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Solutions of the Yang–Mills–Higgs equations in 2+1-dimensional anti-de Sitter space–time

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The solutions of the Bogomolny equation in 2+1-dimensional anti-de Sitter space–time are obtained by using Darboux transformations with both constant spectral parameters and variable ''spectral parameters.'' These solutions give the Yang–Mills–Higgs fields in 2+1-dimensional anti-de Sitter space–time. Some examples in the SU(2) case are considered and qualitative asymptotic behaviors of the solutions as $t\to\infty$ are discussed in detail. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1337799]

I. INTRODUCTION

The Yang–Mills–Higgs fields satisfying the Bogomolny equations in $\mathbf{R}^3$ and $\mathbf{R}^{2,1}$ were widely investigated and the equations are known to be integrable. On the other hand, the Yang–Mills–Higgs fields satisfying the Bogomolny equations in some curved spaces such as the hyperbolic space H^3, the 2+1-dimensional anti-de Sitter space–time are also integrable.[1–3] In the present paper, we consider the solutions of the Bogomolny equation in the 2+1-dimensional anti-de Sitter space–time, the Lax pair of which has been known and the soliton solutions (with constant spectral parameters) were also obtained.[3]

With the Darboux transformation method, we obtain exact multisoliton solutions. Moreover, the ''spectral parameters'' in the construction of Darboux transformation can depend on the space–time variables as in some other problems with dimensions $\geqslant 3$, like the self-dual Yang–Mills equation, modified principal chiral field, the Bolgomolny equation in $\mathbf{R}^3$, etc.[4–8] These kinds of equations are also called breaking soliton equations, since the spectral parameter may not be constant and satisfies an equation whose solution can blow up at finite time.[9,10]

In Secs. II and III the Darboux transformations for $\mathrm{GL}(N,\mathbf{C})$ and $\mathrm{U}(N)$ cases are discussed. As a special case, the general construction of soliton solutions is given in Sec. IV. Then, in Sec. V, some examples of single solitons and multisolitons are considered, with both constant spectral parameters and variable ''spectral parameters.'' Their qualitative asymptotic behavior as $t\to\infty$ is discussed in detail. When the spectral parameters are constants, we find solutions globally defined on the whole 2+1-dimensional anti-de Sitter space–time. When the ''spectral parameters'' are not constants, the solutions derived here are only locally defined in 2+1-dimensional anti-de Sitter space–time.

Our problem is as follows.

Let M be a three-dimensional Lorentz manifold with metric g. A_μ is a gauge potential and Φ is a (scalar) Higgs field, both of which are valued in the Lie algebra of a Lie group G. Hereafter, we always suppose G is a matrix Lie group and the matrices in G are of order N.

The 2+1-dimensional anti-de Sitter space–time is the universal covering space of the hyperboloid

$$U^2+V^2-X^2-Y^2=1 \tag{1.1}$$

in $\mathbf{R}^{2,2}$ with the metric

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$$\mathrm{d}s^2 = -\mathrm{d}U^2 - \mathrm{d}V^2 + \mathrm{d}X^2 + \mathrm{d}Y^2. \tag{1.2}$$

Define

$$r = \frac{1}{U+X}, \quad x = \frac{Y}{U+X}, \quad t = -\frac{V}{U+X}, \tag{1.3}$$

then a part of the $2+1$-dimensional anti-de Sitter space–time with $U+X>0$ is represented by the Poincaré coordinates (r,x,t) with $r>0$ and the metric is

$$\mathrm{d}s^2 = r^{-2}(-\mathrm{d}t^2 + \mathrm{d}r^2 + \mathrm{d}x^2) = r^{-2}(\mathrm{d}r^2 + \mathrm{d}u\,\mathrm{d}v), \tag{1.4}$$

where $u = x+t$, $v = x-t$.

The Yang–Mills–Higgs field in $2+1$-dimensional anti-de Sitter space–time satisfies the Bogomolny equation[1,11]

$$D\Phi = *F, \tag{1.5}$$

or, written in terms of the components,

$$D_\mu \Phi = \frac{1}{2\sqrt{|g|}} g_{\mu\nu} \epsilon^{\nu\alpha\beta} F_{\alpha\beta}, \tag{1.6}$$

where the action of the covariant derivative $D_\mu = \partial_\mu + A_\mu$ on Φ is $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$, $\partial_\mu = \partial/\partial x^\mu$. $\{F_{\mu\nu}\}$ is the curvature corresponding to $\{A_\mu\}$, $F_{\mu\nu} = [D_\mu, D_\nu]$.

With the Poincaré coordinates (1.3), (1.6) becomes

$$D_u \Phi = r F_{ur}, \quad D_v \Phi = -r F_{vr}, \quad D_r \Phi = -2 r F_{uv}. \tag{1.7}$$

It was proposed in Ref. 3 that this system of nonlinear partial differential equations had a Lax pair

$$\begin{gathered} (r D_r + \Phi - 2(\zeta - u) D_u)\psi = 0, \\ \left(2 D_v + \frac{\zeta - u}{r} D_r - \frac{\zeta - u}{r^2}\Phi\right)\psi = 0, \end{gathered} \tag{1.8}$$

where $D_\mu \psi = \partial_\mu \psi + A_\mu \psi$ and ζ is a complex spectral parameter. That is, (1.7) is the integrability condition of the overdetermined system (1.8).

II. DARBOUX TRANSFORMATION IN GL(N,C) CASE

In this section, we consider the case $G = \mathrm{GL}(N,\mathbf{C})$. This is the simplest case because no reduction should be considered. Let

$$\tilde{\psi} = (\zeta - u) R \psi - T \psi, \tag{2.1}$$

where $R(u,v,r)$ and $T(u,v,r)$ are $N \times N$ matrices and R is invertible, then the transformation $\psi \to \tilde{\psi}$ is called a Darboux transformation (of degree one) if there are $(\tilde{A}_\mu, \tilde{\Phi})$ such that

$$\begin{gathered} (r \tilde{D}_r + \tilde{\Phi} - 2(\zeta - u) \tilde{D}_u)\tilde{\psi} = 0, \\ \left(2 \tilde{D}_v + \frac{\zeta - u}{r} \tilde{D}_r - \frac{\zeta - u}{r^2}\tilde{\Phi}\right)\tilde{\psi} = 0 \end{gathered} \tag{2.2}$$

hold. Here $\tilde{D}_\mu = \partial_\mu + \tilde{A}_\mu$.

