS–HIGGS–BOGOMOLNY EQUATIONS AND THEIR LAX PAIRS

Let M be a three dimensional Lorentz manifold with metric $g=(g_{\mu\nu})$. $\{A_\mu \mid \mu=1,2,3\}$ is a gauge potential and Φ is a (scalar) Higgs field, both of which are valued in the Lie algebra of an $N\times N$ matrix Lie group G.

The Yang–Mills–Higgs–Bogomolny equation[1,7] is

$$D\Phi=*F, \tag{2.1}$$

or, written in terms of the components,

a)Electronic mail: zxzhou@guomai.sh.cn

$$D_\mu \Phi = \frac{1}{2\sqrt{|g|}} g_{\mu\nu} \epsilon^{\nu\alpha\beta} F_{\alpha\beta}, \tag{2.2}$$

where the action of the covariant derivative $D_\mu = \partial_\mu + A_\mu$ on Φ is $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$, $\partial_\mu = \partial/\partial x^\mu$. $\{F_{\mu\nu}\}$ is the curvature corresponding to $\{A_\mu\}$, hence $F_{\mu\nu} = [D_\mu, D_\nu]$.

The $2+1$ dimensional anti-de Sitter space–time of constant curvature $-1/\rho^2$ $(\rho > 0)$ is the hyperboloid $U^2 + V^2 - X^2 - Y^2 = \rho^2$ in $\mathbf{R}^{2,2}$ with the metric

$$ds^2 = -dU^2 - dV^2 + dX^2 + dY^2. \tag{2.3}$$

By defining

$$r = \frac{\rho}{U+X} - \rho + 1, \quad x = \frac{Y}{U+X}, \quad t = -\frac{V}{U+X}, \tag{2.4}$$

a part of the $2+1$ dimensional anti-de Sitter space–time with $U+X>0$ is represented by the Poincaré coordinates (r, x, t) with $r > -\rho + 1$ and the metric is

$$ds^2 = \frac{\rho^2}{(r+\rho-1)^2} (-dt^2 + dr^2 + dx^2) = \frac{\rho^2}{(r+\rho-1)^2} (dr^2 + du\, dv), \tag{2.5}$$

where $u = x + t$ and $v = x - t$. Clearly, when $\rho \to +\infty$, the metric on this part of the $2+1$ dimensional anti-de Sitter space–time tends to the flat Minkowski metric on the whole $\mathbf{R}^{2,1}$. In order to consider the change of the solitons with respect to ρ, we only need to consider the solutions in this part.

With the metric (2.5) and the orientation (v, u, r), (2.2) becomes

$$D_u \Phi = \frac{r+\rho-1}{\rho} F_{ur}, \quad D_v \Phi = -\frac{r+\rho-1}{\rho} F_{vr}, \quad D_r \Phi = -\frac{2(r+\rho-1)}{\rho} F_{uv}. \tag{2.6}$$

When $\rho = 1$, it is reduced to

$$D_u \Phi = r F_{ur}, \quad D_v \Phi = -r F_{vr}, \quad D_r \Phi = -2r F_{uv}. \tag{2.7}$$

Reference 4 showed that it had a Lax pair

$$(rD_r + \Phi - 2(\zeta - u) D_u)\psi = 0, \quad \left(2D_v + \frac{\zeta - u}{r} D_r - \frac{\zeta - u}{r^2} \Phi\right)\psi = 0, \tag{2.8}$$

where $D_\mu \psi = \partial_\mu \psi + A_\mu \psi$ and ζ is a complex spectral parameter. That is, (2.7) is the integrability condition of the overdetermined system (2.8).

