Inverse Problems **14** (1998) 1353–1370. Printed in the UK PII: S0266-5611(98)91309-0

Darboux transformations for the twisted $so(p, q)$ system and local isometric immersion of space forms

Zixiang Zhou†

Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

Received 2 February 1998

Abstract. For the n-dimensional integrable system with a twisted $so(p, q)$ reduction, Darboux transformations given by Darboux matrices of degree two are constructed explicitly. These Darboux transformations are applied to the local isometric immersion of space forms with flat normal bundle and linearly-independent curvature normals to give the explicit expression of the position vector. Some examples are given from the trivial solutions and standard imbedding $T^n \to \mathbb{R}^{2n}$.

1. Introduction

The theory of integrable systems has been widely used to study some differential geometric problems such as minimal submanifolds, submanifolds with constant mean curvature, harmonic maps, etc. In particular, the isometric immersion of space form $M_1(K_1)$ of curvature K_1 into space form $M_2(K_2)$ of curvature K_2 has been studied in various papers. If $K_1 \neq K_2$, the nonlinear wave equation and the nonlinear sine–Gordon equation are considered to describe some special problems [1–3, 10, 16]. When $K_1 = K_2$, the local isometric immersion with flat normal bundle and linearly-independent curvature normals was proposed in [17]. That problem has also been dealt with in a purely geometric way [5, 6].

In this paper, we use the Darboux transformation to obtain the explicit expressions of the local isometric immersions of the space forms of the same curvature with flat normal bundle and linearly-independent curvature normals.

In this problem, the Lax pair has a twisted $so(n)$ reduction. When the Lie algebra is $gl(n, \mathbb{C})$ or $sl(n, \mathbb{C})$, there is a systematic construction of Darboux transformations (for the problems discussed later, see [8, 9, 13, 15]), which is now a useful method to obtain explicit solutions of nonlinear integrable partial differential equations. If the Lie algebra is $su(p, q)$, there is also a general algorithm to choose the spectral parameters [12–14, 19] for the Darboux transformation of degree one. In this algorithm, the spectral parameters can take only two mutually conjugate values. As a subalgebra of $su(p, q)$ ($p + q$ even), the $so(p, q)$ ($p + q$ even) problem can be dealt with in a similar way, provided that the real condition can be realized. However, for the $so(p, q)$ ($p + q$ odd) problem, this method is not applicable directly [4, 11] since the spectral parameters should be conjugate and cannot be real (or purely imaginary if written in another way) so that the Darboux transformation is not trivial. It is known that the pure $so(p, q)$ problem can be dealt

† E-mail address: zxzhou@guomai.sh.cn

with by a Darboux transformation of degree two [7, 18]. Here, for the twisted $so(p, q)$ problem (the $so(p, q)$ problem with an additional involution condition), we construct such a Darboux transformation using a limit process so that the Darboux matrix has only two (not four) eigenvalues. For the twisted $so(p, q)$ system, the method in [18] can give the same Darboux matrix under some assumptions, but the method here is more direct and has no assumptions.

Section 2 describes the linear system and the properties of its solutions. In section 3, we obtain explicit expressions of the Darboux transformations of degree two for the twisted $so(p, q)$ reduction. Owing to the isomorphism between $su(1, 1)$ and $so(2, 1)$, we can see that the Darboux transformation for the MKdV and the sine–Gordon equations given by the Darboux matrix here is actually the well known standard Darboux transformation.

Using the above conclusions, section 4 gives the Darboux transformation for the local isometric immersion from $M_n(K)$ to $M_{2n}(K)$ with flat normal bundle and linearly-independent curvature normals, the Lax set of which was proposed in [17]. We present the general expression of the transformation for the position vector of $M_n(K) \to M_{2n}(K)$. Section 4 also gives some interesting examples, including the submanifolds derived from trivial solutions for all $K = 0, 1, -1$ cases and the submanifold derived from the standard torus T^n in $\mathbb{R}^{2n}$.

2. Linear system

Let

$$\mathfrak{g} = so_{\mathrm{ex}}(p, q, r) = \{X \in gl(p+q+r, \mathbb{R}) \mid X^{\mathrm{T}}C + CX = 0\} \tag{1}$$

where

$$C = I_{p,q,r} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}, \underbrace{0, \ldots, 0}_{r}).$$

Clearly, $so_{\mathrm{ex}}(p, q, 0) = so(p, q)$. Here we consider $so_{\mathrm{ex}}(p, q, r)$ instead of $so(p, q)$ for the unified treatment in section 4.

Let σ be a diagonal matrix such that $\sigma^2 = 1$ ($\sigma \neq 1$). Then the transformation $\theta : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto \sigma X \sigma$ is an involution on $\mathfrak{g}$. Hence there is a decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are the $+1$ and -1 eigenspaces of θ, respectively. Moreover, we suppose that there is a maximum commutative subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{g}_1$.

Let

$$\mathfrak{L}(\mathfrak{g}) = \left\{ \sum_{j=0}^{n} X_j \lambda^j \mid X_j \in \mathfrak{g}, j = 0, 1, \ldots, n \right\} \tag{2}$$

be a subalgebra of the loop algebra of $\mathfrak{g}$,

$$\mathfrak{L}^{\sigma}(\mathfrak{g}) = \{v \in \mathfrak{L}(\mathfrak{g}) \mid \sigma v(\lambda)\sigma = v(-\lambda)\} \tag{3}$$

be a subalgebra of $\mathfrak{L}(\mathfrak{g})$.

First, we consider the linear system

$$\Phi_x = U(x, \lambda)\Phi \tag{4}$$

where $U \in \mathfrak{L}^{\sigma}(\mathfrak{g})$. Denote

$$U(x, \lambda) = \sum_{j=0}^{n} U_j(x)\lambda^j \qquad U_j(x) \in \mathfrak{g}_{[j]} \qquad [j] = \begin{cases} 0 & j \text{ even} \\ 1 & j \text{ odd} \end{cases} \tag{5}$$

then $U \in \mathfrak{L}^\sigma(\mathfrak{g})$ if and only if $\sigma U(x,\lambda)\sigma = U(x,-\lambda)$, $(U(x,\lambda))^* C + CU(x,\bar{\lambda}) = 0$. In this paper, we always suppose $\Phi(x,\lambda)$, a solution of (4), is smooth with respect to both x and λ. To discuss the Darboux transformation for (4), we need the following lemmas.

Lemma 1. *If Φ is a solution of (4) with $\lambda = \lambda_0$, then $\sigma\Phi$ is a solution of (4) with $\lambda = -\lambda_0$.*

Lemma 2. *If Φ is a solution of (4) with $\lambda = \lambda_0$, Ψ is a solution of (4) with $\lambda = \bar{\lambda}_0$, then $(\Psi^* C\Phi)_x = 0$.*

Proof. The conclusion follows from

$$\begin{aligned} &\Phi_x = U(x,\lambda_0)\Phi \\ &\Psi_x^* C = -\Psi^* CU(x,\lambda_0). \end{aligned} \tag{6}$$

□

Lemma 3. *Suppose $\mu \in \mathbb{R}$, Φ is a solution of (4) such that $\Phi|_{\lambda=\mu}$ is real, then $\Phi^{\mathrm{T}} C\Phi_\lambda|_{\lambda=\mu}$ is independent of x.*

Proof.

$$\begin{aligned} &(\Phi_\lambda)_x = U(x,\lambda)\Phi_\lambda + U_\lambda(x,\lambda)\Phi \\ &\Phi_x^* C = -\Phi^* CU(x,\lambda) \end{aligned} \tag{7}$$

hold at $\lambda = \mu$, hence

$$(\Phi^* C\Phi_\lambda)_x|_{\lambda=\mu} = \Phi^* CU_\lambda(x,\lambda)\Phi|_{\lambda=\mu} = 0 \tag{8}$$

since CX is antisymmetric for any $X \in \mathfrak{g}$. □

3. Darboux transformations

We say a real symmetric matrix M is semidefinite if for any non-zero vector ξ, $\xi^{\mathrm{T}} M\xi \geqslant 0$ or $\xi^{\mathrm{T}} M\xi \leqslant 0$. The Darboux transformation for (4) can be constructed in the following two ways. The first one is applicable when C is not semidefinite and the second one is applicable when $C\sigma$ is not semidefinite. Since σ is not $\pm I$, these two cases cover all the possible situations.