We should first determine R and T so that (2.1) is a Darboux transformation. For given (A,Φ), $(\tilde{A},\tilde{\Phi})$ and arbitrary matrix function Q, let

$$\Delta_\mu Q = \partial_\mu Q + \tilde{A}_\mu Q - Q A_\mu, \qquad \delta Q = \tilde{\Phi} Q - Q\Phi. \tag{2.3}$$

Expressed in ψ, both equations of (2.2) are polynomials of ζ of degree two. The coefficients of the second, first, and zeroth order of ζ in the two equations of (2.2) lead to

$$\Delta_u R = 0, \quad r\Delta_r R + 2\Delta_u T + \delta R + 2R = 0, \qquad r\Delta_r T + \delta T = 0, \tag{2.4}$$

and

$$\Delta_r R - \frac{1}{r}\delta R = 0, \quad 2\Delta_v R - \frac{1}{r}\Delta_r T + \frac{1}{r^2}\delta T = 0, \qquad \Delta_v T = 0. \tag{2.5}$$

These two systems are equivalent to

$$\Delta_u R = 0, \tag{2.6}$$

$$\Delta_v T = 0, \tag{2.7}$$

$$\Delta_v R = \frac{1}{r}\Delta_r T, \tag{2.8}$$

$$\Delta_r R + \frac{1}{r}\Delta_u T + \frac{1}{r} R = 0, \tag{2.9}$$

$$\Delta_r R - \frac{1}{r}\delta R = 0, \tag{2.10}$$

$$\Delta_r T + \frac{1}{r}\delta T = 0. \tag{2.11}$$

$\tilde{A}_u$ and $\tilde{A}_v$ are solved from (2.6) and (2.7) as

$$\tilde{A}_u = R A_u R^{-1} - (\partial_u R) R^{-1}, \tag{2.12}$$

$$\tilde{A}_v = T A_v T^{-1} - (\partial_v T) T^{-1}, \tag{2.13}$$

while (2.10) and (2.11) lead to

$$\tilde{A}_r = \frac{1}{2}(TA_r - \partial_r T)T^{-1} + \frac{1}{2}(RA_r - \partial_r R)R^{-1} + \frac{1}{2r}(T\Phi T^{-1} - R\Phi R^{-1}), \tag{2.14}$$

$$\tilde{\Phi} = \frac{r}{2}(TA_r - \partial_r T)T^{-1} - \frac{r}{2}(RA_r - \partial_r R)R^{-1} + \frac{1}{2}(T\Phi T^{-1} + R\Phi R^{-1}). \tag{2.15}$$

Now let Z be an $N\times N$ matrix function of (u,v,r), H be a solution of

$$\begin{aligned} &r((\partial_r H)H^{-1}+A_r)+\Phi-2((\partial_u H)H^{-1}+A_u)S=0,\\ &2((\partial_v H)H^{-1}+A_v)+\frac{1}{r}((\partial_r H)H^{-1}+A_r)S-\frac{1}{r^2}\Phi S=0, \end{aligned} \tag{2.16}$$

where $S=HZH^{-1}-u$, then S satisfies

$$\begin{aligned} \partial_r S&=H\left(\partial_r Z-\frac{2}{r}(\partial_u Z)(Z-u)\right)H^{-1}+\frac{2}{r}S+\frac{2}{r}(\partial_u S)S-[A_r,S]+\frac{2}{r}[A_u,S]S-\frac{1}{r}[\Phi,S],\\ \partial_v S&=H\left(\partial_v Z+\frac{1}{2r}(\partial_r Z)(Z-u)\right)H^{-1}-\frac{1}{2r}(\partial_r S)S-[A_v,S]-\frac{1}{2r}[A_r,S]S+\frac{1}{2r^2}[\Phi,S]S. \end{aligned} \tag{2.17}$$

Remark 1: If Z is diagonal and $Z=\mathrm{diag}(\zeta_1,\ldots,\zeta_N)$, then $H=(h_1,\ldots,h_N)$ where h_i is a column solution of the Lax pair (1.8) with $\zeta=\zeta_i(u,v,r)$.

If $T=RS$, then (2.8) and (2.9) hold if and only if

$$\begin{aligned} &\partial_r Z-\frac{2}{r}(\partial_u Z)(Z-u)=0,\\ &\partial_v Z+\frac{1}{2r}(\partial_r Z)(Z-u)=0. \end{aligned} \tag{2.18}$$

Therefore, we have

Theorem 1: *Suppose $R(u,v,r)$ is an arbitrary invertible matrix function. If $Z(u,v,r)$ is an $N\times N$ matrix solution of (2.18) and H is a solution of (2.16) with $S=HZH^{-1}-u$, then $\psi\mapsto\tilde\psi=(\zeta-u)R\psi-RS\psi$ is a Darboux transformation for (1.8).*

If $Z=\mathrm{diag}(\zeta_1,\ldots,\zeta_N)$, then each ζ_i is a solution of

$$\begin{aligned} &\partial_r\zeta-\frac{2}{r}(\zeta-u)\partial_u\zeta=0,\\ &\partial_v\zeta+\frac{1}{2r}(\zeta-u)\partial_r\zeta=0. \end{aligned} \tag{2.19}$$

Apart from the constant solution, the general nonconstant solution of (2.19) is given implicitly by

$$v-\frac{r^2}{\zeta-u}=C(\zeta), \tag{2.20}$$

where C is an arbitrary holomorphic function. We still call ζ_i as a ''spectral parameter.'' However, here the ''spectral parameters'' ζ_i's in Z can be either constant or variable. In the latter case, they are given by (2.20).

Remark 2: Note that the spectral parameter ζ in the Lax pair is still a complex constant. Only ζ_i's in Z can depend on (u,v,r).

According to Theorem 1, we can get the exact solution of (1.7) from a known solution of (1.7) and the corresponding solution of the linear system (2.16). When Z is diagonal, solving (2.16) is equivalent to solving (1.8).

Theorem 1 gives the construction of Darboux transformations of degree one. The Darboux transformations of higher degrees can be obtained by the composition of several Darboux transformations of degree one.

