復旦大學

学 位 论 文

（题 名）

1+1 和 1+2 维的 Darboux 变换

系 （所）：数学所

专 业：数学物理

姓 名：周子翔

攻读学位：理学博士

指导教师：谷超豪教授

完成日期：1989 年 2 月 16 日

原稿用机械打字机写成, 部分公式为手写. 本稿用 LaTeX 重新打印.

Ph.D. Thesis of Fudan University

Darboux transformations in 1+1 and 1+2 dimensions

ZHOU Zixiang

(Institute of Mathematics, Fudan University)

Supervisor: Professor GU Chaohao

February 16, 1989

Contents

Introduction

Since the late sixties, the theory of soliton has developed rapidly. People have found that solitons appear in a lot of fields of mathematics and physics, such as fluid mechanics, differential geometry, nonlinear optics, plasma physics, particle physics etc. Apart from the discovery of many attractive feature of solitary wave, the soliton theory also provides various methods to simplify equations and get explicit soliton solutions, in which there are inverse scattering method, Bäcklund transformation method, Darboux transformation method, Hirota method, prolongation method, Painlevé analysis, and so on. With these methods, a lot of explicit solutions for the equations with mathematical and physical interest are obtained. Therefore, soliton theory has become an important branch of mathematical physics. In this paper, we deal with Darboux transformation, which is one of the most effective method of getting explicit solutions.

The solitary wave was first discovered in nature in 1834 and the KdV equation describing it was found in 1895. However, the significance of the solitary wave in mathematics and physics was not realized until 1960's. In 1965, Zabusky and Kruskal found out by numerical analysis that the solutions of the KdV equation evolved as particles when they interacted. They called these solutions solitons. This fact, together with the discovery of the GGKM method in 1967, greatly stimulated the study of soliton theory. The inverse scattering theory developed rapidly after that. It transforms the problem into solving a linear integral equation, and is an important tool not only to solve Cauchy problems, but also to get explicit expressions of pure solitons.

In 1973, Wahlquist and Estabrook found the Bäcklund transformation for KdV equation and got the soliton solutions. This revealed the importance of the classical transformation for sine-Gordon equation given by Bäcklund. The Bäcklund transformation is a relation between two solutions of an equation or two solutions of different equations, so it gives a new solution of the equation from a known, usually easier solution. Since then, the Bäcklund transformations for a lot of equations have also been found. Most of those Bäcklund transformations transform a solution of an equation to a solution of the same equation. They are called auto-Bäcklund transformations.

After the discovery of Bäcklund transformation, people tried to find some simpler transformations to get soliton solutions since the nonlinear equations in Bäcklund transformation method were still too complicated to be solved in nontrivial cases. [33] showed that the original transformations for Schrödinger equation given by Darboux was effective not only for KdV equation, but also for MKdV equation, sine-Gordon equation etc. They can, in fact, be written as a gauge transformation of the potentials in the Lax pairs. Since then, people have known that many other equations possessing Lax pairs have Darboux transformations from which the Bäcklund transformation in original form can also be obtained [5]. Similar to Bäcklund transformations, Darboux transformations for most equations have auto-Bäcklund property, i.e. they transform a solution of an equation to a solution of the same equation. The Darboux transformation greatly simplifies the procedure of getting explicit solutions. Hence it has become an important method in soliton theory.

The basic idea of the method of Darboux transformation is as follows. For the nonlinear partial differential equation possessing a Lax pair, if one solution of that equation is known, we can construct a Darboux matrix or Darboux operator from the solutions of the Lax pair so that it induces a new solution of that nonlinear equation. Thus, if only we know how to construct the Darboux matrix or Darboux operator, the problem is reduced to a linear problem to solve the Lax pair. This is much simpler than the nonlinear equations in Bäcklund transformation. Also, unlike in the inverse scattering method, we do not need any restriction on the boundary condition of the original solution. Furthermore, we can get a series of solutions of the original nonlinear partial differential equation by a purely algebraic algorithm so long as we know a solution of the Lax pair.

Throughout this paper, every function is assumed to be infinitely differentiable.

In order to see more clearly the advantage of Darboux transformation, we first have a short review of the classical Bäcklund transformation and Darboux transformation, with MKdV equation as an example.

MKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0, \tag{0-1}$$

or equivalently

$$v_t + 2v_x^3 + v_{xxx} = 0 \tag{0-2}$$

($u = v_x$) has a Bäcklund transformation[29]

$$\begin{aligned} v'_x &= v_x + \beta \sin(v' + v) \\ v'_t &= v_t - \beta(2v_{xx}\cos(v' + v) + 2v_x^2 \sin(v' + v) + \beta(v'_x + v_x)) \end{aligned} \tag{0-3}$$

where β is a real parameter.

The integrability condition of equations (0-3) for v' is exactly (0-2). Moreover, if v is a solution of (0-2), then v' is also a solution of (0-2). Hence, from a given solution v of (0-2), we get a new solution v' of (0-2) by solving a simpler equation (0-3). In the simplest case, let $v = 0$, we get the one soliton solution

$$u' = \beta \operatorname{sech}(\beta(x - \beta^2 t))$$

of (0-1).

(0-3) is still nonlinear, and is complicated if u is not very simple. Therefore, people always try to get permutability theorem and nonlinear superposition formula if Bäcklund transformation is known.

In the above example, given a solution v of (0-2), we get a new solution v_1 by Bäcklund transformation with parameter β_1, then get v_{12} from v_1 by Bäcklund transformation with β_2 ($\beta_2 \neq \beta_1$). Likewise, we can get v_2 by Bäcklund transformation with β_2, then get v_{21} from v_2 by Bäcklund transformation with β_1. The permutability theorem is: $v_{12} = v_{21}$

holds for properly chosen integral constants. They can also be expressed by an explicit formula (nonlinear superposition formula) of v, v_1 and v_2:

$$v_{12} = v_{21} = v + 2\tan^{-1}\Big(\frac{\beta_1+\beta_2}{\beta_1-\beta_2}\tan\frac{v_1-v_2}{2}\Big). \tag{0-4}$$

The nonlinear superposition formula simplifies the procedure of getting solutions by Bäcklund transformation. However, we still need to solve a nonlinear integrable equation (Bäcklund transformation) in the first step. What is more, the proof of permutability is very complicated, and usually is carried out for every specific equation.

In order to obtain a simplified way of getting solutions, we need to use Lax pair. In 1968, Lax[19] introduced the equation

$$L_t = [B, L] \tag{0-5}$$

where $L(x,t,u,u_x,\cdots)$, $B(x,t,u,u_x,\cdots)$ are two matrix-valued differential operators with respect to x, which are related to the unknowns $u=(u_1,\cdots,u_s)$. Equation (0-5) is the integrability condition of

$$L\Phi = \lambda\Phi, \quad \Phi_t = B\Phi \tag{0-6}$$

(λ is a parameter), which is called the Lax pair of (0-5).

A lot of useful equations, such as KdV equation, MKdV equation, nonlinear Schrödinger equation equation, Boussinesq equation, N-wave equation, have Lax pairs of this kind. For MKdV equation,

$$\begin{aligned} L &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial - \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix} \\ B &= -4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial^3 - 6\begin{pmatrix} u^2 & u_x \\ u_x & u^2 \end{pmatrix}\partial - 3\begin{pmatrix} 2uu_x & -u_{xx} \\ u_{xx} & 2uu_x \end{pmatrix}. \end{aligned} \tag{0-7}$$

On the other hand, some equations

$$F(x,t,u,u_x,u_t,u_{xx},\cdots) = 0 \tag{0-8}$$

of unknowns $u=(u_1,\cdots,u_s)$ have another type of Lax pair

$$\Phi_x = U[u,x,t,\lambda]\Phi, \quad \Phi_t = V[u,x,t,\lambda]\Phi. \tag{0-9}$$

In other words, (0-8) is equivalent to the integrability condition

$$U_t - V_x + [U,V] = 0 \tag{0-10}$$

of (0-9). Here U, V are $\mathcal{M}_N$-valued rational functions of λ, which depend on u in a certain way (usually they are functions of u and its derivatives). $\mathcal{M}_N$ denotes all $N\times N$ complex matrices. We also write $U(x,t,\lambda)$, $V(x,t,\lambda)$ for $U[u,x,t,\lambda]$, $V[u,x,t,\lambda]$.

Apart from the equations mentioned above, those like sine-Gordon equation, the equation for principal chiral field can also be written as the integrability condition of this type of Lax pair. For MKdV equation,

$$\begin{aligned} U &= \begin{pmatrix} \lambda & u \\ -u & -\lambda \end{pmatrix} \\ V &= \begin{pmatrix} -4\lambda^3 - 2u^2\lambda & -4u\lambda^2 - 2u_x\lambda - 2u^3 - u_{xx} \\ 4u\lambda^2 - 2u_x\lambda + 2u^3 + u_{xx} & 4\lambda^3 + 2u^2\lambda \end{pmatrix}. \end{aligned} \tag{0-11}$$

In order to avoid the nonlinear equation like (0-3), we can make use of another kind of transformation — Darboux transformation.

In 1882, Darboux found out that if ϕ, h are two solutions of the Schrödinger equation

$$-\phi_{xx} + u\phi = \lambda\phi, \tag{0-12}$$

then $\widetilde{\phi} = \phi_x - \phi h_x/h$ is a solution of another Schrödinger equation

$$-\widetilde{\phi}_{xx} + \widetilde{u}\widetilde{\phi} = \lambda\widetilde{\phi} \tag{0-13}$$

where

$$\widetilde{u} = u - 2(\ln h)_{xx}. \tag{0-14}$$

Since Schrödinger equation is a part of the Lax pair of KdV equation, this transformation was applied to KdV equation, then to other equations such as MKdV equation, Sine-Gordon equation, nonlinear Schrödinger equation by [33, 17]. Also, it provided the transformation of the wave function ϕ, which actually leads to the Darboux matrices.

Generally, for the equation (0-8) with Lax pair (0-9), a matrix $G(x,t,\lambda)$ is called a Darboux matrix if there exists $\widetilde{u}$ such that for any solution Φ of (0-8), $\widetilde{\Phi} = G\Phi$ satisfies

$$\widetilde{\Phi}_x = \widetilde{U}\Phi, \quad \widetilde{\Phi}_t = \widetilde{V}\Phi \tag{0-15}$$

where $\widetilde{U} = U[\widetilde{u},x,t,\lambda]$, $\widetilde{V} = V[\widetilde{u},x,t,\lambda]$. In this case, the transformation $(u,\Phi) \to (\widetilde{u},\widetilde{\Phi})$ is called a Darboux transformation.

The transformation of Φ, U, V is similar to the gauge transformation in gauge theory, so it is also called gauge transformation. Therefore, U, V are transformed as the potential (components of the connection) and the zero-curvature condition (0-10) keeps invariant under the transformation, i.e.

$$\widetilde{U} = GUG^{-1} + G_xG^{-1}, \quad \widetilde{V} = GVG^{-1} + G_tG^{-1}, \tag{0-16}$$

and

$$\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U},\widetilde{V}] = 0. \tag{0-17}$$

Thus, $\widetilde{u}$ is another solution of (0-8).

The subject of Chapter 1 is the nondegenerate Darboux matrix of first order, i.e. a Darboux matrix $G(x,t,\lambda) = \lambda R(x,t) - T(x,t)$ where R is invertible, and the poles of U, V are not the eigenvalues of $S = R^{-1}T$. Here, R is useful only when reduction is considered, otherwise, we can choose $R = I$.

For MKdV equation, the Darboux matrix of first order is

$$G = \lambda - \frac{\lambda_0}{1+\sigma^2}\begin{pmatrix} 1-\sigma^2 & 2\sigma \\ 2\sigma & -1+\sigma^2 \end{pmatrix} \tag{0-18}$$

where $\sigma = \eta/\xi$, $(\xi,\eta)^T$ is a solution of the Lax pair with $\lambda = \lambda_0$ and U, V given by (0-11). The new solution of (0-1) given by this Darboux transformation is

$$\widetilde{u} = u + \frac{4\lambda_0\sigma}{1+\sigma^2} \tag{0-19}$$

which come from (0-16). If we eliminate σ in (0-19), the original Bäcklund transformation (0-3) is obtained again.

If the potential decays fast enough at infinity, the Darboux matrix of first order is essentially equivalent to the dressing method [34, 20, 22]. Here, it is also valid for the potentials without boundary condition. In the point of view of inverse scattering theory, if the potential decays fast enough at infinity, the action of Darboux matrix of first order increases or decreases solitons in most cases [27, 25, 30, 3].

A Darboux matrix provides new solutions $\widetilde{u}$, $\widetilde{\Phi}$ from a given solution u of (0-8) and a solution Φ of the corresponding Lax pair (0-9). Moreover, another Darboux matrix can be constructed explicitly from $\widetilde{\Phi}$, and we get new solution $\widetilde{\widetilde{u}}$, $\widetilde{\widetilde{\Phi}}$. Continuing this procedure, we can get a series of solutions of the equation by a purely algebraic way from given solutions u and Φ.

The Darboux matrices for a lot of specific equations have been known since 1970's. After that, the Darboux matrix of first order[22] and higher order[26] in a quite general form were also provided for 2×2 ZS-AKNS system. However, if these Darboux matrices had auto-Bäcklund property was still unknown. This problem was first solved by [10, 8] for KdV, MKdV-SG hierarchies, then by [11, 24] for 2×2 ZS-AKNS system. The Darboux matrix for ZS-AKNS system of arbitrary order was given by [9, 30], with the proof of auto-Bäcklund property in [9].