3.1. Construction of the Darboux transformation when C is not semidefinite

Let $\mu \in \mathbb{R}$. Let H be a real vector solution of (4) with $\lambda = \mu$ which satisfies $H^* CH = 0$. (This is possible due to lemma 2.)

Take $\lambda_1^{(\varepsilon)} = \mu + \sqrt{-1}\varepsilon$, $\lambda_2^{(\varepsilon)} = -\mu - \sqrt{-1}\varepsilon$. Let $h_1^{(\varepsilon)}$ be a vector solution of (4) with $\lambda = \lambda_1^{(\varepsilon)}$ satisfying $h_1^{(0)} = H$ and $h_1^{(\varepsilon)}|_{x=x_0} = H(x_0)$ for some fixed x_0. According to lemma 1, $h_2^{(\varepsilon)} = \sigma h_1^{(\varepsilon)}$ is a solution of (4) with $\lambda = \lambda_2^{(\varepsilon)}$.

Let

$$\Gamma_{jk}^{(\varepsilon)} = \frac{h_j^{(\varepsilon)*} Ch_k^{(\varepsilon)}}{\lambda_k^{(\varepsilon)} - \bar{\lambda}_j^{(\varepsilon)}} \qquad (j,k = 1,2) \tag{9}$$

then

$$G^{(\varepsilon)}(\lambda) = 1 - \sum_{j,k=1}^{2} \frac{1}{\lambda - \bar{\lambda}_k^{(\varepsilon)}} h_j^{(\varepsilon)} (\Gamma^{(\varepsilon)-1})_{jk} h_k^{(\varepsilon)*} C \tag{10}$$

is a Darboux matrix for (4) without considering the $L^\sigma(\mathfrak{g})$ reduction, that is, for any solution Φ of (4), $\widetilde{\Phi} = G^{(\varepsilon)}\Phi$ satisfies

$$\widetilde{\Phi}_x = \widetilde{U}^{(\varepsilon)}(x,\lambda)\widetilde{\Phi} \tag{11}$$

where $\widetilde{U}^{(\varepsilon)} \in \mathfrak{L}(gl(p+q+r,\mathbb{C}))$ [19]. Now we calculate $\Gamma^{(\varepsilon)}$, $G^{(\varepsilon)}$ and their limits as $\varepsilon \to 0$. First,

$$\Gamma^{(\varepsilon)} = \begin{pmatrix} \dfrac{h_1^{(\varepsilon)*}Ch_1^{(\varepsilon)}}{2\sqrt{-1}\varepsilon} & \dfrac{h_1^{(\varepsilon)*}C\sigma h_1^{(\varepsilon)}}{-2\mu} \\ \dfrac{h_1^{(\varepsilon)*}C\sigma h_1^{(\varepsilon)}}{2\mu} & \dfrac{h_1^{(\varepsilon)*}Ch_1^{(\varepsilon)}}{-2\sqrt{-1}\varepsilon} \end{pmatrix}. \tag{12}$$

By lemma 3 and the assumptions $H^*CH = 0$, $\partial h_1^{(\varepsilon)}/\partial\varepsilon|_{x=x_0} = 0$,

$$\lim_{\varepsilon\to 0}\frac{h_1^{(\varepsilon)*}Ch_1^{(\varepsilon)}}{2\sqrt{-1}\varepsilon} = -\sqrt{-1}\lim_{\varepsilon\to 0}\mathrm{Re}\left(h_1^{(\varepsilon)*}C\frac{\partial h_1^{(\varepsilon)}}{\partial\varepsilon}\right) = -\sqrt{-1}\lim_{\varepsilon\to 0}\mathrm{Re}\left(h_1^{(\varepsilon)*}C\frac{\partial h_1^{(\varepsilon)}}{\partial\varepsilon}\right)\Bigg|_{x=x_0} = 0. \tag{13}$$

Hence

$$\Gamma = \lim_{\varepsilon\to 0}\Gamma^{(\varepsilon)} = \frac{1}{2\mu}\begin{pmatrix} 0 & -H^{\mathrm{T}}C\sigma H \\ H^{\mathrm{T}}C\sigma H & 0 \end{pmatrix} = \frac{H^{\mathrm{T}}C\sigma H}{2\mu}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

$$\Gamma^{-1} = \frac{2\mu}{H^{\mathrm{T}}C\sigma H}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{15}$$

$$\begin{aligned} G(\lambda) &= \lim_{\varepsilon\to 0}G^{(\varepsilon)}(\lambda) = 1 - \frac{2\mu}{H^{\mathrm{T}}C\sigma H}\left(\frac{HH^{\mathrm{T}}\sigma}{\lambda+\mu} - \frac{\sigma HH^{\mathrm{T}}}{\lambda-\mu}\right)C \\ &= \frac{1}{\lambda^2-\mu^2}\left(\lambda^2 + \frac{2\lambda\mu}{H^{\mathrm{T}}C\sigma H}[\sigma, HH^{\mathrm{T}}]C + \frac{2\mu^2(\sigma HH^{\mathrm{T}} + HH^{\mathrm{T}}\sigma)C}{H^{\mathrm{T}}C\sigma H} - \mu^2\right). \end{aligned} \tag{16}$$

It is easy to prove that

$$\begin{aligned} &(G(\bar{\lambda}))^*CG(\lambda) = C \\ &\sigma G(\lambda)\sigma = G(-\lambda). \end{aligned} \tag{17}$$

Using these facts, we know that

$$\widetilde{U} = GUG^{-1} + G_xG^{-1} \tag{18}$$

satisfies $\widetilde{U} \in \mathfrak{L}^\sigma(\mathfrak{g})$.

Therefore, the following theorem holds.

Theorem 1. *Suppose C is not semidefinite. Let $\mu \in \mathbb{R}$. Let H be a real vector solution of (4) with $\lambda = \mu$ such that $H^{\mathrm{T}}CH = 0$. Let*

$$\begin{aligned} &G(\lambda) = \frac{1}{\lambda^2-\mu^2}\left(\lambda^2 + \frac{2\lambda\mu}{H^{\mathrm{T}}C\sigma H}[\sigma, HH^{\mathrm{T}}]C + \frac{2\mu^2(\sigma HH^{\mathrm{T}} + HH^{\mathrm{T}}\sigma)C}{H^{\mathrm{T}}C\sigma H} - \mu^2\right) \\ &\widetilde{U} = GUG^{-1} + G_xG^{-1} \end{aligned} \tag{19}$$

then $\widetilde{U} \in \mathfrak{L}^\sigma(\mathfrak{g})$. Moreover, for any solution Φ of (4), $\widetilde{\Phi} = G\Phi$ satisfies $\widetilde{\Phi}_x = \widetilde{U}\widetilde{\Phi}$.

3.2. *Construction of the Darboux transformation when $C\sigma$ is not semidefinite*

When C is semidefinite, theorem 1 is no longer valid, since the required H does not exist in general. Here we discuss the problem when $C\sigma$ is not semidefinite. We use the following transformation to change the problem to a problem dealt with in section 3.1.

Consider the linear system

$$\Phi_x = U(\sqrt{-1}\lambda)\Phi. \tag{20}$$

Let τ be a complex diagonal matrix such that $\tau^2 = \sigma$ and τ has only two eigenvalues 1, $\sqrt{-1}$. Clearly $\tau^*\tau = 1$. Let $V(\lambda) = \tau U(\sqrt{-1}\lambda)\tau^*$, then when λ is real,

$$\begin{aligned}
&(1)\quad \overline{V(\lambda)} = \tau^* U(-\sqrt{-1}\lambda)\tau = \tau^*\sigma U(\sqrt{-1}\lambda)\sigma\tau = \tau U(\sqrt{-1}\lambda)\tau^* = V(\lambda)\\
&(2)\quad (V(\lambda))^* C\sigma = \tau(U(\sqrt{-1}\lambda))^*\tau^* C\sigma = -\tau C U(-\sqrt{-1}\lambda)\tau = -\tau C\sigma U(\sqrt{-1}\lambda)\tau^*\\
&\qquad\qquad = -\sigma C V(\lambda)\\
&(3)\quad V(-\lambda) = \tau U(-\sqrt{-1}\lambda)\tau^* = \tau\sigma U(\sqrt{-1}\lambda)\sigma\tau^* = \sigma V(\lambda)\sigma.
\end{aligned} \tag{21}$$

Hence, $V(\lambda) \in L^\sigma(\mathfrak{g}')$ where

$$\mathfrak{g}' = \{X \in gl(p+q+r, \mathbb{R}) \mid X^{\mathrm{T}} C\sigma + C\sigma X = 0\} \cong so_{\mathrm{ex}}(p', q', r) \tag{22}$$

with $p' + q' = p + q$.