III. DARBOUX TRANSFORMATION IN THE U(N) CASE

When $G=\mathrm{U}(N)$, the Lie algebra consists of all anti-Hermitian matrices. Hence $A_\mu^*=-A_\mu$, $\Phi^*=-\Phi$.

In order to construct Darboux transformation which keeps this reduction, some constraints on ζ_j's and h_j's should be added.

Suppose ψ is a solution of (1.8), ϕ is a solution of (1.8) with $\zeta\to\bar{\zeta}$. Then

$$r\partial_r(\phi^*\psi)-2(\zeta-u)\partial_u(\phi^*\psi)=0,$$
$$2\partial_v(\phi^*\psi)+\frac{\zeta-u}{r}\partial_r(\phi^*\psi)=0. \tag{3.1}$$

It is uniquely solvable for a given initial value of $\phi^*\psi$ at $r=r_0>0$ and $v=v_0$. Hence if $\phi^*\psi|_{r=r_0,v=v_0}=0$, then $\phi^*\psi=0$ identically for $r>0$.

Let ζ_0 be a constant number or a nonconstant solution of (2.19). Take $Z=\mathrm{diag}(\zeta_1,...,\zeta_N)$ with $\zeta_j=\zeta_0$ or $\bar{\zeta}_0$, $H=(h_1,\ldots,h_N)$ where h_j is a column solution of (1.8) with $\zeta=\zeta_j$ such that $\det H\neq 0$ and $h_i^*h_j=0$ for $\zeta_i=\bar{\zeta}_j$. Then, the Darboux transformation given by Theorem 1 keeps the U(N) reduction. That is, $\tilde{A}_\mu^*=-\tilde{A}_\mu$, $\tilde{\Phi}^*=-\tilde{\Phi}$. This is proved similarly to the U(N) reduction for other systems like the AKNS system.

Darboux transformation of higher degree can be obtained by composition of Darboux transformations of degree one. However, when $G=\mathrm{U}(N)$, there is the following special and more explicit construction.

Let $\zeta^{(i)}$ $(i=1,\ldots,r)$ be constant numbers or nonconstant solutions of (2.19), $h^{(i)}$ be a column solution of (1.8) with $\zeta=\zeta^{(i)}$. Consider the composition of r Darboux transformations of degree one. In the ith Darboux transformation, let

$$Z=Z^{(j)}\equiv\mathrm{diag}(\zeta^{(j)},\overline{\zeta^{(j)}},\ldots,\overline{\zeta^{(j)}}),\quad H=H^{(j)}\equiv(h_1^{(j)},\ldots,h_N^{(j)}), \tag{3.2}$$

where $h_1^j=h^{(j)}$ and $h_k^{(j)}$ $(k=2,3,\ldots,n)$ are solutions of (1.8) with $\zeta=\overline{\zeta^{(j)}}$ and satisfy $h_k^{(j)*}h^{(j)}=0$. In this case, the Darboux transformation of degree r can be constructed in the following way, which does not depend on $h_k^{(j)}$ $(j=1,2,\ldots,r;\ k=2,3,\ldots,n)$.

Let

$$\Gamma_{ij}=\frac{h^{(i)*}h^{(j)}}{\overline{\zeta^{(i)}}-\zeta^{(j)}}, \tag{3.3}$$

then $G=(G_{ij})$ with

$$G_{ij}=\prod_{j=1}^{r}(\zeta-\overline{\zeta^{(j)}})\left(1+\sum_{i,j=1}^{r}\frac{h^{(i)}(\Gamma^{-1})_{ij}h^{(j)*}}{\zeta-\overline{\zeta^{(j)}}}\right) \tag{3.4}$$

is a Darboux matrix for (1.8).[6,7]

IV. SOLITON SOLUTIONS

Soliton solutions are obtained in the following way.

Take seed solution $A_\mu=0$ $(\mu=u,v,r)$, $\Phi=0$. Considering the gauge equivalence in (2.12), we can always choose $R=1$ and $T=S$. From (2.12)–(2.15), (2.17), and (2.18), we have

$$\tilde{A}_u=0,\quad \tilde{A}_v=-(\partial_v S)S^{-1}=\frac{1}{2r}\partial_r S, \tag{4.1}$$

$$\tilde{A}_r = -\frac{1}{2}(\partial_r S)S^{-1} = -\frac{1}{r}(\partial_u S + 1), \quad \tilde{\Phi} = -\frac{r}{2}(\partial_r S)S^{-1} = -\partial_u S - 1,$$

and

$$\tilde{F}_{uv} = [\tilde{D}_u, \tilde{D}_v] = \frac{1}{2r}\partial_u \partial_r S,$$

$$\tilde{F}_{ur} = [\tilde{D}_u, \tilde{D}_r] = -\frac{1}{r}\partial_u \partial_u S, \tag{4.2}$$

$$\tilde{F}_{vr} = [\tilde{D}_v, \tilde{D}_r] = -\frac{1}{2r}(\partial_r \partial_r + 2\partial_u \partial_v)S + \frac{1}{2r^2}\partial_r S - \frac{1}{2r^2}[\partial_r S, \partial_u S].$$

Here we always suppose Z is diagonal with $Z = \mathrm{diag}(\zeta_1, \ldots, \zeta_N)$. Then the corresponding h_i's are solved from (1.8) explicitly.

Case 1: ζ_i is a constant.

Then h_i satisfies

$$r\partial_r h_i - 2(\zeta_i - u)\partial_u h_i = 0,$$
$$2\partial_v h_i + \frac{\zeta_i - u}{r}\partial_r h_i = 0. \tag{4.3}$$

Hence

$$h_i = f_i(\omega(\zeta_i)), \tag{4.4}$$

where f_i is an arbitrary holomorphic vector function of $\omega(\zeta_i)$ and

$$\omega(\zeta) = v - \frac{r^2}{\zeta - u}. \tag{4.5}$$

Case 2: ζ_i is not a constant.

According to (2.20), ζ_i satisfies

$$v - \frac{r^2}{\zeta_i - u} = C_i(\zeta_i), \tag{4.6}$$

where C_i is an arbitrary holomorphic function. h_i should be a solution of (4.3) with this ζ_i, which is

$$h_i = f_i(\zeta_i), \tag{4.7}$$

where f_i is an arbitrary holomorphic vector function. The Darboux transformation is also given by $S = HZH^{-1} - u$ with $H = (h_1, \ldots, h_N)$ when $\det H \neq 0$.