When $\rho > 0$, the Yang–Mills–Higgs–Bogomolny equation (2.6) can be derived from (2.7) by substituting $r \to r + \rho - 1$ and $\Phi \to \rho \Phi$. Moreover, since ζ is a constant in (2.8), we can replace ζ by $\rho\zeta$. After the substitution

$$r \to r + \rho - 1, \quad \Phi \to \rho \Phi, \quad \zeta \to \rho \zeta, \tag{2.9}$$

(2.8) leads to the Lax pair of (2.6):

$$\begin{gathered} ((r+\rho-1) D_r + \rho \Phi - 2(\rho\zeta - u) D_u)\psi = 0, \\ \left(2D_v + \frac{\rho\zeta - u}{r+\rho-1} D_r - \frac{\rho(\rho\zeta - u)}{(r+\rho-1)^2} \Phi\right)\psi = 0. \end{gathered} \tag{2.10}$$

It is easy to check directly that the integrability condition of (2.10) is the Yang–Mills–Higgs–Bogomolny equation (2.6).

When $\rho\to+\infty$, the metric (2.5) becomes the standard Minkowski metric

$$ds^2=-dt^2+dr^2+dx^2=dr^2+du\,dv, \tag{2.11}$$

the Yang–Mills–Higgs–Bogomolny equation (2.6) becomes

$$D_u\Phi=F_{ur}, \quad D_v\Phi=-F_{vr}, \quad D_r\Phi=-2F_{uv}, \tag{2.12}$$

and the Lax pair (2.10) becomes

$$\begin{aligned}&(D_r+\Phi-2\zeta D_u)\psi=0,\\&(2D_v+\zeta D_r-\zeta\Phi)\psi=0.\end{aligned} \tag{2.13}$$

Remark 1: If we substitute

$$r\to x, \quad \zeta\to\frac{1}{\lambda}, \quad u\to y+t, \quad v\to y-t, \tag{2.14}$$

then (2.13) is changed to

$$(\lambda D_x-D_t-D_y+\lambda\Phi)\psi=0, \quad (\lambda D_t-\lambda D_y-D_x+\Phi)\psi=0, \tag{2.15}$$

which is just the Lax pair given by Ref. 2.

III. DARBOUX TRANSFORMATIONS

For $\rho\to+\infty$ and $\rho=1$, Refs. 5 and 6 gave the construction of the Darboux matrix separately based on a general method.[8] Here we show that these are the two special cases for general ρ.

For $\rho=1$, the Darboux transformation is given as follows.[6] Let $Z=\mathrm{diag}(\zeta_1,...,\zeta_N)$ be a diagonal matrix that satisfies

$$\partial_r Z-\frac{2(Z-u)}{r}(\partial_u Z)=0, \quad \partial_v Z+\frac{Z-u}{2r}(\partial_r Z)=0, \tag{3.1}$$

$H=(h_1,\ldots,h_N)$ where h_j is a solution of (2.8) with $\zeta=\zeta_j$. Then $G=\zeta-HZH^{-1}$ is a Darboux matrix for (2.8). That is, for any solution ψ of the Lax pair (2.8), $\tilde{\psi}=G\psi$ satisfies

$$(r\tilde{D}_r+\tilde{\Phi}-2(\zeta-u)\tilde{D}_u)\tilde{\psi}=0, \quad \left(2\tilde{D}_v+\frac{\zeta-u}{r}\tilde{D}_r-\frac{\zeta-u}{r^2}\tilde{\Phi}\right)\tilde{\psi}=0, \tag{3.2}$$

where $\tilde{D}_\mu=\partial_\mu+\tilde{A}_\mu$ and $\tilde{\Phi},\tilde{A}_\mu$ are other functions in the Lie algebra of G.