Let $\Psi = \tau\Phi$, then the linear system (20) is changed to

$$\Psi_x = V(\lambda)\Psi. \tag{23}$$

When $C\sigma$ is not semidefinite, we can use theorem 1 to help solve this problem.

Let $G'(\lambda) = \tau G(\sqrt{-1}\lambda)\tau^*$, then $\widetilde{\Psi} = G'(\lambda)\Psi$ satisfies

$$\widetilde{\Psi}_x = \widetilde{V}(\lambda)\Psi$$

where $\widetilde{V}(\lambda) = \tau\widetilde{U}(\sqrt{-1}\lambda)\tau^*$.

Let $\mu \in \mathbb{R}$. Let H' be a real solution of (23) with $\lambda = -\mu$. Then theorem 1 implies that $G'(\lambda)$ can be chosen as

$$G'(\lambda') = \frac{1}{\lambda'^2 - \mu^2}\left(\lambda'^2 - \frac{2\lambda'\mu}{H'^{\mathrm{T}} C\sigma H'}[\sigma, H'H'^{\mathrm{T}}]\sigma C + \frac{2\mu^2(\sigma H'H'^{\mathrm{T}} + H'H'^{\mathrm{T}}\sigma)\sigma C}{H'^{\mathrm{T}} C\sigma H'} - \mu^2\right)$$

since C should be replaced by σC here. However, $\tau^{-1}H'$ is a solution of (20) with $\lambda = -\mu$, i.e. it is a solution of (4) with $\lambda = \sqrt{-1}\mu$. Let $H = \tau^{-1}H'$, then, for $\lambda = -\sqrt{-1}\lambda'$,

$$\begin{aligned}
G(\lambda) &= \tau^{-1}G(\lambda')\tau\\
&= \frac{1}{\lambda^2 + \mu^2}\left(\lambda^2 + \frac{2\sqrt{-1}\lambda\mu}{H^* C H}[\sigma, HH^*]\sigma C - \frac{2\mu^2(\sigma HH^* + HH^*\sigma)\sigma C}{H^* C H} + \mu^2\right).
\end{aligned}$$

Therefore, we have the following.

Theorem 2. *Suppose $C\sigma$ is not semidefinite. Let $\mu \in \mathbb{R}$. Let H be a complex vector solution of (4) with $\lambda = \sqrt{-1}\mu$ such that τH is real and $H^* C\sigma H = 0$. Let*

$$\begin{aligned}
G(\lambda) &= \frac{1}{\lambda^2 + \mu^2}\left(\lambda^2 + \frac{2\sqrt{-1}\lambda\mu}{H^* C H}[\sigma, HH^*]\sigma C - \frac{2\mu^2(\sigma HH^* + HH^*\sigma)\sigma C}{H^* C H} + \mu^2\right)\\
\widetilde{U} &= GUG^{-1} + G_x G^{-1}
\end{aligned} \tag{24}$$

then $\widetilde{U} \in \mathfrak{L}^\sigma(\mathfrak{g})$. Moreover, for any solution Φ of (4), $\widetilde{\Phi} = G\Phi$ satisfies $\widetilde{\Phi}_x = \widetilde{U}\widetilde{\Phi}$.

3.3. Example: the MKdV equation and the sine–Gordon equation

The MKdV equation

$$u_t = u_{xxx} + 6u^2 u_x \tag{25}$$

has a well known Lax pair

$$\begin{aligned}\Psi_x &= \frac{\lambda}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Psi + \begin{pmatrix}0 & u\\ -u & 0\end{pmatrix}\Psi\\ \Psi_t &= \left(\frac{\lambda^3}{2} + \lambda u^2\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Psi + (\lambda^2 u + u_{xx} + 2u^2)\begin{pmatrix}0 & u\\ -u & 0\end{pmatrix}\Psi + \lambda u_x \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Psi.\end{aligned} \tag{26}$$

The real Lie algebra $su(1,1)$ generated by

$$e_1 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad e_2 = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad e_3 = \frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{27}$$

is isomorphic to $so(2,1)$, given by the correspondence

$$e_1 \rightarrow \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \qquad e_2 \rightarrow \begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix} \qquad e_3 \rightarrow \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{28}$$

Now take

$$\mathfrak{g} = so(2,1) \qquad C = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} \qquad \sigma = C.$$

Then,

$$\begin{aligned}\mathfrak{g}_0 &= \mathrm{Span}\left\{\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right\}\\ \mathfrak{g}_1 &= \mathrm{Span}\left\{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\right\}.\end{aligned} \tag{29}$$

Take the Cartan subalgebra

$$\mathfrak{h} = \mathrm{Span}\left\{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\right\} \tag{30}$$

which corresponds to e_1.

Using the correspondence (28), we have the new Lax pair

$$\begin{aligned}\Phi_x &= \begin{pmatrix}0 & 2u & 0\\ -2u & 0 & \lambda\\ 0 & \lambda & 0\end{pmatrix}\Phi\\ \Phi_t &= \begin{pmatrix}0 & 2\lambda^2 u + 2u_{xx} + 4u^3 & -2\lambda u_x\\ -2\lambda^2 u - 2u_{xx} - 4u^3 & 0 & \lambda^3 + 2\lambda u^2\\ -2\lambda u_x & \lambda^3 + 2\lambda u^2 & 0\end{pmatrix}\Phi.\end{aligned} \tag{31}$$

Let $\mu \in \mathbb{R}$. Let $H = (\alpha, \beta, \gamma)^{\mathrm{T}}$ be a real solution of (31) with $\lambda = \mu$ which satisfies $\alpha^2 + \beta^2 - \gamma^2 = 0$, then theorem 1 gives

$$(\lambda^2 - \mu^2)G(\lambda) = \lambda^2 - \frac{2\lambda\mu}{\gamma^2}\begin{pmatrix} 0 & 0 & \alpha\gamma \\ 0 & 0 & \beta\gamma \\ \alpha\gamma & \beta\gamma & 0 \end{pmatrix} + \frac{2\mu^2}{\gamma^2}\begin{pmatrix} \alpha^2 & \alpha\beta & 0 \\ \alpha\beta & \beta^2 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} - \mu^2 \tag{32}$$

$$\widetilde{u} = u - \frac{\mu\alpha}{\gamma}. \tag{33}$$

Let $\theta = \sin^{-1}(\alpha/\gamma)$, then (31) gives $\theta_x = 2u - \mu \sin\theta$. Therefore, (33) is just the same as the standard Darboux transformation for the MKdV equation given by a 2×2 Darboux matrix, which is linear in the spectral parameter, although the 3×3 Darboux matrix here is quadratic in λ.

For the sine–Gordon equation, the x-part of the Lax pair is the same as that of the MKdV equation. Hence, the Darboux transformation (33) is also the standard Darboux transformation for the sine–Gordon equation.

4. Local isometric immersion of space forms with flat normal bundle and linearly-independent curvature normals

[17] gives the Lax sets for the equations describing local isometric immersions with flat normal bundles and linearly-independent curvature normals. Here we apply the above theorems to this problem and give the Darboux transformations of the position vectors.

First, we list some useful conclusions from [17], and rewrite the structure equation in a form to which the Darboux transformation can be easily applied.

Let $M_n(K)$ be an n-dimensional space form of curvature K, where $K = 0, 1, -1$. For the local isometric immersion $M_n(K) \to M_{2n}(K)$ with flat normal bundle and linearly-independent curvature normals, there always exist local coordinates $x = (x_1, \ldots, x_n)$ on $M_n(K)$ and the parallel orthonormal normal vector fields $(e_{n+1}, \ldots, e_{2n})$ such that the first and second fundamental forms are

$$\begin{aligned} \mathrm{I} &= \sum_{i,j=1}^{n} g_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j = \sum_{i=1}^{n} \rho_i^2(x)\,\mathrm{d}x_i^2 \\ \mathrm{II} &= \sum_{i,j,\alpha=1}^{n} \Omega_{ij}^{\alpha}\,\mathrm{d}x_i\,\mathrm{d}x_j e_{n+\alpha} = \sum_{i,j=1}^{n} \rho_i(x)\omega_{i\alpha}(x)\,\mathrm{d}x_i^2 e_{n+\alpha} \end{aligned} \tag{34}$$

where $\omega(x) = (\omega_{ij}(x)) \in O(n)$.