Multisolitons can be obtained by the composition of Darboux transformations of degree one or by (3.3) and (3.4) directly in $\mathrm{U}(N)$ case.

V. EXAMPLES FOR SU(2) CASE

Now we consider the soliton solutions for the simplest non-Abelian group $G = \mathrm{SU}(2)$.

A. Single soliton solutions with constant spectral parameter

Take ζ_0 to be a complex constant which is not real, $Z=\mathrm{diag}(\zeta_0,\overline{\zeta_0})$. Let $\tau=\omega(\zeta_0)$, then

$$H=\begin{pmatrix}\alpha(\tau) & -\overline{\beta(\tau)}\\ \beta(\tau) & \overline{\alpha(\tau)}\end{pmatrix}, \tag{5.1}$$

where α, β are two holomorphic functions. Let $\sigma(\tau)=\beta(\tau)/\alpha(\tau)$, then

$$S=\frac{\zeta_0-\overline{\zeta_0}}{1+|\sigma|^2}\begin{pmatrix}1 & \overline{\sigma}\\ \sigma & |\sigma|^2\end{pmatrix}+\overline{\zeta_0}-u, \tag{5.2}$$

$$\tilde{\Phi}=-\partial_u S-1=\frac{\zeta_0-\overline{\zeta_0}}{(1+|\sigma|^2)^2}\begin{pmatrix}(|\sigma|^2)_u & \overline{\sigma}^2\sigma_u-\overline{\sigma}_u\\ \sigma^2\overline{\sigma}_u-\sigma_u & -(|\sigma|^2)_u\end{pmatrix}, \tag{5.3}$$

and

$$-\mathrm{tr}\,\tilde{\Phi}^2=\frac{8(\mathrm{Im}\,\zeta_0)^2}{(1+|\sigma|^2)^2}|\partial_u\sigma|^2. \tag{5.4}$$

According to (1.1) and (4.5),

$$\tau=\frac{\zeta_0(Y+V)(U+X)-1-Y^2+V^2}{(\zeta_0(U+X)-Y+V)(U+X)}=\frac{\zeta_0(Y+V)+X-U}{\zeta_0(U+X)-Y+V}. \tag{5.5}$$

Denote

$$\xi=\zeta_0(Y+V)+X-U,\qquad \eta=\zeta_0(X+U)-Y+V, \tag{5.6}$$

then both ξ and η cannot be zero anywhere on (1.1) when ζ_0 is not real. Hence τ is a smooth function of U,V,X,Y on (1.1). Moreover,

$$\partial_u\tau=-\frac{r^2}{(\zeta_0-u)^2}=-\frac{1}{\eta^2}. \tag{5.7}$$

Since $\sigma(\tau)$ is a meromorphic function of τ, suppose $\sigma(\tau)=\sigma_1(\tau)/\sigma_2(\tau)$ where $\sigma_1(\tau)$ and σ_2 are two holomorphic functions of τ without common zero. According to (5.4),

$$-\mathrm{tr}\,\tilde{\Phi}^2=\frac{8(\mathrm{Im}\,\zeta_0)^2|\sigma_2(\tau)\partial_\tau\sigma_1(\tau)-\sigma_1(\tau)\partial_\tau\sigma_2(\tau)|^2}{(|\sigma_1(\tau)|^2+|\sigma_2(\tau)|^2)^2}|\eta|^{-4}. \tag{5.8}$$

Hence, $\tilde{\Phi}$ can be extended smoothly to (1.1). Likewise, according to (4.1),

$$\begin{aligned}&-\mathrm{tr}\,\tilde{A}_u^2=0,\quad -\mathrm{tr}\,\tilde{A}_v^2=\frac{8(\mathrm{Im}\,\zeta_0)^2(|\sigma_2(\tau)\partial_\tau\sigma_1(\tau)-\sigma_1(\tau)\partial_\tau\sigma_2(\tau)|^2}{(|\sigma_1(\tau)|^2+|\sigma_2(\tau)|^2)^2}(U+X)^2|\eta|^{-2},\\ &-\mathrm{tr}\,\tilde{A}_r^2=\frac{8(\mathrm{Im}\,\zeta_0)^2(|\sigma_2(\tau)\partial_\tau\sigma_1(\tau)-\sigma_1(\tau)\partial_\tau\sigma_2(\tau)|^2}{(|\sigma_1(\tau)|^2+|\sigma_2(\tau)|^2)^2}(U+X)^2|\eta|^{-4}.\end{aligned} \tag{5.9}$$

Therefore, the solution $(\tilde{\Phi},\tilde{A}_u,\tilde{A}_v,\tilde{A}_r)$ is smooth on (1.1), hence is smooth on the whole $2+1$-dimensional anti-de Sitter space–time.

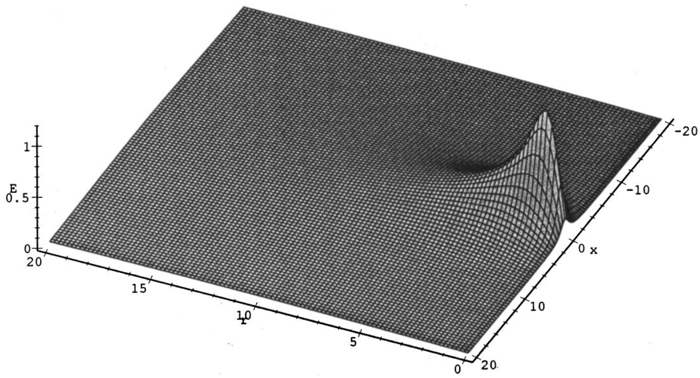

FIG. 1.

The infinity of the $2+1$-dimensional anti-de Sitter space–time includes only $r\to 0$. However, for the parameter space (t,r,x) $(r>0)$ with fixed t, the infinity of the derived half plane contains $r\to 0$ and $r^2+x^2\to\infty$. Here we call a solution localized if $-\operatorname{tr}\tilde{\Phi}^2\to 0$ when $r\to 0$ or $r^2+x^2\to\infty$ for fixed t.