When $\rho>1$, a similar conclusion is obtained by the substitution (2.9) and $Z\to\rho Z$. Hence the Darboux matrix is given by

$$G(r,u,v,\zeta)=\zeta-\frac{u}{\rho}-S(r,u,v), \quad S(r,u,v)=H\left(Z-\frac{u}{\rho}\right)H^{-1}, \tag{3.3}$$

where $Z=\mathrm{diag}(\zeta_1,...,\zeta_N)$ satisfies

$$\partial_r Z-\frac{2(\rho Z-u)}{r+\rho-1}\partial_u Z=0, \quad \partial_v Z+\frac{\rho Z-u}{2(r+\rho-1)}\partial_r Z=0, \tag{3.4}$$

$H=(h_1,\ldots,h_N)$ and h_j is a solution of (2.10) with $\zeta=\zeta_j$. It can be checked that S satisfies

$$(r+\rho-1)(\partial_r S+[A_r,S])-2\rho(\partial_u S+[A_u,S])S+\rho[\Phi,S]-2S=0,$$
$$2(\partial_v S+[A_v,S])+\frac{\rho}{r+\rho-1}(\partial_r S+[A_r,S])S-\frac{\rho^2}{(r+\rho-1)^2}[\Phi,S]S=0. \tag{3.5}$$

By direct computation, we know that for any solution ψ of (2.10), $\tilde{\psi}=G\psi$ satisfies

$$((r+\rho-1)\tilde{D}_r+\rho\tilde{\Phi}-2(\rho\zeta-u)\tilde{D}_u)\tilde{\psi}=0,$$
$$\left(2\tilde{D}_v+\frac{\rho\zeta-u}{r+\rho-1}\tilde{D}_r-\frac{\rho(\rho\zeta-u)}{(r+\rho-1)^2}\tilde{\Phi}\right)\tilde{\psi}=0 \tag{3.6}$$

with $\tilde{D}_\mu=\partial_\mu+\tilde{A}_\mu$ $(\mu=u,v,r)$,

$$\tilde{A}_u=A_u,$$
$$\tilde{A}_v=A_v+\frac{\rho}{2(r+\rho-1)}(\partial_r S+[A_r,S])-\frac{\rho^2}{2(r+\rho-1)^2}[\Phi,S],$$
$$\tilde{A}_r=A_r-\frac{1+\rho(\partial_u S+[A_u,S])}{r+\rho-1},$$
$$\tilde{\Phi}=\Phi-\frac{1+\rho(\partial_u S+[A_u,S])}{\rho}. \tag{3.7}$$

Hence G is really a Darboux matrix for (2.10).

According to (3.4), each $\zeta_j(j=1,\ldots,N)$ is a constant or a nonconstant solution of

$$\partial_r\zeta-\frac{2(\rho\zeta-u)}{r+\rho-1}\partial_u\zeta=0,\quad \partial_v\zeta+\frac{\rho\zeta-u}{2(r+\rho-1)}\partial_r\zeta=0. \tag{3.8}$$

The general nonconstant solution is given implicitly by

$$v-\frac{(r+\rho-1)^2}{\rho\zeta-u}=C_1(\zeta,\rho), \tag{3.9}$$

where C_1 is an arbitrary function, which is meromorphic to ζ and smooth to $\rho\in(0,+\infty)$.

In order to consider the limit for $\rho\to+\infty$, we rewrite (3.9) as

$$v-\frac{(r+\rho-1)^2}{\rho\zeta-u}+\frac{\rho-1}{\zeta}=C(\zeta,\rho). \tag{3.10}$$

Here $C(\zeta,\rho)$ is also an arbitrary function, which is holomorphic to ζ and smooth to ρ. Moreover, suppose that $\lim_{\rho\to+\infty}C(\zeta,\rho)$ exists.

When $\rho=1$, (3.10) becomes

$$v-\frac{r^2}{\zeta-u}=C(\zeta,1), \tag{3.11}$$

which is given by Ref. 6. When $\rho\to+\infty$, (3.10) becomes

$$v-\frac{u}{\zeta^2}-\frac{2r}{\zeta}=C(\zeta,+\infty)-\frac{1}{\zeta}. \tag{3.12}$$

When the group $G=U(N)$, there should be more constraints on ζ_j's and h_j's in the construction of the Darboux matrix. They are

$$\begin{gathered}\zeta_j=\zeta_0 \quad\text{or}\quad \overline{\zeta_0}\ \text{for certain fixed}\ \zeta_0,\\ h_j^* h_k=0 \quad\text{if}\quad \zeta_j\neq\zeta_k,\end{gathered} \tag{3.13}$$

as mentioned in Refs. 5 and 6. If so, after the Darboux transformation, $\tilde{\Phi}\in u(N)$, $\tilde{A}_\mu\in u(N)$ provided that $\Phi\in u(N)$, $A_\mu\in u(N)$.