The ZS-AKNS system is

$$\Phi_x = (\lambda J + P(x,t))\Phi, \quad \Phi_t = \sum_{j=0}^{n} V_{n-j}\lambda^j \Phi \tag{0-20}$$

where J is a constant diagonal matrix with mutually different diagonal entries, P is valued in off-diagonal matrices, V_j's are actually differential polynomials of P. For (0-20), Darboux matrix can be constructed as follows [9].

Given $\lambda_1, \cdots, \lambda_N \in \mathbf{C}$, let h_i be a solution of (0-20) with $\lambda = \lambda_i$, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, $H = (h_1, \cdots, h_N)$. If $\det H \neq 0$, then $G = \lambda - H\Lambda H^{-1}$ is a Darboux matrix of (0-20).

[30] also proved that for ZS-AKNS system, these are the all possible Darboux matrices of form $\lambda - S(x,t)$ if P is integrable with respect to x in whole real line and G is bounded for fixed λ, t.

The natural questions are, what are the Darboux matrices for other integrable systems, and what happens for the solution which does not decay at infinity? Theorem 1 and Theorem 2 in this paper provide a solution to these questions.

In §1.1, we discuss the Darboux matrix for unreduced equation (0-10), i.e. we regard each entry of the coefficients of U, V are independent unknown. (Here U, V can be any $\mathcal{M}_N$-valued rational functions of λ.) Theorem 1 provides all the nondegenerate Darboux matrices of first order, which depend on the solutions of the Lax pair with nonhomogeneous terms. Theorem 2 gives a more constructive way to show that any nondegenerate Darboux matrix of first order can be expressed in the form of [9] or a certain kind of limit of those matrices. Also, Theorem 2 provides all the nondegenerate Darboux matrices of first order explicitly in terms of their initial values and the fundamental solution of the Lax pair.

Unlike in the case of the original Bäcklund transformation, we can obtain the permutability theorem (Theorem 3) for all the equations as (0-10) by twice Darboux transformations (for ZS-AKNS system, see also [26, 11, 13]), while the nonlinear superposition formula is just equivalent to the twice Darboux transformations.

The unreduced equation (0-10) is an equation with a lot of unknowns. However, in practical problems, unknowns are very few, and reductions often appear. In other words, there are usually some relations among the entries of the coefficients of U, V. Also, in many cases, V can be determined by U in some sense. Therefore, if the Darboux matrix for the system (0-10) keeps reduction is an important problem.

One of the most popular reduction is that U, V satisfy $U(-\bar{\lambda}) = -U^*(\lambda)$, $V(-\bar{\lambda}) = -V^*(\lambda)$. The Darboux matrix keeping this kind of reduction is given in Theorem 4. Also, the general form of Darboux matrices which keep the form of the Lax pairs of principal chiral field and Kaup-Newell system is determined in Theorem 5. After that, we detremine the Darboux matrix of first order for some concrete equations: $U(N)$ and $O(2N)$ principal chiral field, Kaup-Newell system and derivative nonlinear Schrödinger equation.

In Chapter 2, 1+2 dimensional problem is discussed. KP equation is one of the simplest integrable equations in 1+2 dimensions. Its Bäcklund transformation was obtaned in 1975 by Hirota and Satsuma. The Bäcklund transformation for some other equations such as 1+3 dimensional sine-Gordon equation, R-gauge Yang-Mills equation were also found. Apart form that, most conclusions in higher dimensions were obtained in the last decade. The inverse scattering transformation for KP equation was got in 1983 [1], and that for 1+2 dimensional ZS-AKNS system was obtained by [7, 2]. Bäcklund transformations for ZS-AKNS system was also known by [4] in 1985, while for a special case — Davey-Stewartson equation, the Bäcklund transformation is got by [21]. As for Darboux transformation, the only known Darboux operator took the original form of Darboux, which was valid for some equations with scalar Lax pair as Kadomtsev-Petviashvili (KP) equation. However, the Darboux operators for the equations with matrix Lax pair, such as Davey-Stewartson equation, N-wave equation, were still unknown. This paper is devoted to finding the Darboux operator for general Lax pair in matrix form.

We discuss the equation of unknowns $u = (u_1, \cdots, u_s)$:

$$F(x, y, t, u, u_x, u_y, u_t, u_{xx}, \cdots) = 0 \tag{0-21}$$

which possesses a Lax pair

$$\Phi_y = U[u, x, y, t, \partial]\Phi, \quad \Phi_t = V[u, x, y, t, \partial]\Phi, \tag{0-22}$$

i.e. it can be written as

$$U_t(\partial) - V_y(\partial) + [U(\partial), V(\partial)] = 0, \tag{0-23}$$

which is the integrability condition of (0-22). In the above equations, $U[u, x, y, t, \partial]$, $V[u, x, y, t, \partial]$ are two $\mathcal{M}_N$-valued differential operators with respect to x, whose coefficients depend on u in a certain way. $U(\partial)$, $V(\partial)$ are simplified notation for $U[u, x, y, t, \partial]$, $V[u, x, y, t, \partial]$. $\partial = \partial/\partial x$.

For example, KP equation

$$u_{xt} = \frac{1}{4}(u_{xxx} + 6uu_x)_x + \frac{3}{4}u_{yy} \tag{0-24}$$

or

$$v_{xt}=\frac{1}{4}(v_{xxx}+6v_xv_{xx})+\frac{3}{4}v_{yy} \tag{0-25}$$

$(u=v_x)$ has

$$U(\partial)=\partial^2+u,\quad V(\partial)=\partial^3+\frac{3u}{2}\partial+\frac{3}{4}(v_y+u_x). \tag{0-26}$$

It is well-known that KP equation has a Bäcklund transformation [29]

$$\begin{array}{l} v'_y=v_y+v'_{xx}+v_{xx}+(v'-v)(v'_x-v_x),\\ v'_t=v_t+v'_{xxx}+\frac{1}{2}v_{xxx}+\frac{3}{2}(v'-v)v'_{xx}+\frac{3}{2}(v'_x-v_x)v'_x\\ +\frac{3}{4}(v'-v)^2(v'_x-v_x)+\frac{3}{2}v_{xy}. \end{array} \tag{0-27}$$

More generally, [18] provided a Bäcklund transformation for the equations connecting with 1+2 dimensional Gelfand-Dikij-Zakharov-Shabat system:

$$U(\partial)=\sum_{j=0}^{m}U_{m-j}\partial^j,$$

$U_0=I$, and U_j's are scalar functions.

For 1+2 dimensional ZS-AKNS system,

$$U(\partial)=J\partial+P$$

(J is a constant diagonal matrix with mutually different diagonal entries, P is valued in off-diagonal matrices), [4] got a Bäcklund transformation in the form of integro-differential equations as (0-27). However, the Bäcklund transformations mentioned above are rather complicated comparing with that of 1+1 dimensional problems except for that of KP equation.

In this paper, we shall discuss 1+2 dimensional Darboux transformation, which can greatly simplify the procedure of getting explicit solutions.

It is known that KP equation has a Darboux operator $\partial-h_x/h$ where h is a solution of the Lax pair corresponding to (0-26). This Darboux operator provides a transformation of the solution[32]

$$\widetilde{u}=u+2(h_x/h)_x. \tag{0-28}$$

For general equation (0-21) with Lax pair (0-22), a differential operator $G(x,y,t,\partial)$ with respect to to x is called a Darboux operator if there exists $\widetilde{u}$ such that for any solution Φ of (0-22), $\widetilde{\Phi}=G(\partial)\Phi$ satisfies

$$\widetilde{\Phi}_y=\widetilde{U}(\partial)\widetilde{\Phi},\quad \widetilde{\Phi}_t=\widetilde{V}(\partial)\widetilde{\Phi}, \tag{0-29}$$

where $\widetilde{U}(\partial)=U[\widetilde{u},x,y,t,\partial]$, $\widetilde{V}(\partial)=V[\widetilde{u},x,y,t,\partial]$.

The transformation of U, V has a similar relation as (0-16), i.e.

$$\begin{array}{l} \widetilde{U}(\partial)G(\partial)=G(\partial)U(\partial)+G_y(\partial),\\ \widetilde{V}(\partial)G(\partial)=G(\partial)V(\partial)+G_t(\partial), \end{array} \tag{0-30}$$

and

$$\widetilde{U}_t(\partial) - \widetilde{V}_y(\partial) + [\widetilde{U}(\partial), \widetilde{V}(\partial)] = 0. \tag{0-31}$$

Nondegenerate Darboux operator of first order is the most fundamental one which is a Darboux operator $G(x,y,t,\partial) = R(x,y,t)(\partial - S(x,y,t))$ with R nondegenerate. This is the main object we want to find. What is more, we can always choose $R - I$ if there is no reduction, i.e. the coefficients of U, V are independent unknowns. In this case, Theorem 6 gives all the nondegenerate Darboux operators of first order for general Lax pair (0-22) in matrix form. That is, $\partial - S$ is a Darboux operator if and only if $S = H_x H^{-1}$ for certain nondegenerate matrix solution H of (0-22).

By virtue of this conclusion, we can get a series of solutions of a nonlinear partial differential equation from a known solution by successive action of Darboux operators as in 1+1 dimensions.

In §2.1, we also discuss the properties of Darboux operator of higher order, and give the permutability theorem (Theorem 7 and Corollary).

As applications, ZS-AKNS system and KP hierarchy are discussed, and the Darboux operators are provided. The result of the former is completely new and that of the latter is a generalization of the conclusion for KP equation. Also, concrete results for Davey-Stewartson equation are given.

If the coefficients of U, V are independent of y, then Theorem 6 leads to Theorem 8, which provides all the nondegenerate Darboux operators of first order for the Lax pair (0-6).

Acknowledgements

The Darboux matrices for principal chiral field was obtained by Prof. Gu Chaohao at the same time.

I would like to express my deep gratitude to Prof. Gu Chaohao for the assistance and advice in the whole period of preparing this paper. Also, I would like to thank Prof. Hu Hesheng, Prof. Li Yisheng, Prof. Tu Guizhang, Prof. Gu Xinshen, Dr. Cheng Yi and Dr. Lin Junmin for helpful advice.

Chapter 1

1+1 Dimensional Darboux Transformation

1.1 General form of Darboux matrices

In this part, Ω is a simply connected domain in $\mathbf{R}^2$, and $\nu_1, \cdots, \nu_l \in \mathbf{C}$.

We discuss the equation (or the system of equations) (0-8):

$$F(x, t, u, u_x, u_t, u_{xx}, \cdots) = 0 \tag{1-1-1}$$

of unknowns $u = (u_1, \cdots, u_s)$ with Lax pair

$$\Phi_x = U[u, x, t, \lambda]\Phi, \quad \Phi_t = V[u, x, t, \lambda]\Phi. \tag{1-1-2}$$

Here U, V are $\mathcal{M}_N$-valued functions which may be differential polynomials of u. Moreover, they are rational functions of the spectral parameter and can be written in the form

$$\begin{aligned} U(x, t, \lambda) &= \sum_{j=0}^{m} U_j(x, t)\lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)}(x, t)(\lambda - \nu_k)^{-j}, \\ V(x, t, \lambda) &= \sum_{j=0}^{n} V_j(x, t)\lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{n_k} V_j^{(k)}(x, t)(\lambda - \nu_k)^{-j} \end{aligned} \tag{1-1-3}$$

provided that $u(x, t)$ is given. (1-1-2) is a Lax pair of (1-1-1) means that (1-1-1) is equivalent to the integrability condition

$$U_t - V_x + [U, V] = 0 \tag{1-1-4}$$

of (1-1-2).

In this section, we discuss the Darboux matrix for Lax pair without reduction, i.e. we regard the entries of U_j, V_j, $U_j^{(k)}$, $V_j^{(k)}$ as independent unknowns. In this case, the general definition of Darboux matrix in the Introduction implies that G is a Darboux matrix is equivalent to that there exist

$$\begin{aligned}\widetilde{U}(x,t,\lambda) &= \sum_{j=0}^{m} \widetilde{U}_j(x,t)\lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} \widetilde{U}_j^{(k)}(x,t)(\lambda-\nu_k)^{-j},\\ \widetilde{V}(x,t,\lambda) &= \sum_{j=0}^{n} \widetilde{V}_j(x,t)\lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{n_k} \widetilde{V}_j^{(k)}(x,t)(\lambda-\nu_k)^{-j}\end{aligned} \tag{1-1-5}$$

such that $\widetilde{\Phi} = G\Phi$ satisfies

$$\widetilde{\Phi}_x = \widetilde{U}\widetilde{\Phi}, \quad \widetilde{\Phi}_t = \widetilde{V}\widetilde{\Phi} \tag{1-1-6}$$

for any fundamental solution Φ of (1-1-2).

Under the Darboux matrix G, (0-16) and (0-17) give

$$\widetilde{U} = GUG^{-1} + G_xG^{-1}, \quad \widetilde{V} = GVG^{-1} + G_tG^{-1} \tag{1-1-7}$$

and

$$\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U},\widetilde{V}] = 0. \tag{1-1-8}$$

Thus, we get another solution of (1-1-4) by the Darboux matrix G.

Now we shall discuss the nondegenerate Darboux matrix of first order, i.e. the Darboux matrix of form $\lambda R(x,t) - T(x,t)$ satisfying the condition

(I): R is invertible, $\nu_1, \cdots, \nu_l$ are not the eigenvalues of $S = R^{-1}T$.

In this case, $G = R(\lambda - S)$, and we may set $R = I$ without loss of generality since R is a trivial Darboux matrix if there is no reduction.

The following theorem provides all the possible nondegenerate Darboux matrices of first order.