Denote

$$R_n(K) = \begin{cases} \mathbb{R}^n & \text{if } K = 0, 1 \\ \mathbb{R}^{n-1,1} & \text{if } K = -1 \end{cases} \tag{35}$$

where $\mathbb{R}^{n-1,1}$ has the metric $x_1^2 + \cdots + x_{n-1}^2 - x_n^2$.

Let $i_n(K) : M_n(K) \to R_{n+1}(K)$ be the standard imbedding given by

$$\begin{aligned} M_n(0) &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\} \\ M_n(1) &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = 1\} \\ M_n(-1) &= \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n,1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1\}. \end{aligned} \tag{36}$$

Then we can consider the immersion

$$\boldsymbol{V} : M_n(K) \to M_{2n}(K) \stackrel{i_{2n}(K)}{\longrightarrow} R_{2n+1}(K). \tag{37}$$

Clearly, $\boldsymbol{V} \in \mathbb{R}^{2n}$ for $K = 0$, $\boldsymbol{V} \cdot \boldsymbol{V} = K$ for $K = \pm 1$. Here '$\cdot$' refers to the inner product in $R_{2n+1}(K)$.

Remark 1. *We imbed $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n+1}$ only for the unification of the three cases.*

The structure equations for such immersions are

$$\begin{aligned} \partial_j \boldsymbol{V}_i &= \Gamma_{ij}^k \boldsymbol{V}_k + \Omega_{ij}^\alpha \boldsymbol{n}_\alpha - K g_{ij} \boldsymbol{V} \\ \partial_i \boldsymbol{n}_\alpha &= -g^{lk} \Omega_{il}^\alpha \boldsymbol{V}_k \end{aligned} \tag{38}$$

where $\boldsymbol{V}_i = \partial_i \boldsymbol{V}$, Γ_{ij}^k's are the Christofel symbols corresponding to the metric (g_{ij}), $\partial_i = \partial/\partial x_i$.

From (34),

$$g_{ij} = \rho_i^2 \delta_{ij} \qquad \Omega_{ij}^\alpha = \rho_i \omega_{i\alpha} \delta_{ij}. \tag{39}$$

Then (38) becomes

$$\begin{aligned} \partial_j \boldsymbol{V}_i &= \frac{\partial_j \rho_i}{\rho_i} \boldsymbol{V}_i + \frac{\partial_i \rho_j}{\rho_j} \boldsymbol{V}_j \qquad (i \neq j) \\ \partial_i \boldsymbol{V}_i &= -\sum_{k \neq i} \frac{\rho_i \partial_k \rho_i}{\rho_k^2} \boldsymbol{V}_k + \frac{\partial_i \rho_i}{\rho_i} \boldsymbol{V}_i + \rho_i \omega_{i\alpha} \boldsymbol{n}_\alpha - K \rho_i^2 \boldsymbol{V} \\ \partial_i \boldsymbol{n}_\alpha &= -\frac{\omega_{i\alpha}}{\rho_i} \boldsymbol{V}_i. \end{aligned} \tag{40}$$

The Gauss–Codazzi equations are the integrability conditions for (40), which are

$$\begin{aligned} &\partial_i \gamma_{ij} + \partial_j \gamma_{ji} = \sum_{l \neq i, l \neq j} \gamma_{li} \gamma_{lj} \qquad (i \neq j) \\ &\partial_k \gamma_{ij} + \gamma_{ik} \gamma_{kj} = 0 \qquad (i \neq j, i \neq k, j \neq k) \\ &\partial_i \gamma_{ji} + \partial_j \gamma_{ij} - \sum_{l \neq i, l \neq j} \gamma_{il} \gamma_{jl} - K \rho_i \rho_j = 0 \qquad (i \neq j) \\ &\partial_j \rho_i + \gamma_{ij} \rho_j = 0 \qquad (i \neq j) \end{aligned} \tag{41}$$

$$\begin{aligned} &\partial_j \omega_{i\alpha} + \gamma_{ij} \omega_{j\alpha} = 0 \qquad (i \neq j) \\ &\partial_i \omega_{i\alpha} = \sum_{k \neq i} \gamma_{ki} \omega_{k\alpha} \end{aligned} \tag{42}$$

where

$$\gamma_{ij} = \begin{cases} -\partial_j \rho_i / \rho_j & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases} \tag{43}$$

Let

$$\boldsymbol{p}_i = \sum_\alpha \omega_{i\alpha} \boldsymbol{n}_\alpha \qquad \boldsymbol{q}_i = \rho_i^{-1} \boldsymbol{V}_i \qquad \boldsymbol{r} = \boldsymbol{V} \tag{44}$$

then (40) becomes

$$\partial_i \boldsymbol{p}_i = \sum_{k\neq i} \gamma_{ki}\boldsymbol{p}_k - \boldsymbol{q}_i \qquad \partial_j \boldsymbol{p}_i = -\gamma_{ij}\boldsymbol{p}_j \qquad (i \neq j)$$
$$\partial_i \boldsymbol{q}_i = \sum_{k\neq i} \gamma_{ik}\boldsymbol{q}_k + \boldsymbol{p}_i - K\rho_i \boldsymbol{r} \qquad \partial_j \boldsymbol{q}_i = -\gamma_{ji}\boldsymbol{q}_j \qquad (i \neq j) \tag{45}$$
$$\partial_i \boldsymbol{r} = \rho_i \boldsymbol{q}_i.$$

For simplicity, we write

$$\boldsymbol{p} = \begin{pmatrix} \boldsymbol{p}_1 \\ \vdots \\ \boldsymbol{p}_n \end{pmatrix} \qquad \boldsymbol{q} = \begin{pmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_n \end{pmatrix}. \tag{46}$$

In [17], a spectral parameter was proposed so that (41) has a Lax set (generalized Lax pair)

$$\partial_i u_i = \sum_{k\neq i} \gamma_{ki} u_k - \lambda v_i \qquad \partial_j u_i = -\gamma_{ij} u_j \qquad (i \neq j)$$
$$\partial_i v_i = \sum_{k\neq i} \gamma_{ik} v_k + \lambda u_i - K\rho_i w \qquad \partial_j v_i = -\gamma_{ji} v_j \qquad (i \neq j) \tag{47}$$
$$\partial_i w = \rho_i v_i.$$

Note that (45) is a special case of (47) for $\lambda = 1$.

Let $\Phi(\lambda) = (u_1, \ldots, u_n, v_1, \ldots, v_n, w)^{\mathrm{T}}$, the above system of equations can be written in matrix form as

$$\partial_i \Phi = (\lambda J_i + [J_i, P])\Phi \tag{48}$$

where

$$J_i = \begin{pmatrix} 0 & -E_{ii} & 0 \\ E_{ii} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & \sum_{i\neq j} \gamma_{ij} E_{ij} & -K\sum_i \rho_i e_i \\ -\sum_{i\neq j} \gamma_{ji} E_{ij} & 0 & 0 \\ \sum_i \rho_i e_i^{\mathrm{T}} & 0 & 0 \end{pmatrix}$$
$$[J_i, P] = \begin{pmatrix} \sum_{j\neq i} \gamma_{ji}(E_{ij} - E_{ji}) & 0 & 0 \\ 0 & \sum_{j\neq i} \gamma_{ij}(E_{ij} - E_{ji}) & -K\rho_i e_i \\ 0 & \rho_i e_i^{\mathrm{T}} & 0 \end{pmatrix}. \tag{49}$$

Here E_{ij} is a constant matrix whose (i, j)th entry is one and the rest of the entries are zero, and e_i is a column matrix whose ith entry is one and the rest of the entries are zero.

If we write a $(2n + 1) \times (2n + 1)$ matrix M as a block matrix

$$\begin{array}{c} \quad n \qquad\quad n \qquad\quad 1 \\ \begin{pmatrix} M_{[1,1]} & M_{[1,2]} & M_{[1,3]} \\ M_{[2,1]} & M_{[2,2]} & M_{[2,3]} \\ M_{[3,1]} & M_{[3,2]} & M_{[3,3]} \end{pmatrix} \begin{matrix} n \\ n \\ 1 \end{matrix} \end{array} \tag{50}$$

then

$$(\rho_i) = P_{[3,1]} \qquad (\gamma_{ij}) = P_{[1,2]} \qquad (\omega_{ij}) = \Phi_{[1,1]}(0)L_\omega \qquad \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \end{pmatrix} = \Phi(1)\boldsymbol{L} \tag{51}$$

where L_ω is an $n \times n$ constant matrix and $\boldsymbol{L}$ is a $(2n + 1) \times 1$ constant matrix.