IV. SOLITON SOLUTIONS IN SU(2) CASE

Single soliton solutions are given by Darboux transformations from the trivial seed solution $\Phi=0$, $A_u=A_v=A_r=0$. In the construction of $S=H(Z-u/\rho)H^{-1}$, $Z=\mathrm{diag}(\zeta_1,\dots,\zeta_N)$, where ζ_j is a constant or a nonconstant solution of (3.8), h_j is a column solution of (2.10) with $\zeta=\zeta_j$.

With the action of the Darboux matrix $G=\zeta-u/\rho-S$, (3.7) gives

$$\tilde{A}_u=0,\quad \tilde{A}_v=\frac{\rho\partial_r S}{2(r+\rho-1)},\quad \tilde{A}_r=-\frac{1+\rho\partial_u S}{r+\rho-1},\quad \tilde{\Phi}=-\frac{1+\rho\partial_u S}{\rho}. \tag{4.1}$$

Here we only consider the case where all ζ_j's are constants. When ζ_j's are nonconstant solutions of (3.8), we can obtain solutions in similar ways. However, in the latter case, solutions may only be defined when t is larger than some constant.[6] Now h_j satisfies

$$\partial_r h_j-\frac{2(\rho\zeta_j-u)}{r+\rho-1}\partial_u h_j=0,\quad \partial_v h_j+\frac{\rho\zeta_j-u}{2(r+\rho-1)}\partial_r h_j=0. \tag{4.2}$$

The general solution is

$$h_j=\omega(\zeta_j), \tag{4.3}$$

where

$$\omega(\zeta)=v-\frac{(r+\rho-1)^2}{\rho\zeta-u}+\frac{\rho-1}{\zeta}. \tag{4.4}$$

When $\rho=1$,

$$\omega(\zeta)=v-\frac{r^2}{\zeta-u}, \tag{4.5}$$

which is the same as the result in Ref. 6. When $\rho\to+\infty$,

$$\omega(\zeta)\to v-\frac{u}{\zeta^2}-\frac{2r}{\zeta}+\frac{1}{\zeta}. \tag{4.6}$$

With the substitution (2.14),

$$\omega(\lambda^{-1})\to(1-\lambda^2)y-(1+\lambda^2)t-2\lambda x+\lambda. \tag{4.7}$$

This coincides with Ref. 5.

When $G=\mathrm{SU}(2)$, the conditions (3.13) should be satisfied. Hence we want $\zeta_1=\zeta_0$, $\zeta_2=\overline{\zeta_0}$ for some $\zeta_0\in\mathbf{C}$ and

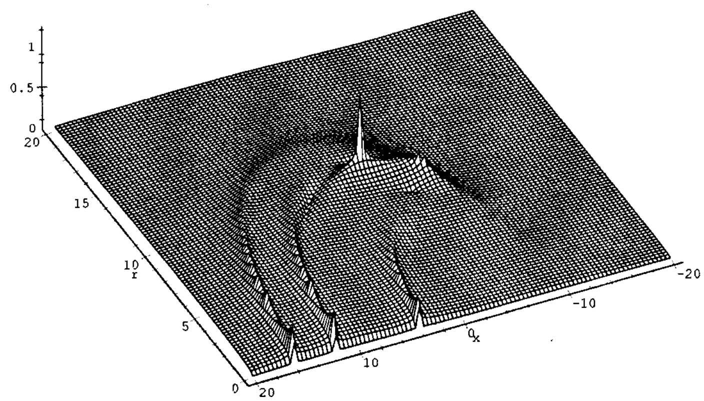

FIG. 1. $\rho=1$.