Theorem 1 *Under the assumption (I), $G(x,t,\lambda) = \lambda - S(x,t)$ is a Darboux matrix of (1-1-2) if and only if $S = K\Gamma K^{-1}$, where Γ is a constant matrix, and K is an $\mathcal{M}_N$-valued nondegenerate solution of the integrable equations*

$$\begin{aligned}K_x &= \sum_{j=0}^{m} U_jK\Gamma^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)}K(\Gamma-\nu_k)^{-1},\\ K_t &= \sum_{j=0}^{n} V_jK\Gamma^j + \sum_{k=1}^{l}\sum_{j=1}^{n_k} V_j^{(k)}K(\Gamma-\nu_k)^{-1}.\end{aligned} \tag{1-1-9}$$

It is required to obtain the explicit solutions of (1-1-4) and the Lax pair (1-1-2). However, it is not easy to get the explicit solution of (1-1-9). In order to derive a series of solutions of (1-1-4) by successive Darboux transformations, we hope to express the Darboux matrix only through the solutions of the Lax pair. This is possible due to the following theorem.

For any open subset D of Ω, let $\mathcal{S}(D)$ be the set of all the matrices $H(x,t)\Lambda H(x,t)^{-1}$ such that $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ is a constant diagonal matrix, $H = (h_1, \cdots, h_N)$ is nondegenerate, where h_i is a solution of (1-1-2) in D with $\lambda = \lambda_i$. Then we have

Theorem 2 *Under the assumption (I), $G(x,t,\lambda)=\lambda-S(x,t)$ is a Darboux matrix of (1-1-2) if and only if (i) $S\in\mathcal{S}(\Omega)$, or (ii) there exist open sets $\Omega_k\subset\Omega$ $(k=1,2,\cdots)$ satisfying $\Omega_1\subset\Omega_2\subset\cdots$ and $\bigcup_{k=1}^{\infty}\Omega_k=\Omega$, and $S_k\in\mathcal{S}(\Omega_k)$ such that for any point $(x,t)\in\Omega_k$, the sequences $\{S_i(x,t)\}$, $\{S_{i,x}(x,t)\}$, $\{S_{i,t}(x,t)\}$ converge to $S(x,t)$, $S_x(x,t)$, $S_t(x,t)$ respectively for $i\geq k$.*

Before the proof of these two theorems, we first derive an equation which a Darboux matrix should satisfy.

For $M\in\mathcal{M}_N$ with eigenvalues different from $\nu_1,\cdots,\nu_l$, define

$$U(M)=\sum_{j=0}^{m}U_jM^j+\sum_{k=1}^{l}\sum_{j=1}^{m_k}U_j^{(k)}(M-\nu_k)^{-1}, \tag{1-1-10}$$

then we have

Lemma 1 *Under the nondegenerate Darboux matrix $G=R(\lambda-S)$,*

$$GUG^{-1}+G_xG^{-1}=\sum_{j=0}^{m}\widetilde{U}_j\lambda^j+\sum_{k=1}^{l}\sum_{j=1}^{m_k}\widetilde{U}_j^{(k)}(\lambda-\nu_k)^{-j}-R\Delta(\lambda-S)^{-1}R^{-1}. \tag{1-1-11}$$

Here

$$\begin{aligned}
&\widetilde{U}_j=R\Big(U_j+\sum_{i=1}^{m-j}[U_{j+i},S]S^{i-1}\Big)R^{-1}+R_xR^{-1}\delta_{j0},\\
&\widetilde{U}_j^{(k)}=R\Big((S-\nu_k)U_j^{(k)}(S-\nu_k)^{-1}-\sum_{i=1}^{m_k-j}[U_{j+i}^{(k)},S](S-\nu_k)^{-i-1}\Big)R^{-1},\\
&\Delta=S_x+[S,U(S)].
\end{aligned} \tag{1-1-12}$$

Therefore, G is a Darboux matrix if and only if

$$S_x+[S,U(S)]=0,\quad S_t+[S,V(S)]=0. \tag{1-1-13}$$

Proof.

$$\begin{aligned}
&GUG^{-1}+G_xG^{-1}\\
&=R(\lambda-S)U(\lambda)(\lambda-S)^{-1}R^{-1}+R_xR^{-1}-RS_x(\lambda-S)^{-1}R^{-1}\\
&=RU(\lambda)R^{-1}-R[S,(U(\lambda)-U(S))(\lambda-S)^{-1}]R^{-1}+R_xR^{-1}-R\Delta(\lambda-S)^{-1}R^{-1}.
\end{aligned} \tag{1-1-14}$$

From the identities

$$\begin{aligned}
&(\lambda^j-S^j)(\lambda-S)^{-1}=\sum_{i=0}^{j-1}S^{j-i-1}\lambda^i,\\
&\Big((\lambda-\nu_k)^{-j}-(S-\nu_k)^{-j}\Big)(\lambda-S)^{-1}=-\sum_{i=1}^{j}(S-\nu_k)^{-j+i+1}(\lambda-\nu_k)^{-i},
\end{aligned} \tag{1-1-15}$$

we get (1-1-11) by direct calculation. **QED.**

Proof of Theorem 1. Suppose K is a nondegenerate solution of (1-1-9), and let $S = K\Gamma K^{-1}$, then

$$K_x K^{-1} = U(S). \tag{1-1-16}$$

Hence

$$S_x = K_x \Gamma K^{-1} - K\Gamma K^{-1} K_x K^{-1} = [U(S), S].$$

This implies $G = \lambda - S$ is a Darboux matrix by Lemma 1.

Conversely, suppose $\lambda - S$ is a Darboux matrix, then

$$S_x + [S, U(S)] = 0, \quad S_t + [S, V(S)] = 0.$$

Since U_t, V_x, UV, VU are all $\mathcal{M}_N$-valued rational functions of λ, we can define $U_t(S)$, $V_x(S)$, $(UV)(S)$, $(VU)(S)$ as (1-1-10). Thus, the identity

$$U_t(\lambda) - V_x(\lambda) + [U(\lambda), V(\lambda)] = 0$$

implies

$$U_t(S) - V_x(S) + (UV)(S) - (VU)(S) = 0. \tag{1-1-17}$$

However,

$$\begin{aligned}
(U(S))_t &= \Big(\sum_{j=0}^{m} U_j S^j\Big)_t + \Big(\sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)} (S-\nu_k)^{-j}\Big)_t \\
&= \sum_{j=0}^{m} U_{j,t} S^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_{j,t}^{(k)} (S-\nu_k)^{-j} + \sum_{j=0}^{m}\sum_{i=0}^{j-1} U_j S^i S_t S^{j-i-1} \\
&\quad - \sum_{k=1}^{l}\sum_{j=1}^{m_k}\sum_{i=0}^{j-1} U_j^{(k)} (S-\nu_k)^{-i-1} S_t (S-\nu_k)^{-j+i} \\
&= U_t(S) + \sum_{j=0}^{m}\sum_{i=0}^{j-1} U_j S^i [V(S), S] S^{j-i-1} \\
&\quad - \sum_{k=1}^{l}\sum_{j=1}^{m_k}\sum_{i=0}^{j-1} U_j^{(k)} (S-\nu_k)^{-i-1} [V(S), S] (S-\nu_k)^{-j+i} \\
&= U_t(S) + \sum_{j=0}^{m} U_j [V(S), S^j] + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)} [V(S), (S-\nu_k)^{-j}] \\
&= U_t(S) + \sum_{j=0}^{m} U_j V(S) S^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)} V(S) (S-\nu_k)^{-j} \\
&\quad - \sum_{j=0}^{m} U_j S^j V(S) + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)} (S-\nu_k)^{-j} V(S) \\
&= U_t(S) + (UV)(S) - U(S)V(S).
\end{aligned}$$

Likewise,

$$V(S)_x = V_x(S) + (VU)(S) - V(S)U(S).$$

Combining these with (1-1-17), we have

$$(U(S))_t - (V(S))_x + [U(S), V(S)] = 0. \tag{1-1-18}$$

Chose any point $(x_0, t_0) \in \Omega$, let $\Gamma = S(x_0, t_0)$. Then (1-1-18) implies that the equation

$$K_x = U(S)K, \quad K_t = V(S)K, \quad K(x_0, t_0) = I \tag{1-1-19}$$

has a unique solution $K$with $\det K \neq 0$ in Ω.

Let $S' = S - K\Gamma K^{-1}$, then

$$S'_x = [U(S), S] - [U(S), K\Gamma K^{-1}] = [U(S), S'], \quad S'_t = [V(S), S']$$

and

$$S'(x_0, t_0) = 0.$$

Hence

$$S = K\Gamma K^{-1}$$

everywhere. (1-1-18) implies (1-1-9). This proves the theorem. **QED.**

Proof of Theorem 2. First suppose $S = H\Lambda H^{-1} \in \mathcal{S}(\Omega)$, where Λ is a constant diagonal matrix, and $H = (h_1, \cdots, h_N)$ with h_i a solution of (1-1-2) as $\lambda = \lambda_i$. Then H is a solution of (1-1-9) with $\Gamma = \Lambda$. Hence $G = \lambda - S$ is a Darboux matrix of (1-1-2) by Theorem 1. On the other hand, if S satisfies (ii) of this theorem, G is also a Darboux matrix by the limit of (1-1-13). This proves the necessity.

Now suppose $\lambda - S$ is a Darboux matrix. From Theorem 1, $S = K\Gamma K^{-1}$ where Γ is a constant matrix and K satisfies (1-1-9).

If Γ is diagonalizable, $\Gamma = T\Lambda H^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$. Let $H = KT$. Then $S = H\Lambda H^{-1}$, and

$$\begin{aligned} H_x &= \sum_{j=0}^{m} U_j H\Lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)} H(\Lambda - \nu_k)^{-j}, \\ H_t &= \sum_{j=0}^{m} V_j H\Lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{n_k} V_j^{(k)} H(\Lambda - \nu_k)^{-j}. \end{aligned}$$

If we write $H = (h_1, \cdots, h_N)$, then h_i is a solution of (1-1-2) with $\lambda = \lambda_i$. Hence $S \in \mathcal{S}(\Omega)$.

Now suppose Γ is not diagonalizable. Choose a constant matrix Θ such that $\Gamma^{(\varepsilon)} = \Gamma + \varepsilon\Theta$ is diagonalizable with eigenvalues different from ν_k $(k = 1, \cdots, l)$ for sufficiently small ε. Let (x_0, t_0) be any point in Ω and $K^{(\varepsilon)}$ be the solution of

$$\begin{aligned} K_x^{(\varepsilon)} &= \sum_{j=0}^{m} U_j K^{(\varepsilon)}(\Gamma^{(\varepsilon)})^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)} K^{(\varepsilon)}(\Gamma^{(\varepsilon)} - \nu_k)^{-j}, \\ K_t^{(\varepsilon)} &= \sum_{j=0}^{m} V_j K^{(\varepsilon)}(\Gamma^{(\varepsilon)})^j + \sum_{k=1}^{l}\sum_{j=1}^{n_k} V_j^{(k)} K^{(\varepsilon)}(\Gamma^{(\varepsilon)} - \nu_k)^{-j}, \\ K^{(\varepsilon)}&(x_0, t_0) = K(x_0, t_0). \end{aligned} \tag{1-1-20}$$

Let $K_k = K^{(1/k)}$, $\Gamma_k = \Gamma^{(1/k)}$, Ω_k be the interior of

$$\bigcap_{\varepsilon \le 1/k} \{(x,t) \in \Omega \,|\, \det K^{(\varepsilon)}(x,t) \neq 0\}.$$

Then, at any given point $(x',t') \in \Theta$, $\det K^{(\varepsilon)}(x',t') \neq 0$ for sufficiently small ε since $K^{(\varepsilon)}$ depends continuously on ε in Ω and $\det K(x',t') \neq 0$. Hence there exists k_0 such that $(x',t') \in \Omega_k$ for $k \ge k_0$. This implies $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$.

By (1-1-20), we have

$$K_k(x,t) \to K(x,t), \quad K_{k,x}(x,t) \to K_x(x,t), \quad K_{k,t}(x,t) \to K_t(x,t)$$

as $k \to \infty$ with $\det K_k(x,t) \neq 0$ in Ω_{k_0} for $k \ge k_0$.

Since Γ_k is diagonalizable, let

$$\Gamma_k = T_k \Lambda_k T_k^{-1}$$

with $\Lambda_k = \mathrm{diag}(\lambda_1^{(k)}, \cdots, \lambda_N^{(k)})$ and let

$$H_k = K_k T_k,$$

then $H_k = (h_1^{(k)}, \cdots, h_N^{(k)})$ where $h_j^{(k)}$ is a solution of (1-1-2) with $\lambda = \lambda_j^{(k)}$, and

$$\begin{aligned} &S_k = H_k \Lambda_k H_k^{-1} = K_k \Gamma_k K_k^{-1} \to K \Gamma K^{-1} = S, \\ &S_{k,x} = [H_{k,x} H_k^{-1}, S_k] = [K_{k,x} K_k^{-1}, S_k] \to [K_x K^{-1}, S] = S_x, \\ &S_{k,t} \to S_t \end{aligned}$$

at $(x,t) \in \Omega_{k_0}$ for $k \ge k_0$. **QED.**

Theorems 1 and 2 provide all the nondegenerate Darboux matrices if we do not consider the reduction. If there is reduction, Darboux matrices will be a part of that.

We can also show that there are Darboux matrices which are not diagonalizable, so in Theorem 2, (ii) is not included in (i) in general. Furthermore, the proof of Theorem 2 provides a constructive way to get nondiagonalizable Darboux matrices only through the solutions of the Lax pair. We repeat it briefly here.