Example 1: If $\zeta_0=\mathrm{i}$, $\tau=\omega(\zeta_0)$, $\sigma(\tau)=\tau$, this is just the localized solution (25) of Ref. 3, and

$$-\operatorname{tr}\tilde{\Phi}^2=\frac{8r^4}{((r^2+x^2-t^2)^2+2x^2+2t^2+1)^2}.$$

Let

$$x=tR\cos\theta,\quad r=tR\sin\theta. \tag{5.10}$$

When t and θ are fixed, $-\operatorname{tr}\tilde{\Phi}^2$ is a function of R only. Its maximum appears at $R=\pm\sqrt{t^2+1}/t$. Hence as $t\to\infty$, the ridge of the solution locates on the circle $x^2+r^2=t^2+1$.

Figures 1 and 2 describe this soliton at $t=0$ and $t=10$, respectively. In Figs. 1 and 2, the vertical axis is $(-\operatorname{tr}\tilde{\Phi}^2)^{1/4}$.

Example 2: $\tau=\omega(\zeta_0)$, $\sigma(\tau)$ is a polynomial of τ of degree k $(k\geqslant 1)$.

If $r\to 0$, then $\tau\to v$, $\partial_v\tau\to 1$ and all the other derivatives of τ (including derivatives of higher orders) tend to zero. According to (5.4), $-\operatorname{tr}\tilde{\Phi}^2\to 0$.

If $r^2+x^2\to\infty$, then (for fixed t)

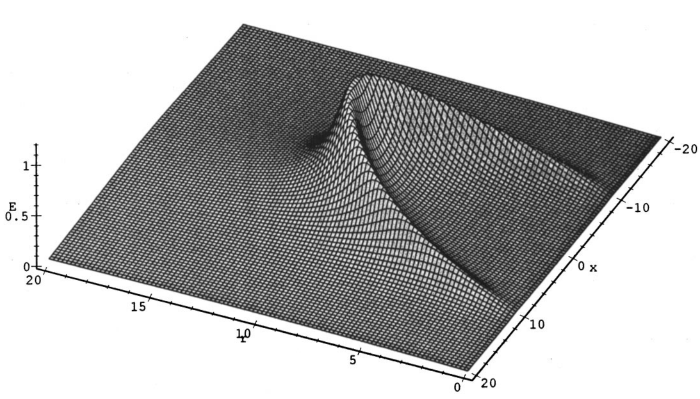

FIG. 2.

$$\tau=\frac{\zeta_0(x-t)-(r^2+x^2-t^2)}{\zeta_0-x-t}\to\infty. \tag{5.11}$$

Moreover, when t fixed, r^2+x^2 and $|\tau|$ are large enough, we have the estimates

$$\begin{aligned}\frac{|\tau|^{k+1}|\partial_\tau\sigma|}{1+|\sigma|^2}&\leqslant C_1\frac{|\tau|^{k+1}|\tau|^{k-1}}{1+|\tau|^{2k}}\leqslant C_1,\\ \left|\frac{\partial_u\tau}{\tau^2}\right|(r^2+x^2)&=\frac{r^2}{|\zeta_0(x-t)-(r^2+x^2-t^2)|^2}(r^2+x^2)\leqslant C_2,\end{aligned} \tag{5.12}$$

where C_1 and C_2 are independent of x and r, but may depend on ζ_0 and t. According to (5.4),

$$-\operatorname{tr}\tilde{\Phi}^2\leqslant\frac{8(\operatorname{Im}\zeta_0)^2C_1^2C_2^2}{|\tau|^{2k-2}}\frac{1}{(x^2+r^2)^2}\to 0. \tag{5.13}$$

Hence the solution is also localized whenever $\sigma(\tau)$ is a nonconstant polynomial of τ.

Now we consider the asymptotic behavior of the solution as $t\to\infty$ for $\zeta_0=i$. The following discussion in this example is qualitative and not rigorous.

Suppose all the roots of $\sigma(\tau)$ are simple roots. Denote $E=-\operatorname{tr}\tilde{\Phi}^2$. By (5.4), when τ is near a root of $\sigma(\tau)$, E may be large. Hence when τ is near a root of $\sigma(\tau)$, perhaps there will be a ridge in the graph of E.

From (4.5), the real and imaginary parts of $\tau=\omega(i)$ are

$$\begin{aligned}\operatorname{Re}\tau&=\frac{x-t+(x+t)(r^2+x^2-t^2)}{1+(x+t)^2},\\ \operatorname{Im}\tau&=\frac{r^2}{1+(x+t)^2}.\end{aligned} \tag{5.14}$$

When t is large and $x+t$ is not very small,

$$\operatorname{Re}\tau\approx\frac{r^2+x^2-t^2}{x+t}. \tag{5.15}$$

For a root ρ of $\sigma(\cdot)$, the points with

$$\frac{r^2+x^2-t^2}{x+t}=\operatorname{Re}\rho \tag{5.16}$$

are on the circle

$$C:\quad r^2+(x-\tfrac{1}{2}\operatorname{Re}\rho)^2=(t+\tfrac{1}{2}\operatorname{Re}\rho)^2. \tag{5.17}$$

On this circle C, for fixed t and $\operatorname{Re}\rho$, $\operatorname{Im}\tau$ can be expressed by x as

$$\operatorname{Im}\tau=\frac{t^2-x^2+\operatorname{Re}\rho(x+t)}{1+(x+t)^2}. \tag{5.18}$$

By computing $(d/dx)\operatorname{Im}\tau$, we know that $\operatorname{Im}\tau$ decreases when x increases if $x\geqslant -t+1$ and $t\geqslant -\operatorname{Re}\rho/2$. Hence it is easy to derive that $|\operatorname{Im}\tau|\leqslant 2$ when $t>|\operatorname{Re}\rho|$ and $x\geqslant 0$. Therefore, when $|\operatorname{Im}\rho|$ is not large, there will be a ridge of E on C.