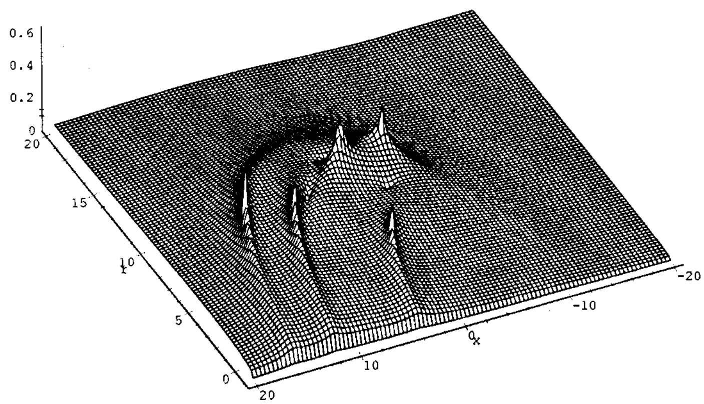

FIG. 2. $\rho=2$.

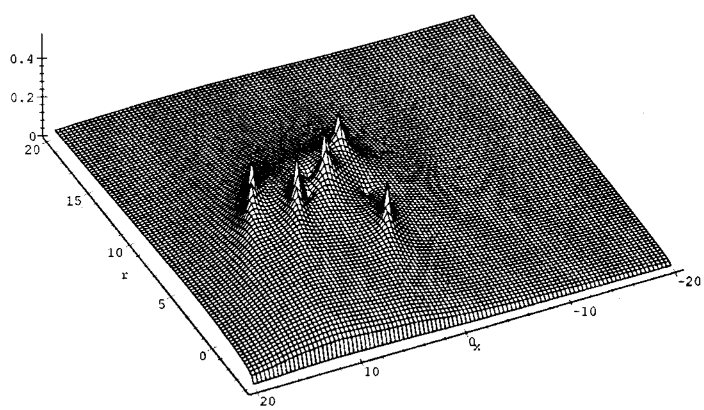

FIG. 3. $\rho=5$.

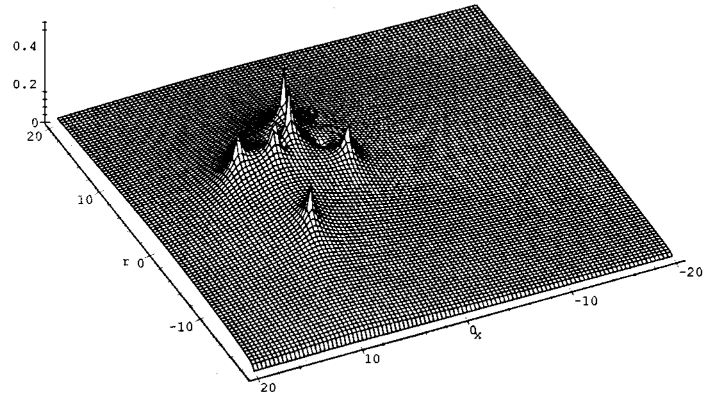

FIG. 4. $\rho=20$.

$$H=\begin{pmatrix} \alpha(\tau) & -\overline{\beta(\tau)} \\ \beta(\tau) & \overline{\alpha(\tau)} \end{pmatrix}, \tag{4.8}$$

where α, β are two holomorphic functions of $\tau=\omega(\zeta_0)$. Let $\sigma(\tau)=\beta(\tau)/\alpha(\tau)$. Then

$$S=\frac{\zeta_0-\overline{\zeta_0}}{1+|\sigma|^2}\begin{pmatrix} 1 & \bar{\sigma} \\ \sigma & |\sigma|^2 \end{pmatrix}+\overline{\zeta_0}-\frac{u}{\rho}, \tag{4.9}$$