For given constant matrix Γ, we can construct a Darboux matrix $\lambda - S$ satisfying $S(x_0,t_0) = \Gamma$ in the following way. Choose a constant matrix Θ such that $\Gamma_k = \Gamma + \Theta/k$ $(k = 1, 2, \cdots)$ is diagonalizable with eigenvalues different from ν_j $(j = 1, 2, \cdots, l)$ for sufficiently large k. Let $\Gamma_k = T_k \Lambda_k T_k^{-1}$ with $\Lambda_k = \mathrm{diag}(\lambda_1^{(k)}, \cdots, \lambda_N^{(k)})$. Let $H_k = (h_1^{(k)}, \cdots, h_N^{(k)})$ satisfy $H_k(x_0,t_0) = T_k$ where $h_j^{(k)}$ is a solution of (1-1-2) with $\lambda = \lambda_j^{(k)}$. Then $S = \lim_{k\to\infty} S_k$ with $S_k = H_k \Lambda_k H_k^{-1}$ gives a Darboux matrix $\lambda - S$. All the Darboux matrices of form $\lambda - S$ can be constructed in this way if there is no reduction.

Example. The equation

$$\mathrm{i} p_t = p_{xx} - 2p^2 q, \quad -\mathrm{i} q_t = q_{xx} - 2pq^2 \tag{1-1-21}$$

admits a Lax pair

$$\begin{aligned} \Phi_x &= \begin{pmatrix} \lambda & p \\ q & -\lambda \end{pmatrix} \Phi, \\ \Phi_t &= \begin{pmatrix} -2\mathrm{i}\lambda^2 + \mathrm{i}pq & -2\mathrm{i}\lambda p - \mathrm{i}p_x \\ -2\mathrm{i}\lambda + \mathrm{i}q_x & 2\mathrm{i}\lambda^2 - \mathrm{i}pq \end{pmatrix} \Phi. \end{aligned} \tag{1-1-22}$$

(1-1-21) has a solution

$$p = \alpha\,\mathrm{sech}(\alpha x)\mathrm{e}^{-\mathrm{i}\alpha^2 t}, \quad q = -\alpha\,\mathrm{sech}(\alpha x)\mathrm{e}^{\mathrm{i}\alpha^2 t}, \tag{1-1-23}$$

which is obtained from zero solution by a Darboux matrix $\lambda - S_0$ with

$$S_0 = \frac{\alpha}{2} \begin{pmatrix} \tanh(\alpha x) & \mathrm{sech}(\alpha x)\mathrm{e}^{-\mathrm{i}\alpha^2 t} \\ \mathrm{sech}(\alpha x)\mathrm{e}^{\mathrm{i}\alpha^2 t} & -\tanh(\alpha x) \end{pmatrix}. \tag{1-1-24}$$

So the fundamental solution for (1-1-22) with p, q given by (1-1-23) is

$$(\lambda - S_0) \begin{pmatrix} \mathrm{e}^{\lambda x - 2\mathrm{i}\lambda^2 t} & 0 \\ 0 & \mathrm{e}^{-\lambda x + 2\mathrm{i}\lambda^2 t} \end{pmatrix}. \tag{1-1-25}$$

As an example, we solve the Darboux matrix $\lambda - S$ with

$$S(0,0) = \Gamma = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Following the procedure given above, let

$$\Gamma^{(\varepsilon)} = \begin{pmatrix} \varepsilon & 1 \\ 0 & 0 \end{pmatrix},$$

then

$$\Gamma^{(\varepsilon)} = T^{(\varepsilon)} \Lambda^{(\varepsilon)} (T^{(\varepsilon)})^{-1}$$

where

$$\Lambda^{(\varepsilon)} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}, \quad T^{(\varepsilon)} = \begin{pmatrix} \dfrac{4\varepsilon^2 - \alpha^2}{2\alpha} & 1 \\ 0 & -\varepsilon \end{pmatrix}.$$

Now $H^{(\varepsilon)} = (h_1^{(\varepsilon)}, h_2^{(\varepsilon)})$, where $h_1^{(\varepsilon)}$, $h_2^{(\varepsilon)}$ are solutions of the Lax pair with $\lambda = \varepsilon$ and $\lambda = 0$ respectively, and $H^{(\varepsilon)}(0,0) = T^{(\varepsilon)}$. Hence, we have

$$H^{(\varepsilon)} = \begin{pmatrix} \varepsilon\Big(\dfrac{2\varepsilon}{\alpha} - \tanh(\alpha x)\Big)\mathrm{e}^{\theta} - \dfrac{\alpha}{2}\mathrm{sech}(\alpha x)\mathrm{e}^{-\mathrm{i}\alpha^2 t - \theta} & -\varepsilon\tanh(\alpha x) + \mathrm{sech}(\alpha x)\mathrm{e}^{-\mathrm{i}\alpha^2 t} \\ -\varepsilon\mathrm{sech}(\alpha x)\mathrm{e}^{\mathrm{i}\alpha^2 t + \theta} + \Big(\varepsilon + \dfrac{\alpha}{2}\tanh(\alpha x)\Big)\mathrm{e}^{-\theta} & -\varepsilon\mathrm{sech}(\alpha x)\mathrm{e}^{-\mathrm{i}\alpha^2 t} - \tanh(\alpha x) \end{pmatrix}$$

(where $\theta = \varepsilon x - 2\mathrm{i}\varepsilon^2 t$) from the fundamental solution (1-1-25). Then, we have

$$\begin{aligned} &H^{(\varepsilon)} \Lambda^{(\varepsilon)} (H^{(\varepsilon)})^{-1} \to \\ &S = \frac{\alpha}{\cosh(\alpha x)((2+\alpha)\cosh(\alpha x) - 2\mathrm{e}^{-\mathrm{i}\alpha^2 t})} \begin{pmatrix} \sinh(\alpha x)\mathrm{e}^{-\mathrm{i}\alpha^2 t} & \mathrm{e}^{-2\mathrm{i}\alpha^2 t} \\ -\sinh^2(\alpha x) & -\sinh(\alpha x) \end{pmatrix} \end{aligned}$$

as $\varepsilon \to 0$. This gives the Darboux matrix $\lambda - S$ satisfying $S(0,0) = \Gamma$, and provides a solution of (1-1-21):

$$\widetilde{p} = \frac{\alpha(\alpha+2)\mathrm{e}^{-\mathrm{i}\alpha^2 t}}{(\alpha+2)\cosh(\alpha x) - 2\mathrm{e}^{-\mathrm{i}\alpha^2 t}}, \quad \widetilde{q} = \frac{\alpha(2\cosh(\alpha x) - (\alpha+2)\mathrm{e}^{\mathrm{i}\alpha^2 t})}{(\alpha+2)\cosh(\alpha x) - 2\mathrm{e}^{-\mathrm{i}\alpha^2 t}}.$$

Now we consider the composition of two Darboux matrices. We have

Theorem 3 *Suppose* $G_\alpha(x,t,\lambda) = \lambda - S_\alpha(x,t)$ $(\alpha = 1,2)$ *are two nondegenerate Darboux matrices for* (U,V)*,* $\det(S_1 - S_2) \neq 0$*. Then*

$$G_{\beta\alpha}(x,t,\lambda) = \lambda - (S_\beta - S_\alpha)S_\beta(S_\beta - S_\alpha)^{-1} \tag{1-1-26}$$

is a Darboux matrix for

$$U_\alpha = G_\alpha U G_\alpha^{-1} + G_{\alpha,x}G_\alpha^{-1}, \quad V_\alpha = G_\alpha V G_\alpha^{-1} + G_{\alpha,t}G_\alpha^{-1}$$

$(\alpha,\beta = 1,2;\ \alpha \neq \beta)$ *and the permutability*

$$G_{21}(\lambda)G_1(\lambda) = G_{12}(\lambda)G_2(\lambda) \tag{1-1-27}$$

holds.

Proof. First suppose $S_\alpha(\alpha = 1,2)$ are diagonalizable. Let $S_\alpha = H_\alpha\Lambda_\alpha H_\alpha^{-1}$ where $\Lambda_\alpha = \mathrm{diag}(\lambda_1^{(\alpha)},\cdots,\lambda_N^{(\alpha)})$ with $\lambda_i^{(\alpha)} \neq \nu_j$ $(i = 1,\cdots,N;\ j = 1,\cdots,l;\ \alpha = 1,2)$, $H_\alpha = (h_1^{(\alpha)},\cdots,h_N^{(\alpha)})$, $h_i^{(\alpha)}$ is a solution of (1-1-2) with $\lambda = \lambda_i^{(\alpha)}$. Under the action of G_α, H_β changes to

$$\begin{aligned} H_{\beta\alpha} &= (G_\alpha(\lambda_1^{(\beta)})h_1^{(\beta)},\cdots,G_\alpha(\lambda_N^{(\beta)})h_N^{(\beta)}) \\ &= H_\beta\Lambda_\beta - H_\alpha\Lambda_\alpha H_\alpha^{-1}H_\beta \\ &= (S_\beta - S_\alpha)H_\beta \end{aligned} \tag{1-1-28}$$

$(\alpha,\beta = 1,2;\ \alpha \neq \beta)$. Therefore,

$$S_{\beta\alpha} = H_{\beta\alpha}\Lambda_\beta H_{\beta\alpha}^{-1} = (S_\beta - S_\alpha)S_\beta(S_\beta - S_\alpha)^{-1} \tag{1-1-29}$$

gives a Darboux matrix $G_{\beta\alpha}(\lambda) = \lambda - S_{\beta\alpha}$ for U_α, V_α. Moreover,

$$\begin{aligned} &G_{\beta\alpha}(\lambda)G_\alpha(\lambda) \\ &= \{\lambda - (S_\beta - S_\alpha)S_\beta(S_\beta - S_\alpha)^{-1}\}(\lambda - S_\alpha) \\ &= \lambda^2 - (S_\beta^2 - S_\alpha^2)(S_\beta - S_\alpha)^{-1}\lambda + (S_\beta - S_\alpha)S_\beta(S_\beta - S_\alpha)^{-1}S_\alpha. \end{aligned} \tag{1-1-30}$$

Noticing that

$$S_2(S_2 - S_1)^{-1}S_1 + S_1(S_1 - S_2)^{-1}S_2 = 0, \tag{1-1-31}$$

we know (1-1-30) is symmetric for S_α, S_β. Hence

$$G_{21}(\lambda)G_1(\lambda) = G_{12}(\lambda)G_2(\lambda). \tag{1-1-32}$$

For nondiagonalizable $G_\alpha = \lambda - S_\alpha(x,t)$, we have nondegenerate Darboux matrices $G_\alpha^{(k)}(\lambda) = \lambda - S_\alpha^{(k)}$ such that $S_\alpha^{(k)}$ is diagonalizable, $\det(S_1^{(k)} - S_2^{(k)}) \neq 0$, and $G_\alpha^{(k)} \to G_\alpha$ as $k \to \infty$. From the above discussion,

$$G_{\beta\alpha}^{(k)} = \lambda - (S_\beta^{(k)} - S_\alpha^{(k)})S_\beta^{(k)}(S_\beta^{(k)} - S_\alpha^{(k)})^{-1} \tag{1-1-33}$$

is a Darboux matrix for $U_\wedge(k)$, $V_\alpha^{(k)}$, which are transformed from U, V by $G_\alpha^{(k)}$, and

$$G_{21}^{(k)}G_1^{(k)} = G_{12}^{(k)}G_2^{(k)}. \tag{1-1-34}$$

Taking the limit in (1-1-33) and (1-1-34), we derive the conclusions in the theorem. **QED.**

This theorem shows that the permutability holds for any two generic Darboux matrices of first order.

1.2 Darboux matrices for reduced problems

In the last section, we considered the problem without reduction. However, there are reductions in many concrete problems. In the present section, we shall first discuss two basic reductions, then apply them to some equations of physical interest.

1.2.1 Two general reductions

First, we consider the following reduction: $U(x,t,\lambda)$, $V(x,t,\lambda)$ satisfy

$$U(-\varepsilon\bar{\lambda}) = -CU^\dagger(\lambda)C, \quad V(-\varepsilon\bar{\lambda}) = -CV^\dagger(\lambda)C \tag{1-2-1}$$

where C is a diagonal matrix with diagonal entries ± 1, "$\dagger$" implies complex conjugate transpose, $\varepsilon = \pm 1$.

The Darboux matrix keeping this reduction can be constructed in the following way.

Theorem 4 *Given $\mu \in \mathbf{C}$ ($\mu, -\varepsilon\bar{\mu}$ are not the poles of U, V), let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ with $\lambda_i = \mu$ or $(-\varepsilon\bar{\mu})$. Let h_i be a solution of (1-1-2) with $\lambda = \lambda_i$ in $\Omega' \subset \Omega$ such that $h_1, \cdots, h_N$ are linearly independent and $h_i^\dagger C h_j = 0$ if $\lambda_i = -\varepsilon\bar{\lambda}_j$. Let $H = (h_1, \cdots, h_N)$, $S = H\Lambda H^{-1}$. Moreover, let R be an invertible $\mathcal{M}_N$-valued function such that $R^\dagger CRC$ is a constant number. Then, in Ω', under the Darboux matrix $G = R(\lambda - S)$, $\widetilde{U}, \widetilde{V}$ (given by (1-1-7)) satisfy $\widetilde{U}(-\varepsilon\bar{\lambda}) = -C\widetilde{U}^\dagger(\lambda)C$, $\widetilde{V}(-\varepsilon\bar{\lambda}) = -C\widetilde{V}^\dagger(\lambda)C$.*

Proof. If $\lambda_i = -\varepsilon\bar{\lambda}_j$, then

$$(h_i^\dagger C h_j)_x = h_i^\dagger U^\dagger(\lambda_i) C h_j + h_i^\dagger C U(\lambda_j) h_j = 0$$

by (1-2-1). Similarly, $(h_i^\dagger C h_j)_t = 0$. Hence $h_i^\dagger C h_j = 0$ is possible.