For the unification of the three cases $K = 0, \pm 1$, we apply theorem 2 to (48). Notice that (45) for $(\boldsymbol{p}_i, \boldsymbol{q}_i, \boldsymbol{r})$ is a special case of (47) for (u_i, v_i, w) when $\lambda = 1$. Moreover, $\omega_{i\alpha}$ satisfy the same equations (in (42)) as u_i do in (47) with $\lambda = 0$.

For $K = 0, 1, -1$, $\lambda J_i + [J_i, P] \in \mathfrak{L}^\sigma(\mathfrak{g})$ which corresponds to

$$\begin{aligned} C &= \mathrm{diag}(\underbrace{1, \ldots, 1}_n, \underbrace{1, \ldots, 1}_n, K) \\ \sigma &= \mathrm{diag}(\underbrace{1, \ldots, 1}_n, \underbrace{-1, \ldots, -1}_n, -1). \end{aligned} \tag{52}$$

Hence

$$C\sigma = \mathrm{diag}(\underbrace{1, \ldots, 1}_n, \underbrace{-1, \ldots, -1}_n, -K) \tag{53}$$

is not always semidefinite. Therefore, we can use theorem 2.

Let μ be a real number. Let $H = (\xi_1, \ldots, \xi_n, \sqrt{-1}\eta_1, \ldots, \sqrt{-1}\eta_n, \sqrt{-1}\zeta)$ be a solution of (47) with $\lambda = \sqrt{-1}\mu$ such that $\sum_i \xi_i^2 = \sum_i \eta_i^2 + K\zeta^2$. Then τH is real and $H^* C\sigma H = 0$. Written explicitly, the components of H satisfy

$$\begin{aligned} &\partial_i \xi_i = \sum_{k\neq i} \gamma_{ki}\xi_k + \mu\eta_i \qquad \partial_j \xi_i = -\gamma_{ij}\xi_j \qquad (i \neq j) \\ &\partial_i \eta_i = \sum_{k\neq i} \gamma_{ik}\eta_k + \mu\xi_i - K\rho_i\zeta \qquad \partial_j \eta_i = -\gamma_{ji}\eta_j \qquad (i \neq j) \\ &\partial_i \zeta = \rho_i \eta_i. \end{aligned} \tag{54}$$

The Darboux matrix in theorem 2 is

$$\begin{aligned} G(\lambda) = \frac{1}{\lambda^2 + \mu^2} &\left(\lambda^2 + \mu^2 + \frac{2\lambda\mu}{\Delta} \begin{pmatrix} 0 & -\xi\eta^{\mathrm{T}} & -K\zeta\xi \\ \eta\xi^{\mathrm{T}} & 0 & 0 \\ \zeta\xi^{\mathrm{T}} & 0 & 0 \end{pmatrix} \right. \\ &\left. - \frac{2\mu^2}{\Delta} \begin{pmatrix} \xi\xi^{\mathrm{T}} & 0 & 0 \\ 0 & \eta\eta^{\mathrm{T}} & K\eta\zeta \\ 0 & \zeta\eta^{\mathrm{T}} & K\zeta^2 \end{pmatrix} \right) \end{aligned} \tag{55}$$

where $\xi = (\xi_1, \ldots, \xi_n)^{\mathrm{T}}$, $\eta = (\eta_1, \ldots, \eta_n)^{\mathrm{T}}$, $\Delta = \sum_{l=1}^{n} \xi_l^2$.

The corresponding geometric quantities are given by

$$\widetilde{P} = P - \frac{2\sqrt{-1}\mu}{H^* C H}[\sigma, HH^*]\sigma C \qquad \widetilde{\omega} = G(0)_{[1,1]}\omega \tag{56}$$

$$(\widetilde{\boldsymbol{p}}_1, \ldots, \widetilde{\boldsymbol{p}}_n, \widetilde{\boldsymbol{q}}_1, \ldots, \widetilde{\boldsymbol{q}}_n, \widetilde{\boldsymbol{r}})^{\mathrm{T}} = G(1)(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n, \boldsymbol{r})^{\mathrm{T}}. \tag{57}$$

Written explicitly, they are

$$\begin{aligned}
&\widetilde{\rho}_i = \rho_i - \frac{2\mu\zeta\xi_i}{\Delta} \qquad \widetilde{\gamma}_{ij} = \gamma_{ij} + \frac{2\mu\xi_i\eta_j}{\Delta} \qquad \widetilde{\omega}_{i\alpha} = \omega_{i\alpha} - \sum_j \frac{2\xi_i\xi_j}{\Delta}\omega_{j\alpha} \\
&\widetilde{\boldsymbol{p}}_i = \boldsymbol{p}_i - \sum_j \frac{b\mu\xi_i\xi_j}{\Delta}\boldsymbol{p}_j - \sum_j \frac{b\xi_i\eta_j}{\Delta}\boldsymbol{q}_j - \frac{Kb\zeta\xi_i}{\Delta}\boldsymbol{r} \\
&\widetilde{\boldsymbol{q}}_i = \boldsymbol{q}_i + \sum_j \frac{b\eta_i\xi_j}{\Delta}\boldsymbol{p}_j - \sum_j \frac{b\mu\eta_i\eta_j}{\Delta}\boldsymbol{q}_j - \frac{Kb\mu\zeta\eta_i}{\Delta}\boldsymbol{r} \\
&\widetilde{\boldsymbol{r}} = \boldsymbol{r} + \sum_j \frac{b\zeta\xi_j}{\Delta}\boldsymbol{p}_j - \sum_j \frac{b\mu\zeta\eta_j}{\Delta}\boldsymbol{q}_j - \frac{Kb\mu\zeta^2}{\Delta}\boldsymbol{r}
\end{aligned} \tag{58}$$

where $b = 2\mu/(1+\mu^2)$.

Using (44) and (40), we have

$$\begin{aligned}
\widetilde{\boldsymbol{V}} = \sum_j \frac{b\zeta\xi_j}{\Delta\rho_j}\frac{\partial^2 \boldsymbol{V}}{\partial x_j^2} + \sum_j \frac{b\zeta}{\Delta\rho_j^2}\left(-\xi_j\partial_j\rho_j + \sum_{k\neq j}\xi_k\partial_j\rho_k - \mu\rho_j\eta_j\right)\frac{\partial \boldsymbol{V}}{\partial x_j} \\
+\left(1 - \frac{Kb\mu\zeta^2}{\Delta} + \sum_j \frac{Kb\zeta\rho_j\xi_j}{\Delta}\right)\boldsymbol{V}.
\end{aligned} \tag{59}$$

Therefore, we have the following general procedure to get the local isometric immersion $M_n(K) \to M_{2n}(K)$ with flat normal bundle and linearly-independent curvature normals. Since we consider the linear system (47), the condition $\rho_i \neq 0$ can be removed temporarily, provided that $\widetilde{\rho}_i \neq 0$ for the derived submanifold.

(1) Suppose we know a solution of (41) and can solve the linear system (47) to get the fundamental solution $\Phi(x_1, \ldots, x_n, \lambda)$.

(2) Let $\lambda = \sqrt{-1}\mu$ and get a solution (ξ, η, ζ) of (54) such that

$$(\xi, \sqrt{-1}\eta, \sqrt{-1}\zeta)^{\mathrm{T}} = \Phi(x_1, \ldots, x_n, \sqrt{-1}\mu)C \tag{60}$$

where C is a constant matrix.

(3) Using (58), one gets the position vector $\widetilde{\boldsymbol{r}}$ and the corresponding quantities. When $\rho_i \neq 0$, (59) gives a more direct answer. Moreover, the corresponding solution of (47) is $\widetilde{\Phi}(x_1, \ldots, x_n, \lambda) = G(x_1, \ldots, x_n, \lambda)\Phi(x_1, \ldots, x_n, \lambda)$.

These three steps give the new solution of (41) and (45). When $\widetilde{\rho}_i \neq 0$, the solution represents a real geometric immersion.