$$\widetilde{\Phi}=-\partial_u S-\frac{1}{\rho}=\frac{\zeta_0-\overline{\zeta_0}}{(1+|\sigma|^2)^2}\begin{pmatrix} (|\sigma|^2)_u & \bar{\sigma}^2\sigma_u-\overline{\sigma_u} \\ \sigma^2\overline{\sigma_u}-\sigma_u & -(|\sigma|^2)_u \end{pmatrix} \tag{4.10}$$

and

$$-\operatorname{tr}\widetilde{\Phi}^2=\frac{8(\operatorname{Im}\zeta_0)^2}{(1+|\sigma|^2)^2}|\partial_u\sigma|^2. \tag{4.11}$$

When $\sigma(z)$ is a given meromorphic function of z which is independent of ρ, then by (4.6) and (4.5),

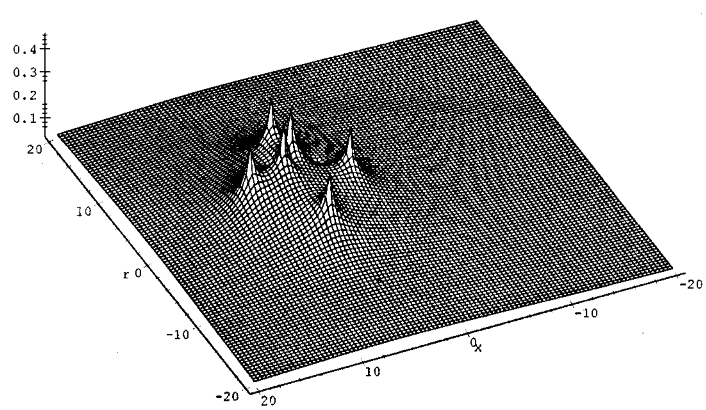

FIG. 5. $\rho=+\infty$.

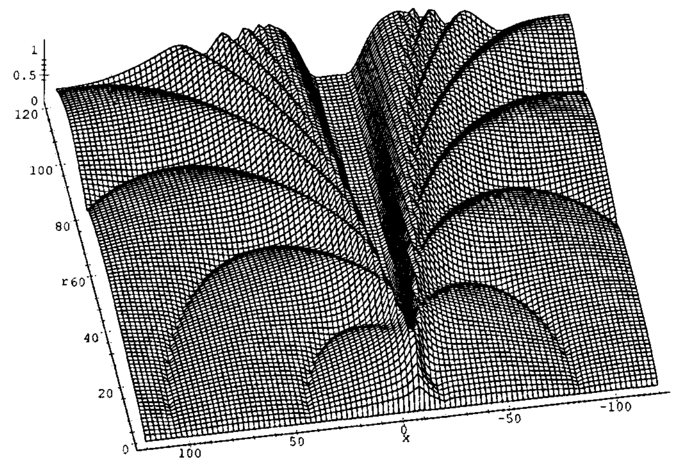

FIG. 6. $\rho=1$.

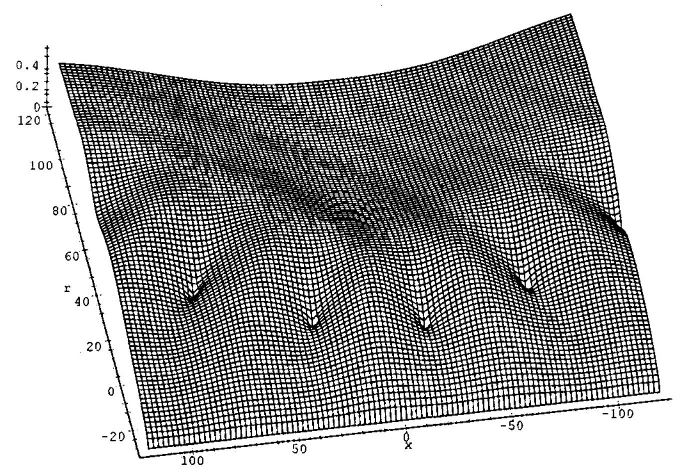

FIG. 7. $\rho=30$.