By the definition of S,

$$Sh_j = \lambda_j h_j,$$

which implies

$$h_i^\dagger (S^\dagger C - \varepsilon CS) h_j = (\bar{\lambda}_i - \varepsilon h_j) h_i^\dagger C h_j.$$

Noticing that

$$\bar{\lambda}_i - \varepsilon\lambda_j = \bar{\mu} - \varepsilon\mu$$

if $\lambda_i = \lambda_j$, and

$$h_i^\dagger C h_j = 0$$

if $\lambda_i \neq \lambda_j$, we have

$$h_i^\dagger(S^\dagger C - \varepsilon CS)h_j = (\bar{\mu} - \varepsilon\mu)h_i^\dagger Ch_j. \tag{1-2-2}$$

Hence

$$S^\dagger C - \varepsilon CS = (\bar{\mu} - \varepsilon\mu)C \tag{1-2-3}$$

since $\{h_i(x,t)\}$ forms a basis of $\mathbf{R}^N$.

It is easy to see that

$$\Lambda^2 = (\mu - \varepsilon\bar{\mu})\Lambda + \varepsilon|\mu|^2. \tag{1-2-4}$$

Hence

$$S^2 = H\Lambda^2 H^{-1} = (\mu - \varepsilon\bar{\mu})S + \varepsilon|\mu|^2 \tag{1-2-5}$$

and

$$S^\dagger CS = (\varepsilon CSC + \bar{\mu} - \varepsilon\mu)CS = |\mu|^2 C. \tag{1-2-6}$$

Let

$$U'(\lambda) = (\lambda - S)U(\lambda)(\lambda - S)^{-1} - S_x(\lambda - S)^{-1}, \tag{1-2-7}$$

then

$$\begin{aligned}&U'(-\varepsilon\bar{\lambda}) + CU'^\dagger(\lambda)C\\ &= C(\bar{\lambda} - S^\dagger)^{-1}\{-(\bar{\lambda} - S^\dagger)C(\varepsilon\bar{\lambda} + S)CU^\dagger(\lambda)C + (\bar{\lambda} - S^\dagger)CS_x\\ &+U^\dagger(\lambda)(\bar{\lambda} - S^\dagger)C(\varepsilon\bar{\lambda} + S) - S_x^\dagger C(\varepsilon\bar{\lambda} + S)\}(\varepsilon\bar{\lambda} + S)^{-1} = 0\end{aligned} \tag{1-2-8}$$

by (1-2-3) and (1-2-6).

On the other hand,

$$\widetilde{U}(\lambda) = RU'(\lambda)R^{-1} + R_x R^{-1}, \tag{1-2-9}$$

thus

$$\begin{aligned}&\widetilde{U}(-\varepsilon\bar{\lambda}) + C\widetilde{U}^\dagger(\lambda)C\\ &= C(R^\dagger)^{-1}\{-R^\dagger CRCU'^\dagger(\lambda) + R^\dagger CR_x C + U'^\dagger(\lambda)R^\dagger CRC + R_x^\dagger CRC\}CR^{-1}\\ &= 0\end{aligned} \tag{1-2-10}$$

since $R^\dagger CRC$ is a constant number. This proves the theorem. **QED.**

Remark 1 *If $C = I$, we can find $\{h_i\}$ such that $h_i^\dagger h_j = 0$ for $\lambda_i \neq \lambda_j$. Therefore, these h_i's are linearly independent in whole Ω, and G is globally defined.*

Remark 2 *If ν_k, U_j, V_j, $U_j^{(k)}$, $V_j^{(k)}$ are all real, and*

$$U(-\lambda) = -CU^T(\lambda)C, \quad V(-\lambda) = -CV^T(\lambda)C,$$

then the Darboux matrix constructed above keeps the reduction provided μ is chosen to be real number, and h_i be real function.

Under reduction (1-2-1), we can get a more explicit expression of Darboux matrix. Choose specifically $\Lambda = \mathrm{diag}(-\varepsilon\bar{\mu}, -\varepsilon\bar{\mu}, \cdots, -\varepsilon\bar{\mu}, \mu)$. Let $h = (\xi_1, \cdots, \xi_N)^T$ be a solution of (1-1-2) as $\lambda = \mu$. Without loss of generality, we can choose

$$H = \begin{pmatrix} K & \xi \\ -\eta^T & \xi_N \end{pmatrix}$$

by Theorem 4, where

$$\begin{aligned}
&K = (K_{ij})_{1\le i,j\le N-1},\\
&\xi = (\xi_1,\cdots,\xi_{N-1})^T, \quad \eta = (\eta_1,\cdots,\eta_{N-1}^T) = \frac{K^T C'\bar{\xi}}{c_N\bar{\xi}_N},\\
&C = \mathrm{diag}(c_1,\cdots,c_N), \quad C' = \mathrm{diag}(c_1,\cdots,c_{N-1})
\end{aligned}$$

and $\det K \neq 0$, $\xi_N \neq 0$. ($C = \mathrm{diag}(c_1,\cdots,c_N)$, $C' = \mathrm{diag}(c_1,\cdots,c_{N-1})$.) Then

$$H^{-1} = \begin{pmatrix} 1 & -K^{-1}\xi \\ \xi_N^{-1}\eta^T & 1 \end{pmatrix}\begin{pmatrix} (K+\xi\eta^T/\xi_N)^{-1} & 0 \\ 0 & (\xi_N+\eta^T K^{-1}\xi)^{-1} \end{pmatrix}.$$

Using the fact that

$$(1+ab^T)^{-1} = 1 - \frac{ab^T}{1+a^T b}$$

for any vectors a and b, we have

$$\begin{aligned}
(K+\xi\eta^T/\xi_N)^{-1} &= K^{-1}\Big(1 - \frac{\xi\xi^\dagger C'}{\xi^\dagger C'\xi + c_N|\xi_N|^2}\Big)\\
(\xi_N+\eta^T K^{-1}\xi)^{-1} &= \frac{c_N\bar{\xi}_N}{\xi^\dagger C'\xi + c_N|\xi_N|^2}.
\end{aligned}$$

Hence by direct calculation,

$$S = -\varepsilon\bar{\mu} + \frac{\mu+\varepsilon\bar{\mu}}{\xi^\dagger C'\xi + c_N|\xi_N|^2}\begin{pmatrix} \xi\xi^\dagger C' & c_N\bar{\xi}_N\xi \\ \xi_N\xi^\dagger C' & c_N|\xi_N|^2 \end{pmatrix},$$

or, $S = (S_{ij})_{1\le i,j\le N}$ where

$$S_{ij} = -\varepsilon\bar{\mu}\delta_{ij} + \frac{(\mu+\varepsilon\bar{\mu})c_j\xi_i\bar{\xi}_j}{\sum_{k=1}^N c_k|\xi_k|^2}.$$

This gives the Darboux matrix

$$\begin{aligned}
&G = (G_{ij})_{1\le i,j\le N},\\
&G_{ij} = (\lambda+\varepsilon\bar{\mu})\delta_{ij} - \frac{(\mu+\varepsilon\bar{\mu})c_j\xi_i\bar{\xi}_j}{\sum_{k=1}^N c_k|\xi_k|^2}.
\end{aligned} \tag{1-2-11}$$

Now we discuss another type of reduction. Fix a constant diagonal matrix J with mutually different diagonal entries. Let

$$\begin{aligned}
&\mathcal{A}_1 = \{\lambda P \,|\, P \in C^\infty(\Omega,\mathcal{M}_N)\},\\
&\mathcal{A}_2 = \{\lambda^2 J + \lambda P \,|\, P \in C^\infty(\Omega,\mathcal{M}_N), \text{ and } P \text{ is off-diagonal}\}.
\end{aligned}$$

We know that for principal chiral field, U is in $\mathcal{A}_1$, while for Kaup-Newell system, U is in $\mathcal{A}_2$. These two systems will be discussed later. Here we shall determine the forms of the Darboux matrix which keeps these reductions.

A matrix $G = R(\lambda - S)$ with R invertible has a natural action on $\mathcal{A}_1$ and $\mathcal{A}_2$, i.e.

$$D_G U = GUG^{-1} + G_x G^{-1}.$$

Then, we have

Theorem 5 *(1) $D_G(\mathcal{A}_1) \subset \mathcal{A}_1$ if and only if RS is a constant matrix and (1-1-13) holds.*

(2) $D_G(\mathcal{A}_2) \subset \mathcal{A}_2$ if and only if R is a diagonal matrix, RS is a constant matrix, and (1-1-13) holds.

Proof. (1) From (1-1-12), (1-1-13),

$$\widetilde{U}_1 = RPR^{-1} = \widetilde{P}, \tag{1-2-12}$$

$$\widetilde{U}_0 = R[P,S]R^{-1} + R_x R^{-1} = 0, \tag{1-2-13}$$

$$S_x + [S, PS] = 0. \tag{1-2-14}$$

Combining the last two equations, we have

$$(RS)_x = 0.$$

This proves the sufficiency. The converse is also true by (1-2-12), (1-2-13) and (1-2-14).

(2) (1-1-12) and (1-1-13) give

$$\widetilde{U}_2 = RPR^{-1} = J, \tag{1-2-15}$$

$$\widetilde{U}_1 = R(P + [J,S])R^{-1} = \widetilde{P}, \tag{1-2-16}$$

$$\widetilde{U}_0 = R([P,S] + [J,S]S)R^{-1} + R_x R^{-1} = 0, \tag{1-2-17}$$

$$S_x + [S, JS^2 + PS] = 0. \tag{1-2-18}$$

(1-2-15) implies R is diagonal. Eliminating the term $[S, JS + P]$ in (1-2-17), (1-2-18), we get $(RS)_x = 0$. The converse is also true by (1-2-15) to (1-2-18). **QED.**

Using this theorem, it is easier to determine the Darboux matrices for those concrete problems.

1.2.2 $U(N)$ principal chiral field

In order to unify the notation, x, t are used as the light-cone coordinates of $\mathbf{R}^{1,1}$ (then $x \pm t$ are space and time coordinates). Let $g : \mathbf{R}^{1,1} \to U(N)$, then the equation for principal chiral field (i.e. harmonic map from $\mathbf{R}^{1,1}$ to $U(N)$) is

$$(g_x g^{-1})_t + (g_t g^{-1})_x = 0. \tag{1-2-19}$$

Write

$$P = g_x g^{-1}, \quad Q = g_t g^{-1}, \tag{1-2-20}$$

then (1-2-19), (1-2-20) are equivalent to

$$P_t + Q_x = 0, \quad P_t - Q_x + [P, Q] = 0, \tag{1-2-21}$$

which possesses a Lax pair

$$\Phi_x = \lambda P\Phi, \quad \Phi_t = \frac{\lambda}{2\lambda - 1} Q\Phi. \tag{1-2-22}$$

According to Theorem 4 and Theorem 5, the Darboux matrix can be constructed as follows.

Choose $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ where λ_i can be either μ or $\bar{\mu}$ (μ is a complex number which is not real). Let $H = (h_1, \cdots, h_N)$ such that h_i is a solution of (1-2-22) with $\lambda = \lambda_i$, $h_i^\dagger h_j = 0$ if $\lambda_i \neq \lambda_j$, and $\det H \neq 0$. Let $= H\Lambda^{-1}H^{-1}$, $S = R^{-1}$, then Theorem 4 implies that $G = \lambda R - I$ keeps $u(N)$ reduction, and (1-1-12), (1-1-13) give the transformation

$$\widetilde{P} = RPR^{-1} = P - R_x, \quad \widetilde{Q} = (2 - R)Q(2 - R)^{-1} = Q + R_t. \tag{1-2-23}$$

Moreover, from (1-2-3), (1-2-6), we have

$$(R - 1)^\dagger (R - 1) = (S^\dagger)^{-1}(S^\dagger - 1)(S - 1)S^{-1} = |1 - \mu^{-1}|^2. \tag{1-2-24}$$

Hence the corresponding new principal chiral field (harmonic map) is

$$\widetilde{g} = (R - 1)g\Lambda(1 - \Lambda)^{-1}. \tag{1-2-25}$$

Here the right multiplier $\Lambda(1 - \Lambda)^{-1}$ makes $\widetilde{g}$ in $U(N)$.

Remark 3 *Instead of the group $U(N)$, we can also get the Darboux matrix for $SU(N)$ principal chiral field in the same way because of (1-2-25).*

1.2.3 $O(2N)$ principal chiral field

In this case, the equation of principal chiral field and the corresponding Lax pair are still (1-2-21) and (1-2-22), while $P, Q \in o(2N)$, i.e. $P^T = -P$, $Q^T = -Q$.