When $|\operatorname{Im}\rho|\geqslant 1$, E is large on C only when $\operatorname{Im}\tau\approx\operatorname{Im}\rho$. If t is large and $x+t$ is not very small,

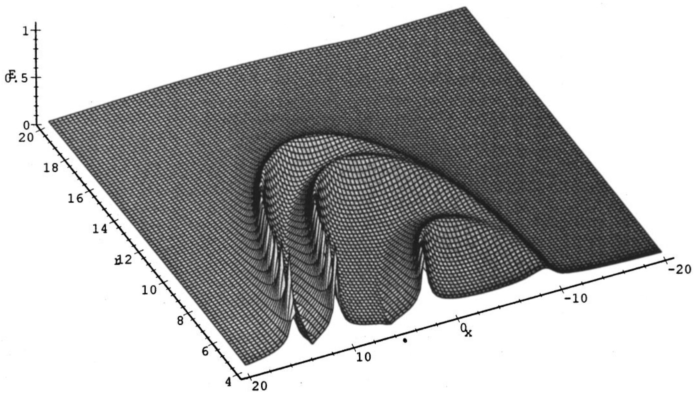

FIG. 3.

$$\mathrm{Im}\,\tau \approx \frac{t-x+\mathrm{Re}\,\rho}{x+t}. \tag{5.19}$$

The equation

$$\frac{t-x+\mathrm{Re}\,\rho}{x+t} = \mathrm{Im}\,\rho \tag{5.20}$$

has a unique solution

$$x = \frac{-(\mathrm{Im}\,\rho-1)t+\mathrm{Re}\,\rho}{\mathrm{Im}\,\rho+1}. \tag{5.21}$$

If $\mathrm{Im}\,\rho \gg 1$, then when t is large enough, x satisfies $-t \leqslant x \leqslant t+\mathrm{Re}\,\rho$. Hence there exists unique $r>0$ such that $(x,r)\in C$. This means that when $\mathrm{Im}\,\rho \gg 1$, there will be a peak rather than a ridge. If $\mathrm{Im}\,\rho \ll -1$, then when t is large enough, $x<-t$. Hence there does not exist $r>0$ such that $(x,r)\in C$, that is, there is neither ridge nor peak in the graph of E.

The above-presented discussion on the graph of E is summarized as follows. As $t\to\infty$, a root ρ with $|\mathrm{Im}\,\rho| \ll 1$ corresponds to a ridge, a root ρ with $\mathrm{Im}\,\rho \gg 1$ corresponds to a peak, and a root ρ with $\mathrm{Im}\,\rho \ll -1$ corresponds to nothing.

Figure 3 ($t=10$) shows the solution for

$$\sigma(\tau) = (\tau-2)(\tau-6)(\tau+6), \tag{5.22}$$

which has three real roots. It is plotted for $r \geqslant 4$ because the ridge is perpendicular to $r=0$ and the figures cannot be plotted well near $r=0$. Figure 4 shows its local behavior for a part of the region with $0 \leqslant r \leqslant 4$.

Figure 5 ($t=10$) shows the solution for

$$\sigma(\tau) = (\tau-2)(\tau-6)(\tau+6)(\tau-2i)(\tau-6i)(\tau+6i), \tag{5.23}$$

which has three real roots and three purely imaginary roots, but one imaginary root has the negative imaginary part. The solution has three ridges and two peaks.

Figure 6 ($t=10$) shows the solution for

$$\sigma(\tau) = (\tau-2-2i)(\tau-6-6i)(\tau+6-4i), \tag{5.24}$$

which has no real roots. In Fig. 6, there are three peaks.

Figure 7 ($t=10$) shows the solution for

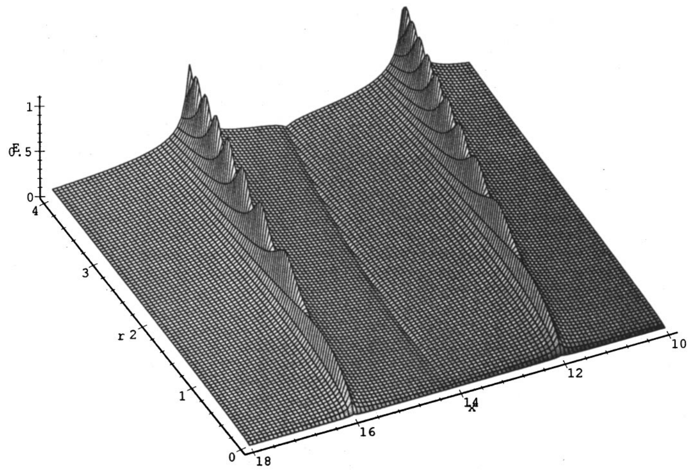

FIG. 4.

$$\sigma(\tau)=(\tau-2-2i)(\tau-6-6i)(\tau+6+6i), \tag{5.25}$$

which has no real roots, and only two roots have positive imaginary parts. In Fig. 7, there are only two peaks.

In all these figures (Figs. 3–7), the vertical axis is $(-\mathrm{tr}\,\tilde{\Phi}^2)^{1/8}$.

Example 3: $\zeta_0=i,\ \tau=\omega(i),\ \sigma(\tau)=\sin(\mu\tau)$ where μ is a real constant.

If $r\to 0$, then $\tau\to v$, hence $E\to 0$. When the point $(x,r)\to\infty$ along the straight line $r=kx+b$ $(k,b$ are real constants), (5.14) gives

$$\begin{aligned}\mathrm{Im}\,\tau&=\frac{(kx+b)^2}{1+(x+t)^2}\to k^2,\\ \mathrm{Re}\,\tau&=\frac{x-t+(x+t)((kx+b)^2+x^2-t^2)}{1+(x+t)^2}\sim(k^2+1)x.\end{aligned} \tag{5.26}$$

Denote $\mu\tau=p+qi$ where p and q are real, then

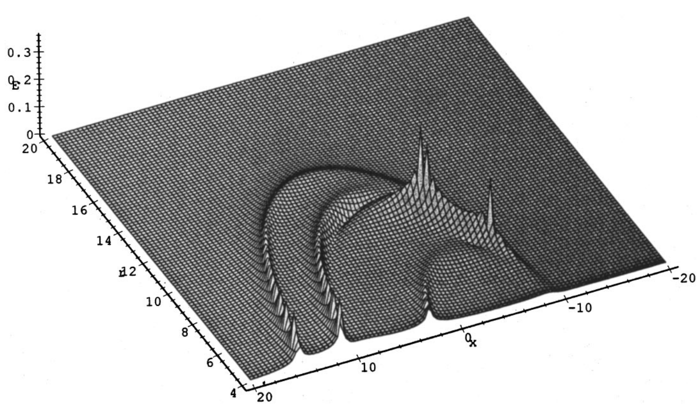

FIG. 5.