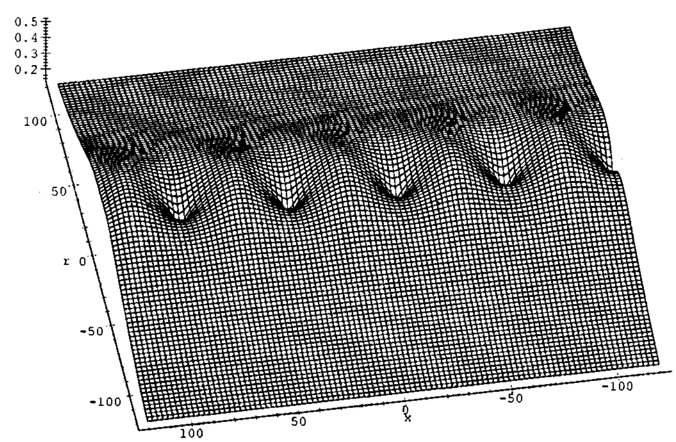

FIG. 8. $\rho=+\infty$.

$$\sigma(\tau)|_{\rho\to+\infty}=\sigma\left(v-\frac{u}{\zeta_0^2}-\frac{2r}{\zeta_0}+\frac{1}{\zeta_0}\right), \quad \sigma(\tau)|_{\rho=1}=\sigma\left(v-\frac{r^2}{\zeta_0-u}\right). \tag{4.12}$$

Hence when $\rho\to+\infty$ and $\rho=1$, the solutions tend to the soliton solutions in the Minkowski and anti-de Sitter space–time, respectively.

These are single soliton solutions. Each solution depends on a complex constant ζ_0 and a meromorphic function σ. Multi-soliton solutions can be constructed by successive Darboux transformations.[5,6] For simplicity, here we only consider the change of single soliton solutions with respect to ρ.

Example 1: $\sigma(\tau)$ is a polynomial of τ without multiple zero. In this case, the solutions are always localized. When $\rho=1$, the behavior of the asymptotic solution as $t\to\infty$ varies according to the roots of $\sigma(\tau)$.[6] Suppose τ_0 is a root of $\sigma(\tau)$. Then (1) if $|\mathrm{Im}\,\tau_0|\ll 1$, it corresponds to a ridge in the graph of $-\mathrm{tr}\tilde{\Phi}^2$; (2) if $\mathrm{Im}\,\tau_0\gg 1$, it corresponds to a peak; (3) if $\mathrm{Im}\,\tau_0\ll -1$, it corresponds to nothing. However, when $\rho\to+\infty$, each root of $\sigma(\tau)$ corresponds to a peak.[5] Figures 1–5 show the change of the solution with respect to ρ for fixed $t=10$, where $\zeta_0=2i$,

$$\sigma(\tau)=(\tau-2)(\tau-6)(\tau+6)(\tau-2i)(\tau-6i)(\tau+6i). \tag{4.13}$$

In these figures the vertical axis is $(-\mathrm{tr}\tilde{\Phi}^2)^{1/16}$.

Example 2: $\sigma(\tau)=\sin(\tau/20)$. For both finite and infinite ρ, the solution is always nonlocalized. For finite ρ, it behaves in a very complicated manner. However, for infinite ρ, (4.12) shows that the solution is invariant if (x,r) is changed to (x',r') with $\mathrm{Re}[(1-\zeta_0^{-2})(x'-x)-2\zeta_0^{-1}(r'-r)]=40\pi k$ (k is an arbitrary integer). Hence the solution is periodic in one direction. Figures 6–8 show this solution for $\rho=1,30,+\infty$ with $t=10$, $\zeta_0=2i$. In these figures the vertical axis is $(-\mathrm{tr}\tilde{\Phi}^2)^{1/8}$.

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