Since real μ leads to trivial transformation, we can not simply use Remark 2 after Theorem 4. Alternatively, we choose μ to be complex, and only want R to be real. The Darboux matrix is given as follows. Let $\mu = (\mu_1 + \mathrm{i}\mu_2)^{-1}$ ($\mu_2 \neq 0$), and

$$J = \begin{pmatrix} \mu_1 I & \mu_2 I \\ -\mu_2 I & \mu_1 I \end{pmatrix} \tag{1-2-26}$$

(I is $N \times N$ identity matrix). Choose a nondegenerate real matrix F such that $F^T F$ has the form

$$\begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}$$

for some $N \times N$ matrix W. (This is obviously true if F is an orthogonal matrix.) Let

$$R_0 = FJF^{-1}, \tag{1-2-27}$$

then R_0 is proportional to an orthogonal matrix since

$$J^T F^T F J = (\mu_1^2 + \mu_2^2) \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}$$

and

$$R_0^R R_0 = (F^T)^{-1} J^T F^T F J F^{-1} = \mu_1^2 + \mu_2^2. \tag{1-2-28}$$

Therefore, R_0 can be written as

$$R_0 = K\Lambda K^{-1} \tag{1-2-29}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_{2N})$ with $\lambda_i = \mu$ or $\bar{\mu}$, $K = (k_1, \cdots, k_{2N})$ satisfies $k_i^\dagger k_j = 0$ $(\lambda_i \neq \lambda_j)$. Let $h_i(x,t)$ be the solution of

$$h_{i,x} = \lambda_i P h_i, \quad h_{i,t} = \frac{\lambda_i}{2\lambda_i - 1} Q h_i, \quad h_i(x_0, t_0) = k_i, \tag{1-2-30}$$

$H = (h_1, \cdots, h_{2N})$, then $h_i^\dagger h_j = 0$ if $\lambda_i \neq \lambda_j$, and $\det H \neq 0$.

By the conclusion for $U(2N)$ principal chiral field, we only need to prove that $R = H\Lambda^{-1}H^{-1}$ takes real value.

Write $R = A + \mathrm{i}B$. Let $Z = \{(x,t) \in \Omega \,|\, B(x,t) = 0\}$. Then $(x_0, t_0) \in Z$. Near $(x_1, t_1) \in Z$, (1-2-23) implies

$$\begin{aligned} &A_x A - B_x B = [P, A], \quad A_x B + B_x A = [P, B], \\ &A_t(A-2) - B_t B = -[Q, A], \quad A_t B + B_t(A-2) = -[Q, B]. \end{aligned} \tag{1-2-31}$$

Eliminating A_x, A_t:

$$\begin{aligned} &B_x(A + BA^{-1}B) = APA^{-1}B - BP, \\ &B_t(A - 2 + B(A-2)^{-1}B) = BQ - (A-2)Q(A-2)^{-1}B. \end{aligned} \tag{1-2-32}$$

Since $A + BA^{-1}B$, $A - 2 + B(A-2)^{-1}B$ are nondegenerate near (x_1, t_1), B is zero near (x_1, t_1) by the uniqueness of the solution of this group of equations. Hence Z is open. This implies $B = 0$ identically, or R takes real value.

Till now, we have proved that this Darboux matrix keeps $o(2N)$ reduction. Explicitly, the transformation of P, Q is still

$$\widetilde{P} = P - R_x, \quad \widetilde{Q} = Q + R_t \tag{1-2-33}$$

while the transformation of g is

$$\widetilde{g} = (R-1)g|1 - \mu^{-1}|^{-1} \tag{1-2-34}$$

which guarantees that $\widetilde{g} \in O(2N)$.

1.2.4 Kaup-Newell system

Kaup-Newell system is

$$\Phi_x = \begin{pmatrix} \mathrm{i}\lambda^2 & \mathrm{i}\lambda p \\ \mathrm{i}\lambda q & -\mathrm{i}\lambda^2 \end{pmatrix} \Phi, \quad \Phi_t = \sum_{j=1}^{2n} V_{2n-j}\lambda^j \phi \tag{1-2-35}$$

where

$$V_{2k} = \begin{pmatrix} a_{2k} & 0 \\ 0 & -a_{2k} \end{pmatrix}, \quad V_{2k+1} = \begin{pmatrix} 0 & b_{2k+1} \\ c_{2k+1} & 0 \end{pmatrix},$$

$$
\begin{aligned}
b_{2k+1} &= pa_{2k} - \frac{\mathrm{i}}{2} b_{2k-1,x}, \\
c_{2k+1} &= qa_{2k} + \frac{\mathrm{i}}{2} c_{2k-1,x}, \quad (0 \le k \le n-1), \\
a_{2k,x} &= -\frac{1}{2}(pc_{2k-1,x} + qb_{2k-1,x}), \quad (1 \le k \le n-1).
\end{aligned}
\tag{1-2-36}
$$

The equations given by (1-2-35) are

$$
p_t = -\mathrm{i} b_{2n-1,x}, \quad q_t = -\mathrm{i} c_{2n-1,x}. \tag{1-2-37}
$$

Here $\{a_{2k}, b_{2k+1}, c_{2k+1}\}$ which are determined by (1-2-36) are all differential polynomials of p, q, which will also be written as $a_{2k}[p,q]$, $b_{2k+1}[p,q]$, $c_{2k+1}[p,q]$. Moreover, the Darboux matrix of second order is already known [35].

Here we shall construct the Darboux matrix of first order, the composition of two of which gives the Darboux matrix in [35].

Let

$$
\Lambda = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \quad H = \begin{pmatrix} \xi & -\xi \\ \eta & \eta \end{pmatrix}, \quad \sigma = \eta/\xi \tag{1-2-38}
$$

where $(\xi, \eta)^T$ is a solution of (1-2-35). Then

$$
H\Lambda H^{-1} = \mu \begin{pmatrix} 0 & 1/\sigma \\ \sigma & 0 \end{pmatrix}. \tag{1-2-39}
$$

By Theorem 5, we choose $S = H\Lambda H^{-1}$,

$$
RS = -\mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} \sigma & 0 \\ 0 & 1/\sigma \end{pmatrix},
$$

then

$$
G = \begin{pmatrix} \sigma & 0 \\ 0 & 1/\sigma \end{pmatrix} \lambda - \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1-2-40}
$$

is a Darboux matrix keeping the form of U in (1-2-35), which induces

$$
\widetilde{p} = p\sigma^2 + 2\mu\sigma, \quad \widetilde{q} = q/\sigma^2 - 2\mu/\sigma. \tag{1-2-41}
$$

Now we prove $\widetilde{V} = V[\widetilde{p}, \widetilde{q}]$, i.e. the Darboux matrix keeps the form of V invariant, which means $\widetilde{p}, \widetilde{q}$ satisfy the same equation as p, q. To prove this, we use the technique in [8].

By (1-2-38) and (1-2-35),

$$
\sigma_x = \mathrm{i}\lambda q - 2\mathrm{i}\lambda^2\sigma - \mathrm{i}\lambda p\sigma^2, \quad \sigma_t = c - 2a\sigma - b\sigma^2 \tag{1-2-42}
$$

where a, b, c are entries of V:

$$
V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},
$$

therefore, $V = V[\widetilde{p}, \widetilde{q}]$ follows from the next lemma.

Lemma 2 *Given $\mu \in \mathbf{C}$ and functions $p(x), q(x), \sigma(x)$ satisfying $\sigma_x = \mathrm{i}\lambda q - 2\mathrm{i}\lambda^2\sigma - \mathrm{i}\lambda p\sigma^2$. Let $\widetilde{p}, \widetilde{q}$ be defined as (1-2-41), V_k be given by (1-2-36) with fixed integral constants. Let $\widetilde{V} = GVG^{-1} + G_1G^{-1}$ where*

$$G_1 = \lambda(c - 2a\sigma - b\sigma^2)\begin{pmatrix} 1 & 0 \\ 0 & -1/\sigma^2 \end{pmatrix}.$$

Then $\widetilde{V} = V[\widetilde{p}, \widetilde{q}]$.

Proof. Let

$$\begin{array}{l} \hat{p}(x,t) = p - \mathrm{i}tb_{2n-1}[p,q], \\ \hat{q}(x,t) = q - \mathrm{i}tc_{2n-1}[p,q], \\ \hat{\sigma}(x,t) = \sigma + t(c - 2a\sigma - b\sigma^2), \\ \hat{G}(x,t) = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & 1/\hat{\sigma} \end{pmatrix}\lambda - \mu\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{array}$$

then

$$\hat{U}_t = V_x - [U, V]$$

at $t = 0$ by the recursion formula (1-2-36). However,

$$\widetilde{\hat{U}} = \hat{G}\hat{U}\hat{G}^{-1} + \hat{G}_x\hat{G}^{-1}, \quad \widetilde{\hat{V}} = \hat{G}\hat{V}\hat{G}^{-1} + \hat{G}_t\hat{G}^{-1}$$

give

$$\widetilde{\hat{U}}_t - \widetilde{\hat{V}}_x + [\widetilde{\hat{U}}, \widetilde{\hat{V}}] = 0$$

at $t = 0$. By the definition of $\widetilde{V}$ and G_1, we have

$$\widetilde{\hat{U}}_t|_{t=0} - \widetilde{V}_x + [\widetilde{U}, \widetilde{V}] = 0.$$

This implies $\widetilde{V}$ satisfies the same recursion relations as $V[\widetilde{p}, \widetilde{q}]$.

Now we use induction to complete the proof. Clearly, $\widetilde{a}_0 = a_0[\widetilde{p}, \widetilde{q}]$, $\widetilde{b}_1 = b_1[\widetilde{p}, \widetilde{q}]$, $\widetilde{c}_1 = c_1[\widetilde{p}, \widetilde{q}]$. Suppose $\widetilde{a}_{2k} = a_{2k}[\widetilde{p}, \widetilde{q}]$, $\widetilde{b}_{2k+1} = b_{2k+1}[\widetilde{p}, \widetilde{q}]$, $\widetilde{c}_{2k+1} = c_{2k+1}[\widetilde{p}, \widetilde{q}]$, then, by (1-2-36), $\widetilde{a}_{2k+2} - a_{2k+2}[\widetilde{p}, \widetilde{q}] = \Delta_k$ is a constant for given p, q, σ, which is also a differential polynomial of p, q, σ. Furthermore, from (1-2-42), we can suppose

$$\Delta_k = \Delta_k(p, p_x, \cdots, p^{(r)}, q, q_x, \cdots, q^{(s)}, \sigma)$$

where the superscripts refer to the order of the derivative with respect to x. So

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\Delta_k = \sum_{j=0}^{r}\frac{\partial\Delta_k}{\partial p^{(j)}}p^{(j+1)} + \sum_{j=0}^{s}\frac{\partial\Delta_k}{\partial q^{(j)}}q^{(j+1)} + \frac{\partial\Delta_k}{\partial\sigma}(\mathrm{i}\lambda q - 2\mathrm{i}\lambda^2\sigma - \mathrm{i}\lambda p\sigma^2). \tag{1-2-43}$$

We know from this equation that the partial derivatives of Δ_k with respect to the derivatives of p, q of highest order are zero, which implies Δ_k is independent of p, q, σ by (1-2-43). Now choose particularly $p = q = \sigma = 0$, we have $\Delta_k = 0$, and $\widetilde{b}_{2k+3} = b_{2k+3}[\widetilde{p}, \widetilde{q}]$, $\widetilde{c}_{2k+3} = c_{2k+3}[\widetilde{p}, \widetilde{q}]$ by (1-2-36). Therefore, $\widetilde{V}_k = V_k[\widetilde{p}, \widetilde{q}]$ for any k. **QED.**

Next, we shall get the Darboux matrix for a concrete equation — derivative nonlinear Schrödinger equation [16], which describes, for example, the propagation of circular polarized nonlinear Alfvén waves in plasmas.

The equation is

$$\mathrm{i}p_t + p_{xx} + \mathrm{i}\varepsilon(|p|^2 p)_x = 0 \quad (\varepsilon = \pm 1) \tag{1-2-44}$$

which corresponds to

$$\begin{aligned} U &= \begin{pmatrix} \mathrm{i}\lambda^2 & \mathrm{i}p\lambda \\ -\mathrm{i}\varepsilon\bar{p}\lambda & -\mathrm{i}\lambda^2 \end{pmatrix}, \\ V &= \begin{pmatrix} -2\mathrm{i}\lambda^4 - \mathrm{i}\varepsilon|p|^2\lambda^2 & -2\mathrm{i}p\lambda^3 - (p_x + \mathrm{i}\varepsilon|p|^2 p)\lambda \\ 2\mathrm{i}\varepsilon\bar{p}\lambda^3 - (\varepsilon\bar{p}_x - \mathrm{i}|p|^2\bar{p})\lambda & 2\mathrm{i}\lambda^4 + \mathrm{i}\varepsilon|p|^2\lambda^2 \end{pmatrix}. \end{aligned} \tag{1-2-45}$$

In this case,

$$\begin{aligned} &(|\sigma|^2)_x = 2\mathrm{Re}\{\mathrm{i}\varepsilon p\sigma(\bar{\mu} - \varepsilon\mu|\sigma|^2)\} + 2\mathrm{i}|\sigma|^2(\bar{\mu}^2 - \mu^2), \\ &(|\sigma|^2)_t = -2\mathrm{Re}\{2\mathrm{i}\varepsilon p\sigma(\bar{\mu}^3 - \varepsilon\mu^3|\sigma|^2) + \sigma(\varepsilon p_x + \mathrm{i}|p|^2 p)(\bar{\mu} - \varepsilon\mu|\sigma|^2)\} \\ &\qquad + \{4\mathrm{i}|\sigma|^2(\mu^2 + \bar{\mu}^2) + 2\mathrm{i}\varepsilon|p|^2|\sigma|^2\}(\mu^2 - \bar{\mu}^2). \end{aligned} \tag{1-2-46}$$

If $\varepsilon = 1$, choose μ to be real. If $\varepsilon = -1$, choose μ to be purely imaginary. In both cases, we can want $|\sigma|^2 = 1$. Hence, the Darboux matrix (1-2-40) transforms a solution of (1-2-44) to a solution of itself by (1-2-41).