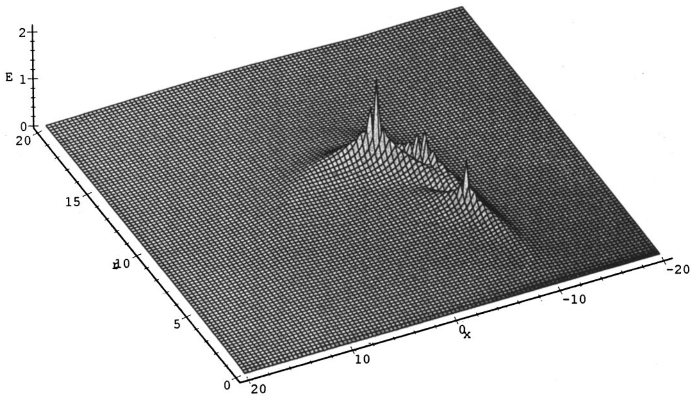

FIG. 6.

$$E=\frac{16\mu^2(\cosh(2q)+\cos(2p))}{(\cosh(2q)-\cos(2p)+2)^2}\,\frac{r^4}{(1+(x+t)^2)^2}\sim\frac{16\mu^2k^4(\cosh(2\mu k^2)+\cos(2\mu(k^2+1)x))}{(\cosh(2\mu k^2)-\cos(2\mu(k^2+1)x)+2)^2} \tag{5.27}$$

as $x^2+r^2\to\infty$. Hence the solution is bounded, but not localized in our definition [on the half (r,x,t) space]. However, as is shown, E tends to zero at the infinity of the $2+1$-dimensional anti-de Sitter space–time $(r=0)$.

This solution is shown in Fig. 8 $(t=10)$.

B. Single soliton solutions with nonconstant ''spectral parameter''

In this case, ζ_0 should satisfy

$$v-\frac{r^2}{\zeta_0-u}=C(\zeta_0). \tag{5.28}$$

S, $\tilde{\Phi}$ are given by (5.2) and (5.3), and $-\operatorname{tr}\tilde{\Phi}^2$ is given by (5.4). However, in these expressions, ζ_0 is no longer a constant.

Contrary to the case where ζ_0 is a constant, here the solutions are defined only on the half (r,x,t) space. In general, they cannot be extended to the whole $2+1$-dimensional anti-de Sitter space–time.

Example 4: $C(\zeta_0)=C_0$ (constant), then

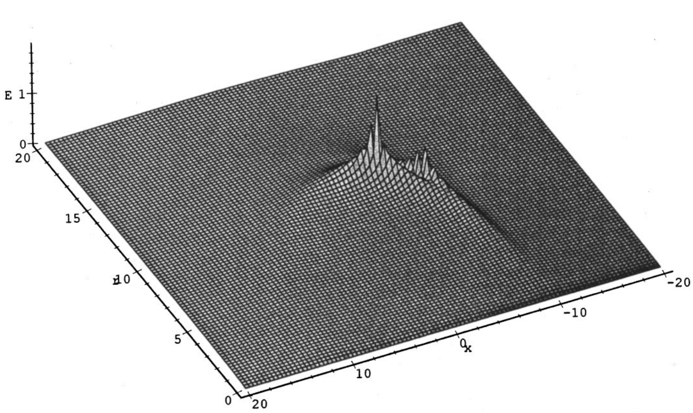

FIG. 7.

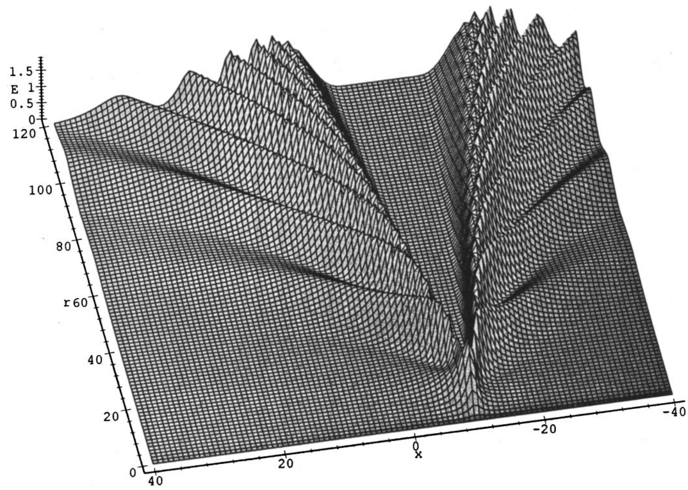

FIG. 8.

$$\zeta_0 = u - \frac{r^2}{C_0 - v}. \tag{5.29}$$

Let

$$H = \begin{pmatrix} \alpha(\zeta_0) & -\overline{\beta(\zeta_0)} \\ \beta(\zeta_0) & \overline{\alpha(\zeta_0)} \end{pmatrix},$$

where α and β are holomorphic functions of ζ_0 and $\sigma = \beta(\zeta_0)/\alpha(\zeta_0)$, then the Darboux matrix S is also given by (5.2). For example, if $\sigma(\zeta) = \zeta$ and $C_0 = i$, then $-\operatorname{tr}\tilde{\Phi}^2$ is completely the same as that in Example 1.

Example 5: $C(\zeta_0) = \zeta_0 + C_0$ where $C_0 = \alpha + \beta i$, α and β are real constants with $\beta \neq 0$, $\sigma(\zeta_0) = \zeta_0$.

Then

$$\zeta_0^2 - (u + v - C_0)\zeta_0 + uv + r^2 - C_0 u = 0. \tag{5.30}$$

The criteria of this quadratic equation is

$$\begin{aligned} \Delta &= (u + v - C_0)^2 - 4(uv + r^2 - C_0 u) \\ &= (u - v + C_0)^2 - 4r^2 \\ &= (2t + \alpha)^2 - 4r^2 - \beta^2 + 2\beta(2t + \alpha)i. \end{aligned} \tag{5.31}$$

When $t > -\alpha/2$, the imaginary part of Δ is never zero. Hence we can choose

$$\zeta_0 = \frac{u + v - C_0 + \sqrt{(u - v + C_0)^2 - 4r^2}}{2}, \tag{5.32}$$

where the square root takes the specific branch in the upper half-plane.