For $\widetilde{\boldsymbol{r}}$, the corresponding solution of (47) is known. Hence, we can repeat steps (2) and (3) to get another solution $\widetilde{\widetilde{\boldsymbol{r}}}$. Continuing this process, a series of immersions are obtained by an algebraic algorithm and one should solve a system of linear ordinary differential equations only once in step (1).

Now we give some examples. The simplest examples are the totally geodesic submanifolds. However, since their second fundamental forms are zero, the Darboux transformation here is not applicable (also, their curvature normals are linearly dependent). Therefore, we seek other solutions as seed solutions of the Darboux transformation.

4.1. Solutions derived from the trivial solutions

We take the seed solution of (41) as $\rho_i = 0$, $\gamma_{ij} = 0$, $\omega_{i\alpha} = \delta_{i\alpha}$. Although this does not correspond to a real geometric object, and (59) is not valid, we can still get $(\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}}, \widetilde{\boldsymbol{r}})$ from (58) and then get the non-trivial local immersion $\widetilde{\boldsymbol{r}}$.

To obtain a solution, we first solve (45) to get $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r})$ and solve (54) to get (ξ_i, η_i, ζ), then normalize the first fundamental form by changing coordinates $\{x_i\}$. After that, the last equation of (58) gives the position vector of the immersion.

First, we solve (45) to get

$$\begin{aligned} \boldsymbol{p}_i &= \boldsymbol{E}_i \cos x_i + \boldsymbol{F}_i \sin x_i \\ \boldsymbol{q}_i &= \boldsymbol{E}_i \sin x_i - \boldsymbol{F}_i \cos x_i \\ \boldsymbol{r} &= \boldsymbol{R} \end{aligned} \tag{61}$$

where $\boldsymbol{E}_i$, $\boldsymbol{F}_i$, $\boldsymbol{R}$ $(i = 1, \ldots, n)$ are constant vectors in $\mathbb{R}_{2n+1}(K)$ such that

$$\begin{gathered} \boldsymbol{E}_i \cdot \boldsymbol{E}_j = \delta_{ij} \qquad \boldsymbol{E}_i \cdot \boldsymbol{F}_j = 0 \qquad \boldsymbol{F}_i \cdot \boldsymbol{F}_j = \delta_{ij} \qquad \boldsymbol{E}_i \cdot \boldsymbol{R} = \boldsymbol{F}_i \cdot \boldsymbol{R} = 0 \\ \boldsymbol{R} \cdot \boldsymbol{R} = K \qquad (\text{if } K = \pm 1). \end{gathered} \tag{62}$$

4.1.1. $K = 0$. The solution of (54) is

$$\xi_i = A_i\, \mathrm{e}^{\mu x_i} \qquad \eta_i = A_i\, \mathrm{e}^{\mu x_i} \qquad \zeta = C \tag{63}$$

where A_i and C are non-zero real constants.

From (58),

$$\widetilde{\rho}_i = -\frac{2\mu C A_i\, \mathrm{e}^{\mu x_i}}{\sum_k A_k^2\, \mathrm{e}^{2\mu x_k}}. \tag{64}$$

Let

$$z_i = \frac{2 C A_i\, \mathrm{e}^{\mu x_i}}{\sum_k A_k^2\, \mathrm{e}^{2\mu x_k}} \tag{65}$$

then

$$\widetilde{I} = \sum_i \widetilde{\rho}_i^2\, \mathrm{d}x_i^2 = \sum_i \mathrm{d}z_i^2. \tag{66}$$

Choose $\boldsymbol{R} = 0$, then

$$\begin{aligned} \widetilde{\boldsymbol{r}} &= \sum_i \frac{\mu}{1+\mu^2} z_i (\boldsymbol{E}_i \cos x_i + \boldsymbol{F}_i \sin x_i) - \sum_i \frac{\mu^2}{1+\mu^2} z_i (\boldsymbol{E}_i \sin x_i - \boldsymbol{F}_i \cos x_i) \\ &= \sum_i \frac{\mu z_i}{\sqrt{1+\mu^2}} (\boldsymbol{E}_i \cos \widetilde{x}_i + \boldsymbol{F}_i \sin \widetilde{x}_i) \end{aligned} \tag{67}$$

where

$$\widetilde{x}_i = x_i + \mathrm{tg}^{-1}\mu = \frac{1}{\mu} \ln\left(\frac{2C}{A_i} \frac{z_i}{\sum_k z_k^2}\right) + \mathrm{tg}^{-1}\mu. \tag{68}$$

This local immersion is only defined for $(z_1, \ldots, z_n)$ with $A_i C z_i > 0$.

For $n = 2$ and $A_i = 1$, $C > 0$, let $z_1 = r\cos\phi$, $z_2 = r\sin\phi$,

$$\begin{aligned} \boldsymbol{k}_1 &= \frac{\boldsymbol{E}_1 + \mu \boldsymbol{F}_1}{\sqrt{1+\mu^2}} \qquad & \boldsymbol{k}_2 &= \frac{\boldsymbol{F}_1 - \mu \boldsymbol{E}_1}{\sqrt{1+\mu^2}} \\ \boldsymbol{k}_3 &= \frac{\boldsymbol{E}_2 + \mu \boldsymbol{F}_2}{\sqrt{1+\mu^2}} \qquad & \boldsymbol{k}_4 &= \frac{\boldsymbol{F}_2 - \mu \boldsymbol{E}_2}{\sqrt{1+\mu^2}} \end{aligned}$$

then $\{\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4\}$ is also an orthonormal frame of $\mathbb{R}^4$, and

$$\begin{aligned}\widetilde{\boldsymbol{r}} = \frac{\mu}{\sqrt{1+\mu^2}}\Bigg(& r\cos\phi\cos\left(\frac{1}{\mu}\ln\left(\frac{2C}{r}\cos\phi\right)\right)\boldsymbol{k}_1 \\ &+ r\cos\phi\sin\left(\frac{1}{\mu}\ln\left(\frac{2C}{r}\cos\phi\right)\right)\boldsymbol{k}_2 \\ &+ r\sin\phi\cos\left(\frac{1}{\mu}\ln\left(\frac{2C}{r}\sin\phi\right)\right)\boldsymbol{k}_3 + r\sin\phi\sin\left(\frac{1}{\mu}\ln\left(\frac{2C}{r}\sin\phi\right)\right)\boldsymbol{k}_4\Bigg) \\ &\left(r > 0, 0 < \phi < \frac{\pi}{2}\right). \end{aligned} \tag{69}$$

Written in terms of the coordinates (x_1, x_2)

$$\widetilde{\boldsymbol{r}} = \frac{2\mu C}{\sqrt{1+\mu^2}} \frac{1}{\mathrm{e}^{2\mu x_1} + \mathrm{e}^{2\mu x_2}} (\mathrm{e}^{\mu x_1}\cos x_1 \boldsymbol{k}_1 + \mathrm{e}^{\mu x_1}\sin x_1 \boldsymbol{k}_2 + \mathrm{e}^{\mu x_2}\cos x_2 \boldsymbol{k}_3 + \mathrm{e}^{\mu x_2}\sin x_2 \boldsymbol{k}_4). \tag{70}$$

Remark 2. *When μC is finite and $\mu \to 0$, except for $(0, 0, 0, 0)$, each point on the submanifold (70) tends to the standard torus in $\mathbb{R}^4$:*

$$\boldsymbol{r} = \mu C(\cos x_1 \boldsymbol{k}_1 + \sin x_1 \boldsymbol{k}_2 + \cos x_2 \boldsymbol{k}_3 + \sin x_2 \boldsymbol{k}_4). \tag{71}$$

The coefficient is μC because the metric on the surface is

$$I = \mu^2 C^2 \sum_{i=1}^{2} \mathrm{d}x_i^2.$$

When C is finite and $\mu \to \infty$, the pointwise limit of the submanifold given by (69) is

$$\boldsymbol{r} = r\cos\phi \boldsymbol{k}_1 + r\sin\phi \boldsymbol{k}_3 \tag{72}$$

which is a plane in $\mathbb{R}^4$.

Remark 3. *In the coordinate $\{z_i\}$, (45) or (40) is no longer satisfied, because those equations depend on the special coordinate $\{x_i\}$.*

Figures 1–3 show the manifold with $\mu = -0.2$, $C = 1$. In figure 1, the three axes are (r_1, r_2, r_3) if $\boldsymbol{r}$ is written as $\boldsymbol{r} = \sum_{j=1}^{4} r_j \boldsymbol{k}_j$. Figure 2 is the corresponding contour plot of figure 1 on the (r_1, r_2) plane. In figure 3, the axes are $(r_1, r_2, \pm\sqrt{r_3^2 + r_4^2})$. The dark spiral in figure 2 represents the boundary of the manifold.