As the simplest example, let $p = a$ (constant), then we can choose

$$\begin{aligned} \xi &= -\mathrm{i}a\mu(A\mathrm{e}^{\theta} + B\mathrm{e}^{-\theta}), \\ \eta &= \mathrm{i}\mu^2(A\mathrm{e}^{\theta} + B\mathrm{e}^{-\theta}) - \nu(A\mathrm{e}^{\theta} - B\mathrm{e}^{-\theta}) \end{aligned}$$

where $|\mu| < |a|$,

$$\nu = \sqrt{|a|^2|\mu|^2 - \mu^4}, \quad \theta = \nu x - (2\nu\mu^2 + \varepsilon\nu|a|^2)t,$$

and A, B are two complex numbers satisfying

$$\mathrm{Re}((\nu + \mathrm{i}\mu^2)\bar{A}B) = 0.$$

The new solution is

$$p = -\frac{\mu^2}{a} - \frac{\nu^2}{a\mu^2}\Big(\frac{A\mathrm{e}^{\theta} - B\mathrm{e}^{-\theta}}{A\mathrm{e}^{\theta} + B\mathrm{e}^{-\theta}}\Big)^2.$$

Chapter 2

1+2 Dimensional Darboux Transformation

2.1 General form of Darboux operators

In the introduction, the following Lax pair in 1+2 dimensions was introduced:

$$\Phi_y = U(x, y, t, \partial)\Phi, \quad \Phi_t = V(x, y, t, \partial)\Phi. \tag{2-1-1}$$

Here $(x, y, t) \in \Omega$ (simply connected domain in $\mathbf{R}^3$),

$$\begin{aligned} U(x, y, t, \partial) &= \sum_{j=0}^{m} U_{m-j}(x, y, t)\partial^j, \\ V(x, y, t, \partial) &= \sum_{j=0}^{n} U_{n-j}(x, y, t)\partial^j \end{aligned} \tag{2-1-2}$$

are differential operators with respect to x whose coefficients $U_j, V_j \in C^\infty(\Omega, \mathcal{M}_N)$, $\partial = \partial/\partial x$.

We say that (2-1-1) is integrable if, for any $(x_0, y_0, t_0) \in \Omega$ and any smooth vector-valued function $\Phi_0(x)$ defined near x_0, there exists a solution Φ of (2-1-1) in some neighbourhood of (x_0, y_0, t_0) such that $\Phi(x, y_0, t_0) = \Phi_0(x)$.

If (2-1-1) is integrable,

$$U_t(\partial)\Phi - V_y(\partial)\Phi + [U(\partial), V(\partial)]\Phi = 0. \tag{2-1-3}$$

Lemma 3 *If (2-1-1) is integrable, then*

$$U_t(\partial) - V_y(\partial) + [U(\partial), V(\partial)] = 0. \tag{2-1-4}$$

Proof. If not, (2-1-3) would give a nontrivial ordinary differential equation (with respect to x) of Φ. This is a contradiction to the solvability of (2-1-1) for any initial data. **QED.**

Remark 4 *The condition in the definition of integrability can be weakened. In stead of the existence of the solution in a whole neighbourhood W, we can require that the solution only exists in one of the subsets*

$$W \cap \{(x, y, t) \,|\, (y - y_0)\xi \geq 0,\, (t - t_0)\eta \geq 0\}, \quad \xi, \eta = \pm 1.$$

If we define the integrability like that, all the conclusions below still hold.

Define

$$\mathcal{D}_N(\Omega) = \Big\{ \sum_{j=0}^{s} A_j \partial^j \,\Big|\, A_j \in C^\infty(\Omega, \mathcal{M}_N),\, r \text{ is a nonnegative integer} \Big\}.$$

In this chapter, we deal with the equation

$$F(x, y, t, u, u_x, u_y, u_t, u_{xx}, \cdots) = 0 \tag{2-1-5}$$

of unknowns $u = (u_1, \cdots, u_s)$ in Ω which admits the Lax pair (2-1-1), i.e. (2-1-5) is equivalent to (2-1-4) where the coefficients of $U(\partial)$, $V(\partial)$ may be differential polynomials of u.

This section is devoted to the equation (2-1-4) without reduction, i.e. the entries of U_j, V_j are independent unknowns. Then, the definition of Darboux operator in the introduction implies that $G(x, y, t, \partial)$ is a Darboux operator if there exist $\widetilde{U}(\partial), \widetilde{V}(\partial) \in \mathcal{D}_N(\Omega)$ such that $\widetilde{\Phi} = G(\partial)\Phi$ satisfies

$$\widetilde{\Phi}_y = \widetilde{U}(\partial)\widetilde{\Phi}, \quad \widetilde{\Phi}_t = \widetilde{V}(\partial)\widetilde{\Phi} \tag{2-1-6}$$

for any solution Φ of (2-1-4).

In this case, $\widetilde{U}(\partial), \widetilde{V}(\partial)$ satisfy

$$G_y(\partial) = \widetilde{U}(\partial)G(\partial) - G(\partial)U(\partial), \quad G_t(\partial) = \widetilde{V}(\partial)G(\partial) - G(\partial)V(\partial) \tag{2-1-7}$$

and

$$\widetilde{U}_t(\partial) - \widetilde{V}_y(\partial) + [\widetilde{U}(\partial), \widetilde{V}(\partial)] = 0. \tag{2-1-8}$$

From now on, we consider the nondegenerate Darboux operator of first order, that is, a Darboux operator $G(x, y, t, \partial) = R(x, y, t)(\partial - S(x, y, t))$ with R nondegenerate. Since we do not consider reduction here, we can always choose $R = I$.

Nondegenerate Darboux operators of first order can be constructed explicitly as follows.

Theorem 6 *$\partial - S(x, y, t)$ is a Darboux operator of (2-1-1) if and only if $S = H_x H^{-1}$ for some $N \times N$ nondegenerate matrix solution H of (2-1-1).*

We first prove two lemmas.

Lemma 4 *For any $M \in C^\infty(\Omega, \mathcal{M}_N)$, there is a differential polynomial $M^{(j)}$ of M such that $\partial_j \Psi = M^{(j)}\Psi$ for any Ψ satisfying $\Psi_x = M\Psi$.*

Proof. Define

$$M^{(0)} = 1, \quad M^{(j+1)} = M^{(j)}_x + M^{(j)}M \quad (j \geq 0), \tag{2-1-9}$$

then

$$\partial^j \Psi = M^{(j)}\Psi \tag{2-1-10}$$

by induction. **QED.**

By this lemma, we can define

$$U(M) = \sum_{j=0}^{m} U_{m-j}M^{(j)} \tag{2-1-11}$$

and clearly

$$U(\partial)\Psi = U(M)\Psi \tag{2-1-12}$$

for any Ψ satisfying $\Psi_x = M\Psi$.

Lemma 5 *$\partial - S$ is a Darboux operator of (2-1-1) if and only if S satisfies*

$$S_y + [S, U(S)] = (U(S))_x, \quad S_t + [S, V(S)] = (V(S))_x. \tag{2-1-13}$$

Proof. First suppose $\partial - S$ is a Darboux operator of (2-1-1). Choose a fundamental solution matrix Ψ of $\Psi_x = S\Psi$, then (2-1-6) implies

$$S_y\Psi = (\partial - S)U(S)\Psi = (U(S))_x\Psi - [S, U(S)]\Psi,$$

which lead to (2-1-13).

Conversely, suppose S is a solution of (2-1-13). Define

$$\widetilde{U}(\partial) = \sum_{j=0}^{m} \widetilde{U}_{m-j}\partial^j \tag{2-1-14}$$

where $\widetilde{U}_j$'s are defined inductively by

$$\begin{aligned} &\widetilde{U}_0 = U_0, \\ &\widetilde{U}_{j+1} = U_{j+1} + U_{j,x} - SU_j + \sum_{k=0}^{j} C^{m-j}_{m-k}\widetilde{U}_k\partial^{j-k}S. \end{aligned} \tag{2-1-15}$$

Then,

$$S_y - (\partial - S)U(\partial) + \widetilde{U}(\partial)(\partial - S) \in C^\infty(W, \mathcal{M}_N).$$

However, for the fundamental solution matrix Ψ of $\Psi_x = S\Psi$, (2-1-13) gives

$$\Big(S_y - (\partial - S)U(\partial) + \widetilde{U}(\partial)(\partial - S)\Big)\Psi = 0.$$

This means

$$S_y - (\partial - S)U(\partial) + \widetilde{U}(\partial)(\partial - S) = 0$$

as a matrix. **QED.**

Proof of Theorem 6. Suppose H is an $N\times N$ nondegenerate matrix solution of (2-1-1), $S=H_xH^{-1}$. (2-1-1) leads to

$$S_y = H_{xy}H^{-1} - H_xH^{-1}H_yH^{-1} = (U(S))_x + [U(S), S],$$

which means $\partial - S$ is a Darboux operator by Lemma 5.

Conversely, suppose $G(\partial) = \partial - S(x,y,t)$ is a Darboux operator of (2-1-1), we need to find a solution H of (2-1-1) such that $S = H_xH^{-1}$, or equivalently, we need to solve

$$H_x = SH, \quad H_y = U(\partial)H, \quad H_t = V(\partial)H. \tag{2-1-16}$$

Again, this is equivalent to

$$H_x = SH, \quad H_y = U(S)H, \quad H_t = V(S)H \tag{2-1-17}$$

by Lemma 4. Therefore, we only need to verify the integrability condition of (2-1-17).

Let Ψ be a fundamental solution matrix of $\Psi_x = S\Psi$. By (2-1-7),

$$(\Psi_y - U(\partial)\Psi)_x = (S\Psi)_y - \partial U(\partial)\Psi = S(\Psi_y - U(\partial)\Psi),$$

thus,

$$\begin{aligned}&(V_y(\partial) + V(\partial)U(\partial))\Psi = (V(\partial)\Psi)_y - V(\partial)(\Psi_y - U(\partial)\Psi)\\ &= (V(S)\Psi)_y - V(S)(\Psi_y - U(S)\Psi) = V(S)_y\Psi + V(S)U(S)\Psi.\end{aligned}$$

Similarly,

$$(U_t(\partial) + U(\partial)V(\partial))\Psi = U(S)_t\Psi + U(S)V(S)\Psi.$$

By the integrability condition (2-1-4),

$$U(S)_t - V(S)_y + [U(S), V(S)] = 0 \tag{2-1-18}$$

since $\det\Psi \neq 0$.

Lemma 5 implies that another two integrability conditions $H_{xy} = H_{yx}$, $H_{xt} = H_{tx}$ hold. Combining this with (2-1-18), we know (2-1-17) is integrable. Therefore, (2-1-17) has an $N\times N$ nondegenerate matrix solution. **QED.**

This theorem implies that any nondegenerate Darboux operator of first order can be determined only by an $N\times N$ matrix solution of the Lax pair.

Analogous to 1+1 dimensional case, the permutability theorem also holds. here.

We first consider the Darboux operator of higher order, i.e. a differential operator

$$G(\partial) = \sum_{j=0}^{r} G_{r-j}\partial^j, \quad G_0 = I$$

satisfying

$$G_y(\partial) = \widetilde{U}(\partial)G(\partial) - G(\partial)U(\partial), \quad G_t(\partial) = \widetilde{V}(\partial)G(\partial) - G(\partial)V(\partial)$$

for some $\widetilde{U}(\partial), \widetilde{V}(\partial) \in \mathcal{D}_N(\Omega)$.

The Darboux operator of r-th order can be constructed as follows.

Theorem 7 *Given r $N\times N$ nondegenerate matrix solutions $H_1,\cdots,H_r$ of (2-1-1), let F_j be the block matrix $(\partial^{\alpha-1}H_\beta)_{1\le\alpha,\beta\le j}$. Suppose* $\det F_r\neq 0$*, then*

(1) There exists a unique differential operator of r-th order

$$G(H_1,\cdots,H_r,\partial)=\sum_{j=0}^{r}G_{r-j}\partial^j,\quad G_0=I \tag{2-1-19}$$

satisfying $G(H_1,\cdots,H_r,\partial)H_i=0$ $(i=1,\cdots,r)$.

(2) For any permutation π of $(1,\cdots,r)$,

$$G(H_{\pi(1)},\cdots,H_{\pi(r)},\partial)=G(H_1,\cdots,H_r,\partial). \tag{2-1-20}$$

(3) If $\det F_{r-1}\neq 0$*, then*

$$G(H_1,\cdots,H_r,\partial)=G(G(H_1,\cdots,H_{r-1},\partial)H_r,\partial)G(H_1,\cdots,H_{r-1},\partial).$$

(4) G is a Darboux operator of r-th order.

Proof. (1): $\det F_r\neq 0$ implies that the group of linear algebraic equations

$$\sum_{j=0}^{r-1}G_{r-j}\partial^jH_i=-\partial^rH_i\quad(i=1,\cdots,r)$$

has a unique solution $G_1,\cdots,G_r$, which determines $G(\partial)$.

(2): holds because of (1).