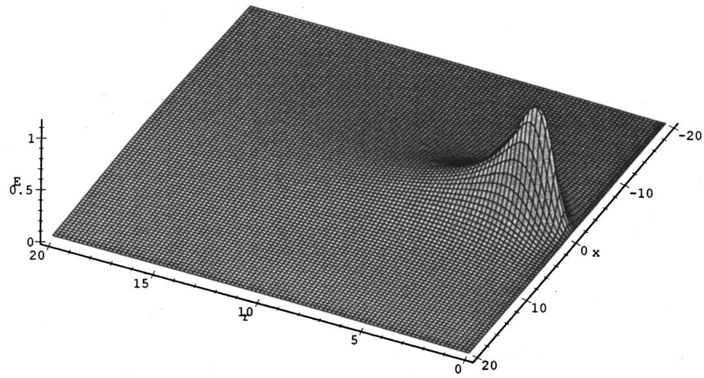

FIG. 9.

If $r \rightarrow 0$, then $\zeta_0 \rightarrow u$, $\partial_u \zeta_0 \rightarrow 1$ and all the other derivatives of ζ_0 tend to zero. Hence by (5.4), $-\operatorname{tr}\tilde{\Phi}^2 \rightarrow 0$.

If $x^2 + r^2 \rightarrow \infty$ (t fixed),

$$\frac{|\zeta_0|}{\sqrt{x^2+r^2}} = \frac{1}{\sqrt{x^2+r^2}} \left| \frac{2x - C_0 + \sqrt{(2t+C_0)^2 - 4r^2}}{2} \right| \rightarrow 1,$$

$$|\partial_u \zeta_0| = \frac{1}{2} \left| 1 + \frac{2t + C_0}{\sqrt{(2t+C_0)^2 - 4r^2}} \right| \leqslant C_3, \quad \frac{|\operatorname{Im} \zeta_0|}{r} \leqslant 2, \tag{5.33}$$

where C_3 is independent of x and r, but may depend on t, α, and β. According to (5.4) for $\sigma(\zeta_0) = \zeta_0$, $-\operatorname{tr}\tilde{\Phi}^2 \rightarrow 0$ as $x^2 + r^2 \rightarrow \infty$. Hence the solution is also localized. However, it cannot be extended to the whole $2+1$-dimensional anti-de Sitter space–time smoothly because of the condition $t > -\alpha/2$.

This soliton is shown in Fig. 9 ($t=1$) and Fig. 10 ($t=10$) for $\alpha=0$ and $\beta=2$.

Example 6: $C(\zeta_0) = \zeta_0 + C_0$, $C_0 = \alpha + \beta i$ ($\beta \neq 0$) as previously, $\sigma(\zeta_0)$ is a polynomial of ζ_0 of degree k.

Then similar to (5.12), we have the estimates

$$\frac{|\zeta_0|^{k+1} |\partial_{\zeta_0} \sigma|}{1 + |\sigma|^2} \leqslant C_1 \frac{|\zeta_0|^{k+1} |\zeta_0|^{k-1}}{1 + |\zeta_0|^{2k}} \leqslant C_1, \tag{5.34}$$

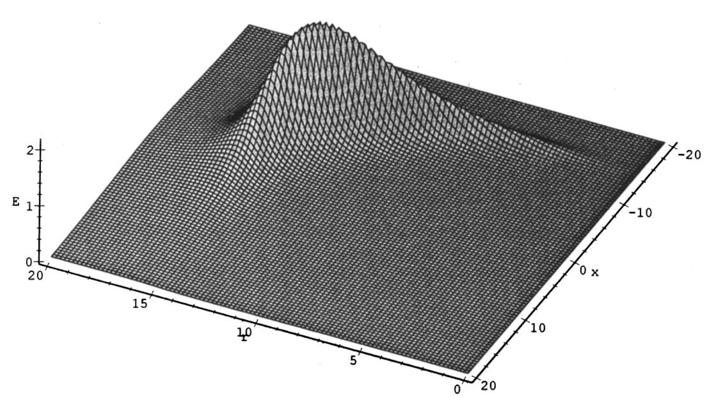

FIG. 10.

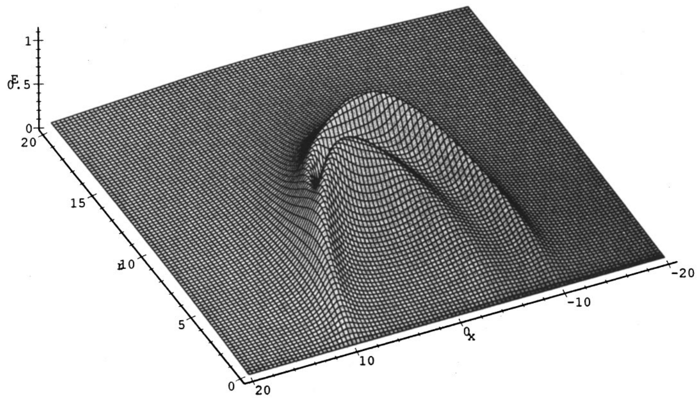

FIG. 11.

$$\left|\frac{\partial_u \zeta_0}{\zeta_0^2}\right|(r^2+x^2)\leqslant 2C_3$$

as $x^2+r^2\to\infty$ by (5.33). From (5.4), when $x^2+r^2\to\infty$,

$$-\operatorname{tr}\tilde{\Phi}^2\leqslant\frac{128C_1^2C_3^2}{|\zeta_0|^{2k-2}}\,\frac{r^2}{(r^2+x^2)^2}\to 0. \tag{5.35}$$

The solution is localized. As in the last example, this solution cannot be extended to the whole $2+1$-dimensional anti-de Sitter space–time.

C. Double soliton solutions

Double soliton solutions are obtained by Darboux transformations of degree two. Here we only show the following simple case for constant spectral parameters. In more complicated cases the double soliton solutions can also be derived similarly.

Example 7: Let $\zeta_1=i$, $\zeta_2=5+2i$, $\tau_j=\omega(\zeta_j)$, $\sigma_j(\tau_j)=\tau_j$ $(j=1,2)$. The double soliton is shown in Fig. 11 $(t=10)$.

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