4.1.2. $K = 1$. The solution of (54) is

$$\xi_i = A_i\,\mathrm{e}^{\mu x_i} + B_i\,\mathrm{e}^{-\mu x_i} \qquad \eta_i = A_i\,\mathrm{e}^{\mu x_i} - B_i\,\mathrm{e}^{-\mu x_i} \qquad \zeta = C \tag{73}$$

where A_i, B_i, C are non-zero real constants with $C^2 = 4\sum_i A_i B_i$.

The corresponding

$$\widetilde{\rho}_i = -\frac{2\mu C(A_i\,\mathrm{e}^{\mu x_i} + B_i\,\mathrm{e}^{-\mu x_i})}{\sum_k (A_k\,\mathrm{e}^{\mu x_k} + B_k\,\mathrm{e}^{-\mu x_k})^2}. \tag{74}$$

Let

$$z_i = \frac{A_i\,\mathrm{e}^{\mu x_i} - B_i\,\mathrm{e}^{-\mu x_i}}{C} \tag{75}$$

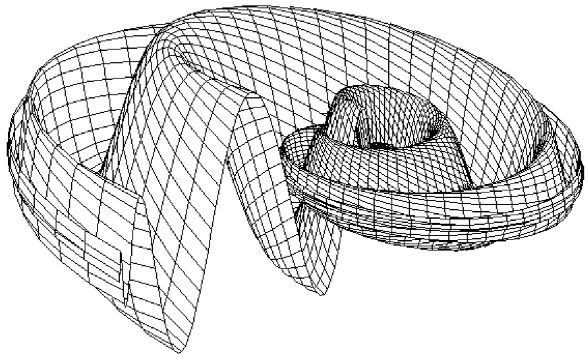

Figure 1.

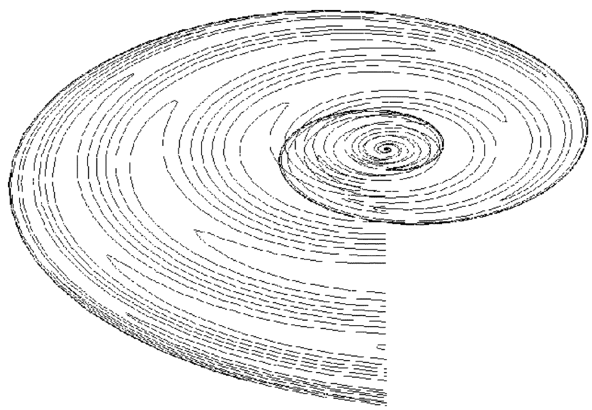

Figure 2.

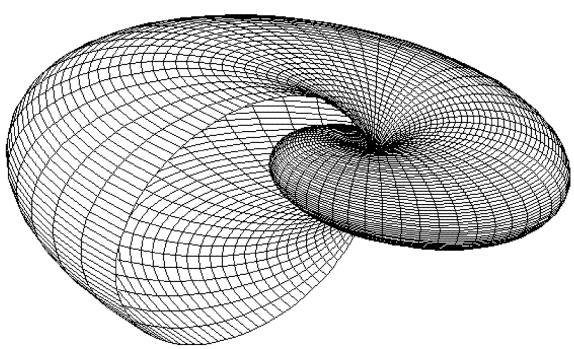

Figure 3.

then

$$\widetilde{I} = \sum_i \widetilde{\rho}_i^2 \, \mathrm{d}x_i^2 = \frac{4\sum_i \mathrm{d}z_i^2}{(\sum_k z_k^2 + 1)^2} \tag{76}$$

$$\widetilde{\boldsymbol{r}} = \left(1 - \frac{b\mu}{\sum_k z_k^2 + 1}\right)\boldsymbol{R} + \frac{b\sqrt{z_i^2 + C_i}}{\sum_k z_k^2 + 1}(\boldsymbol{E}_i \cos x_i + \boldsymbol{F}_i \sin x_i)$$
$$- \frac{b\mu z_i}{\sum_k z_k^2 + 1}(\boldsymbol{E}_i \sin x_i - \boldsymbol{F}_i \cos x_i)$$
$$= \left(1 - \frac{2\mu^2}{1+\mu^2}\frac{1}{\sum_k z_k^2 + 1}\right)\boldsymbol{R} + \frac{2\mu\sqrt{(1+\mu^2)z_i^2 + C_i}}{(1+\mu^2)(\sum_k z_k^2 + 1)}(\boldsymbol{E}_i \cos \widetilde{x}_i + \boldsymbol{F}_i \sin \widetilde{x}_i) \tag{77}$$

with $C_i = 4A_iB_i/C^2$ (therefore, $\sum_i C_i = 1$),

$$\widetilde{x}_i = \frac{1}{\mu}\ln\left(\frac{C}{2A_i}(z_i + \sqrt{z_i^2 + C_i})\right) + \mathrm{tg}^{-1}\frac{\mu z_i}{\sqrt{z_i^2 + C_i}}. \tag{78}$$

It is clear that this immersion is smooth for $|z| < \infty$. It can be verified that at $|z| = \infty$ (by changing coordinates) it is not smooth.

4.1.3. $K = -1$. The solution of (54) is

$$\xi_i = A_i\,\mathrm{e}^{\mu x_i} - B_i\,\mathrm{e}^{-\mu x_i} \qquad \eta_i = A_i\,\mathrm{e}^{\mu x_i} + B_i\,\mathrm{e}^{-\mu x_i} \qquad \zeta = C \tag{79}$$

where A_i, B_i, C are non-zero real constants with $C^2 = 4\sum_i A_iB_i$. Then

$$\widetilde{\rho}_i = -\frac{2\mu C(A_i\,\mathrm{e}^{\mu x_i} - B_i\,\mathrm{e}^{-\mu x_i})}{\sum_k (A_k\,\mathrm{e}^{\mu x_k} - B_k\,\mathrm{e}^{-\mu x_k})^2}. \tag{80}$$

Let

$$z_i = \frac{C(A_i\,\mathrm{e}^{\mu x_i} + B_i\,\mathrm{e}^{-\mu x_i})}{\sum_k (A_k\,\mathrm{e}^{\mu x_k} + B_k\,\mathrm{e}^{-\mu x_k})^2} \tag{81}$$

then

$$\widetilde{I} = \sum_i \widetilde{\rho}_i^2\,\mathrm{d}x_i^2 = \frac{4\sum_i \mathrm{d}z_i^2}{(1 - \sum_k z_k^2)^2} \tag{82}$$

$$\widetilde{\boldsymbol{r}} = \left(1 + \frac{2\mu^2}{1+\mu^2}\frac{\sum_k z_k^2}{1 - \sum_k z_k^2}\right)\boldsymbol{R} + \frac{2\mu\sqrt{(1+\mu^2)z_i^2 - C_i(\sum_k z_k^2)^2}}{(1+\mu^2)(1 - \sum_k z_k^2)}(\boldsymbol{E}_i \cos \widetilde{x}_i + \boldsymbol{F}_i \sin \widetilde{x}_i) \tag{83}$$

with $C_i = 4A_iB_i/C^2$ (also $\sum_i C_i = 1$),

$$\widetilde{x}_i = \frac{1}{\mu}\ln\left(\frac{C}{2A_i}\frac{z_i + \sqrt{z_i^2 - C_i(\sum_k z_k^2)^2}}{\sum_k z_k^2}\right) + \mathrm{tg}^{-1}\frac{\mu z_i}{\sqrt{z_i^2 - C_i(\sum_k z_k^2)^2}}. \tag{84}$$

Now $M_n(-1)$ is the disk $\sum_k z_k^2 < 1$, but the immersion (83) can only be defined in a region of $M_n(-1)$. For example, when all $A_i > 0$, $B_i > 0$, it is defined in $C_i(\sum_k z_k^2)^2 < z_i^2$, which contains the centre $(0, 0, \ldots, 0)$ of the disk.

4.2. Solutions derived from the torus $T^n \to \mathbb{R}^{2n}$

Let

$$\boldsymbol{r} = \sum_{i=1}^{n} (\cos x_i \boldsymbol{E}_i + \sin x_i \boldsymbol{F}_i) \tag{85}$$

where $\boldsymbol{E}_i$ and $\boldsymbol{F}_i$ are constant vectors satisfying

$$\boldsymbol{E}_i \cdot \boldsymbol{E}_j = \delta_{ij} \qquad \boldsymbol{E}_i \cdot \boldsymbol{F}_j = 0 \qquad \boldsymbol{F}_i \cdot \boldsymbol{F}_j = \delta_{ij}. \tag{86}$$

This is the standard torus T^n in $\mathbb{R}^{2n}$, whose $g_{ij} = \delta_{ij}$, $\omega_{i\alpha} = \delta_{i\alpha}$, and the corresponding

$$\begin{aligned} \boldsymbol{p}_i &= -\cos x_i \boldsymbol{E}_i - \sin x_i \boldsymbol{F}_i \\ \boldsymbol{q}_i &= -\sin x_i \boldsymbol{E}_i + \cos x_i \boldsymbol{F}_i. \end{aligned} \tag{87}$$

Solving (54) with $\sum_i \xi_i^2 = \sum_i \eta_i^2$, we have

$$\xi_i = A_i \, \mathrm{e}^{\mu x_i} \qquad \eta_i = A_i \, \mathrm{e}^{\mu x_i} \qquad \zeta = \sum_{k=1}^{n} \frac{A_k}{\mu} \, \mathrm{e}^{\mu x_k} + C \tag{88}$$

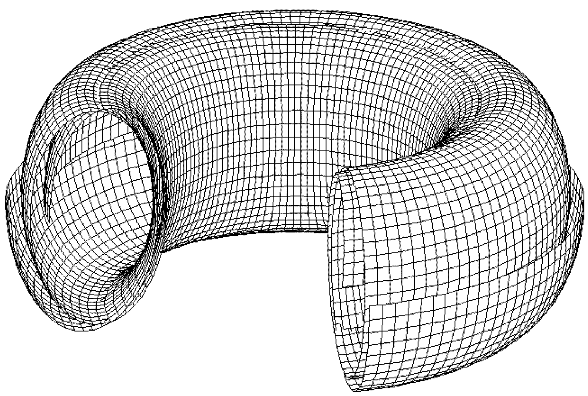

Figure 4.

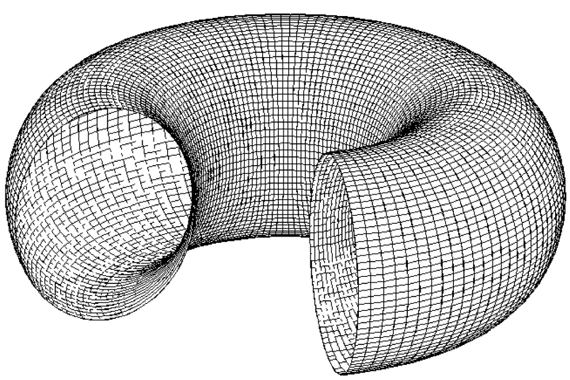

Figure 5.

where A_i and C are real constants. Then (58) gives

$$\widetilde{\boldsymbol{r}} = \sum_{i=1}^{n}(\cos x_i \boldsymbol{E}_i + \sin x_i \boldsymbol{F}_i) - \frac{b}{\Delta}\left(\sum_{k=1}^{n}\frac{A_k}{\mu}\,\mathrm{e}^{\mu x_k} + C\right) \times \sum_{i=1}^{n} A_i\,\mathrm{e}^{\mu x_i}((\cos x_i - \mu\sin x_i)\boldsymbol{E}_i + (\sin x_i + \mu\cos x_i)\boldsymbol{F}_i) \tag{89}$$

$$\widetilde{\rho}_i = 1 - \frac{2A_i\,\mathrm{e}^{\mu x_i}}{\Delta}\left(\sum_{k=1}^{n} A_k\,\mathrm{e}^{\mu x_k} + \mu C\right) \tag{90}$$

where

$$\Delta = \sum_k A_k^2\,\mathrm{e}^{2\mu x_k} \qquad b = \frac{2\mu}{1+\mu^2}. \tag{91}$$

This submanifold is defined in the region where $\widetilde{\rho}_i \neq 0$ for all $1 \leqslant i \leqslant n$.

By the local change of coordinates

$$z_i = x_i - \frac{2A_i\,\mathrm{e}^{\mu x_i}}{\mu\Delta}\left(\sum_{k=1}^{n} A_k\,\mathrm{e}^{\mu x_k} + \mu C\right) \tag{92}$$

the metric is changed to

$$\widetilde{I} = \sum_{i=1}^{n} \mathrm{d}z_i^2.$$

Figure 4 shows this submanifold (surface) for $n = 2$, $A_i = 1$, $C = 1$, $\mu = -0.2$. Writing $\boldsymbol{r} = r_1\boldsymbol{E}_1 + r_2\boldsymbol{F}_1 + r_3\boldsymbol{E}_2 + r_4\boldsymbol{F}_2$, then the coordinates are taken as $((2+r_1)r_3, (2+r_1)r_4, r_2)$. Figure 5 is the corresponding figure for the standard torus.

Acknowledgments

The paper was supported by Chinese National Research Project 'Nonlinear Science', Chinese National Science Foundation for Youth, and Fok Ying-Tung Education Foundation of China. The author would like to thank Professors C H Gu, H S Hu, Y L Xin, J X Hong, X G Huang and Drs J M Lin and Q Ding for encouragement and helpful discussions.

References

[1] Ablowitz M J, Beals R and Tenenblat K 1986 On the solution of the generalized wave and generalized sine–Gordon equation *Stud. Appl. Math.* **74** 177

[2] Barbosa J L, Ferreira W and Tenenblat K 1996 Submanifolds of constant sectional curvature in pseudo-Riemannian manifolds *Ann. Global Anal. Geo.* **14** 381–401

[3] Beals R and Tenenblat K 1991 An intrinsic generalization for the wave and sine–Gordon equation *Differential Geometry (Pitman Monograph 52)* (Harlow: Longman) pp 25–46

[4] Cieśliński J 1995 An algebraic method to construct the Darboux matrix *J. Math. Phys.* **36** 5670–706

[5] Dajczer M and Tojeiro R 1994 Submanifolds with nonparallel first normal bundle *Can. Math. Bull.* **37** 330–7

[6] Dajczer M and Tojeiro R 1995 Isometric immersions and the generalized Laplace and elliptic sine–Gordon equations *J. Reine Angew. Math.* **467** 109–47

[7] Gerdjikov V S 1987 The Zakharov-Shabat dressing method and the representation theory of the semisimple Lie algebras *Phys. Lett.* **126A** 184–8

[8] Gu C H 1989 On the Darboux form of Bäcklund transformations *Integrable Systems (Nankai Lectures on Math. Phys.)* (Singapore: World Scientific) pp 162–8

[9] Gu C H 1992 On the interaction of solitons for a class of integrable systems in the space-time $\mathbb{R}^{n+1}$ *Lett. Math. Phys.* **26** 199–209

[10] Gu C H and Hu H S 1993 Explicit solutions to the intrinsic generalization for the wave and sine–Gordon equations *Lett. Math. Phys* **29** 1–11

[11] Gu C H and Zhou Z X 1990 Explicit form of Bäcklund transformations for $GL(N)$, $U(N)$ and $O(2N)$ principal chiral fields *Nonlinear Evolution Equations: Integrability and Spectral Methods* (Manchester: Manchester University Press) pp 115–23

[12] Gu C H and Zhou Z X 1994 On Darboux transformations for soliton equations in high dimensional space-time *Lett. Math. Phys.* **32** 1–10

[13] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Heidelberg: Springer)

[14] Novikov S, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons* (New York: Consultants Bureau)

[15] Sattinger D H and Zurkowski V D 1987 Gauge theory of Bäcklund transformations II *Physica* **26D** 225–50

[16] Tenenblat K 1985 Bäcklund's theorem for submanifolds of space forms and a generalized wave equation *Boll Soc. Brasil Mat.* **16** 67–92

[17] Terng C L 1996 Soliton equations, Kac–Moody algebras and differential geometry *J. Diff. Geom.* **45** 407–45

[18] Zakharov V E and Mikhailov A V 1980 On the integrability of classical spinor models in two-dimensional space-time *Commun. Math. Phys.* **74** 21–40

[19] Zhou Z X 1996 Soliton solutions for some equations in $1+2$ dimensional hyperbolic $su(N)$ AKNS system *Inverse Problems* **12** 89–109