(3): Let

$$G(H_1,\cdots,H_{r-1},\partial)=\sum_{j=0}^{r-1}g_{r-1-j}\partial^j,$$

then

$$\sum_{j=0}^{r-2}g_{r-1-j}\partial^jH_i=-\partial^{r-1}H_i,$$

or

$$(g_{r-1},\cdots,g_1)F_{r-1}=-(\partial^{r-1}H_1,\cdots,\partial^{r-1}H_{r-1}).$$

Hence,

$$\begin{aligned}&\det(G(H_1,\cdots,H_{r-1},\partial)H_r)\\&=\det(\partial^{r-1}H_r-(\partial^{r-1}H_1,\cdots,\partial^{r-1}H_{r-1})F_{r-1}^{-1}(H_r^T,\cdots,\partial^{r-2}H_r^T)^T)\\&=\det F_r/\det F_{r-1}\neq 0.\end{aligned}$$

Considering the definition of $G(\partial)$,

$$\begin{aligned}&G(G(H_1,\cdots,H_{r-1},\partial)H_r,\partial)G(H_1,\cdots,H_{r-1},\partial)H_r=0,\\&G(G(H_1,\cdots,H_{r-1},\partial)H_i=0\quad(1\le i\le r-1).\end{aligned}$$

Therefore, (3) holds by (1).

(4): Since $\det F_r \neq 0$, there exists a permutation π of $(1, \cdots, r)$ such that

$$\det(\partial^{\alpha-1} H_{\pi(\beta)})_{1\le\alpha,\beta\le k} \neq 0$$

for all $k \le r$. Then, (2) and (3) lead to

$$\begin{aligned} G(H_1, \cdots, H_r, \partial) = G(G(H_{\pi(1)}, \cdots, H_{\pi(r-1)}, \partial)H_{\pi(r)}, \partial)\cdot \\ \cdot G(G(H_{\pi(1)}, \cdots, H_{\pi(r-2)}, \partial)H_{\pi(r-1)}, \partial)\cdots \\ \cdot G(G(H_{\pi(1)}, \partial)H_{\pi(2)}, \partial)G(H_{\pi(1)}, \partial). \end{aligned}$$

Each term on the right hand side is a Darboux operator of first order. Hence $G(\partial)$ is a Darboux operator of r-th order. **QED.**

By the symmetry of the Darboux operator of higher order, we can get the permutability theorem immediately. (It can also be obtained by direct calculation).

Corollary 1 *Let H_1, H_2 be $N \times N$ nondegenerate matrix solutions of (2-1-1),*

$$\det(H_{1,x}H_1^{-1} - H_{2,x}H_2^{-1}) \neq 0,$$

then

$$G(G(H_1, \partial)H_2, \partial)G(H_1, \partial) = G(G(H_2, \partial)H_1, \partial)G(H_2, \partial).$$

2.2 Darboux operator for ZS-AKNS system and KP hierarchy

1+2 dimensional ZS-AKNS system is

$$\Phi_y = J\Phi_x + P\Phi, \quad \Phi_t = \sum_{j=0}^{n} V_{n-j}\partial^j \Phi, \tag{2-2-1}$$

where $J(t)$ is a diagonal matrix with mutually different diagonal entries, $P(x, y, t)$ is valued in off-diagonal matrices.

The integrability condition of (2-2-1) is

$$[J, V_{j+1}^A] = V_{j,y}^A - JV_{j,x}^A - [P, V_j]^A + \sum_{k=0}^{j-1} C_{n-k}^{n-j}(V_k\partial^{j-k}P)^A, \tag{2-2-2}$$

$$V_{j,y}^D - JV_{j,x}^D = [P, V_j]^D - \sum_{k=0}^{j-1} C_{n-k}^{n-j}(V_k\partial^{j-k}P)^D + J_t\delta_{j,n-1}, \tag{2-2-3}$$

$$P_t = V_{n,y}^A - JV_{n,x}^A - [P, V_n]^A + \sum_{k=0}^{n-1}(V_k\partial^{n-k}P)^A. \tag{2-2-4}$$

Here the superscripts D and A represent the diagonal and off-diagonal parts of a matrix.

Unlike in 1+1 dimensional case, V_j's are usually not differential polynomials of P. Therefore, we regard (2-2-3), (2-2-4) as a group of differential equations of unknowns P, V_j^D $(j = 0, 1, \cdots, n)$, while V_j^A $(j = 1, \cdots, n)$ are determined by (2-2-2).

The Darboux operator given by Theorem 6 transforms a solution of (2-2-3), (2-2-4) to a solution of itself, that is,

Proposition 1 *Suppose* $\{P, V_j^D (j = 0, 1, \cdots, N)\}$ *is a solution of (2-2-3), (2-2-4),* $\{V_j^A\}$ *is determined by (2-2-2), H is an* $N \times N$ *nondegenerate matrix solution of (2-2-1). Then* $\widetilde{\Phi} = (\partial - H_x H^{-1})\Phi$ *satisfies*

$$\widetilde{\Phi}_y = \widetilde{U}(\partial)\widetilde{\Phi}, \quad \widetilde{\Phi}_t = \widetilde{V}(\partial)\widetilde{\Phi} \tag{2-2-5}$$

where

$$\widetilde{U}(\partial) = J\partial + \widetilde{P},$$

$$\widetilde{V}(\partial) = \sum_{j=0}^{n} \widetilde{V}_{n-j}\partial^j, \quad \widetilde{V}_0 = V_0,$$

$$\widetilde{P} = P + [J, H_x H^{-1}], \tag{2-2-6}$$

$$\widetilde{V}_{j+1}^D = V_{j+1}^D + V_{j,x}^D - (H_x H^{-1} V_j)^D + \sum_{k=0}^{j} C_{n-k}^{n-j} (\widetilde{V}_k \partial^{j-k}(H_x H^{-1}))^D, \tag{2-2-7}$$

$$[J, \widetilde{V}_{j+1}^A] = \widetilde{V}_{j,y}^A - J\widetilde{V}_{j,x}^A - [\widetilde{P}, \widetilde{V}_j]^A + \sum_{k=0}^{j-1} C_{n-k}^{n-j} (\widetilde{V}_k \partial^{j-k} \widetilde{P})^A. \tag{2-2-8}$$

Moreover, $\{\widetilde{P}, \widetilde{V}_j^D\ (j = 0, 1, \cdots, n)\}$ *is also a solution of (2-2-3), (2-2-4).*

Proof. By direct calculation, (2-2-6) is true. Let

$$\widetilde{V}(\partial) = \sum_{j=0}^{n} V_{n-j}\partial^j, \quad \widetilde{V}'(\partial) = \sum_{j=0}^{n} V'_{n-j}\partial^j$$

where $\{\widetilde{V}_j\}$ is defined by (2-2-7), (2-2-8), $\{V'_j\}$ is defined by

$$\begin{aligned} &V'_0 = V_0, \\ &V'_{j+1} = V_{j+1} + V_{j,x} - H_x H^{-1} V_j + \sum_{k=0}^{j} C_{n-k}^{n-j} V'_k \partial^{j-k}(H_x H^{-1}), \end{aligned} \tag{2-2-9}$$

(see (2-1-15)). Then

$$\widetilde{U}_t(\partial) - V'_y(\partial) + [\widetilde{U}(\partial), V'(\partial)] = 0$$

implies

$$[J, V'^A_{j+1}] = V'^A_{j,y} - JV'^A_{j,x} - [\widetilde{P}, V'_j]^A + \sum_{k=0}^{j-1} C_{n-k}^{n-j} (V'_k \partial^{j-k} \widetilde{P})^A. \tag{2-2-10}$$

We use induction to prove $\widetilde{V}(\partial) = V'(\partial)$. Clearly, $\widetilde{V}_0 = V'_0$. Suppose $\widetilde{V}_j = V'_j$ for $j \le l$, then (2-2-7) and (2-2-9) implies $\widetilde{V}_{l+1}^D = V'^D_{l+1}$, while (2-2-8) and (2-2-10) imply $\widetilde{V}_{l+1}^A = V'^A_{l+1}$. Hence $\widetilde{V}(\partial) = V'(\partial)$, and

$$\widetilde{V}(\partial)(\partial - H_x H^{-1}) = (\partial - H_x H^{-1})V(\partial) - (H_x H^{-1})_t$$

by the definition of $V'(\partial)$. This implies $\widetilde{\Phi}$ is a solution of (2-2-5), and $\{P, V_j^D\,(j = 0,1,\cdots,n)\}$ is a solution of (2-2-3) and (2-2-4). **QED.**

The proposition provides a general expression of Darboux operator for ZS-AKNS system. Next, we shall apply this to the Davey-Stewartson hierarchy, i.e. the system with

$$J = V_0 = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & u \\ \varepsilon\bar{u} & 0 \end{pmatrix}\ (\varepsilon = \pm 1), \quad V_j = \begin{pmatrix} v_j & w_j \\ \varepsilon\bar{w}_j & \bar{v}_j \end{pmatrix}. \tag{2-2-11}$$

Clearly, $\{V_j\}$ is characterized by $\bar{V}_j = \sigma V_j \sigma^{-1}$ where

$$\sigma = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}.$$

The corresponding equations come from (2-2-3) and (2-2-4):

$$\begin{aligned} u_t &= D_- w_n + u(v_n - \bar{v}_n) + \sum_{k=0}^{n-1} v_k \partial^{n-k} u, \\ D_- v_j &= \varepsilon(u\bar{w}_j - \bar{u}w_j) - \varepsilon\sum_{k=0}^{j-1} C_{n-k}^{n-j} w_k \partial^{j-k}\bar{u}, \end{aligned} \tag{2-2-12}$$

where $\{w_j\}$ is determined by

$$\mathrm{i}w_{j+1} = D_- w_j + u(v_j - \bar{v}_j) + \sum_{k=0}^{j-1} C_{n-k}^{n-j} v_k \partial^{j-k} u \tag{2-2-13}$$

which corresponds to (2-2-2). Here

$$D_\pm = \frac{\partial}{\partial y} \pm \frac{\mathrm{i}}{2}\frac{\partial}{\partial x}.$$

If $(\xi,\eta)^T$ is a solution of (2-2-1) and (2-2-11), $(\varepsilon\bar{\eta},\bar{\xi})^T$ is also a solution of it. Let

$$H = \begin{pmatrix} \xi & \varepsilon\bar{\eta} \\ \eta & \xi \end{pmatrix}. \tag{2-2-14}$$

Then, after the action of the Darboux operator $\partial - S = \partial - H_x H^{-1}$, (2-2-6) leads to

$$\widetilde{u} = u + \mathrm{i}\varepsilon\frac{\xi\bar{\eta}_x - \bar{\eta}\xi_x}{|\xi|^2 - \varepsilon|\eta|^2}. \tag{2-2-15}$$

Since H satisfies $\bar{H} = \sigma H \sigma^{-1}$, we have $\bar{S} = \sigma S \sigma^{-1}$. Therefore,

$$\bar{\widetilde{V}}(\partial - \bar{S}) = (\partial - \bar{S})\bar{V} - \bar{S}_x$$

implies

$$\bar{\widetilde{V}} = \sigma V \sigma^{-1}.$$

Hence the Darboux transformation keeps the reduction (2-2-11). $\{\widetilde{u}, \widetilde{v}_j\,(j=0,1,\cdots,n)\}$ ($\{\widetilde{v}_j\}$ is given by (2-2-7)) is another solution of (2-2-12).

The usual Davey-Stewartson equation corresponds to $N=2$, and

$$V_0=\frac{\mathrm{i}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad V_1=\begin{pmatrix}0 & u\\ \varepsilon\bar{u} & 0\end{pmatrix},\quad V_2=\begin{pmatrix}v_2 & -\mathrm{i}D_+u\\ \mathrm{i}\varepsilon D_-\bar{u} & \bar{v}_2\end{pmatrix}. \tag{2-2-16}$$

The equation is

$$\mathrm{i}u_t=u_{yy}-\frac{1}{4}u_{xx}+\mathrm{i}u(v_2-\bar{v}_2),\quad D_-v_2=\mathrm{i}\varepsilon D_+|u|^2. \tag{2-2-17}$$

If we denote

$$v=2\varepsilon|u|^2+\mathrm{i}(v_2-\bar{v}_2),$$

then

$$\mathrm{i}u_t=u_{yy}-\frac{1}{4}u_{xx}-2\varepsilon|u|^2u+uv,\quad v_{yy}+\frac{1}{4}v_{xx}=\varepsilon(|u|^2)_{xx}. \tag{2-2-18}$$

This is the Davey-Stewartson equation in the usually form which describes a 3-dimensional wave packet on the surface of water with finite depth [6].

Since u,v determine v_2 up to a real function, it is easy to know from the fact for (u,v_2) that the Darboux operator transforms a solution of (2-2-18) to a solution of itself. The explicit expression is

$$\widetilde{u}=u+\mathrm{i}\varepsilon\frac{\xi\bar{\eta}_x-\bar{\eta}\xi_x}{|\xi|^2-\varepsilon|\eta|^2},\quad \widetilde{v}=v-(\ln(|\xi|^2-\varepsilon|\eta|^2))_{xx}. \tag{2-2-19}$$

The simplest solution for $\varepsilon=-1$ can be obtained from $u=v=0$. Due to (2-2-11), (2-2-16), any two solutions of

$$\phi_y=\frac{\mathrm{i}}{2}\phi_x,\quad \phi_t=\frac{\mathrm{i}}{2}\phi_{xx}$$

(corresponding to ξ and $\bar{\eta}$ respectively) provide a new solution of (2-2-18).

Especially, let

$$\xi=\exp\Big(\sigma x+\frac{\mathrm{i}}{2}\sigma y+\frac{\mathrm{i}}{2}\sigma^2t\Big),\quad \eta=\exp\Big(-\bar{\sigma}x+\frac{\mathrm{i}}{2}\bar{\sigma}y-\frac{\mathrm{i}}{2}\bar{\sigma}^2t\Big)$$

with $\sigma=\alpha-\mathrm{i}\beta$, then we get the solution of stationary wave

$$\widetilde{u}=(\beta+\mathrm{i}\alpha)\mathrm{sech}(2\alpha x+\beta y)\exp(\mathrm{i}(\alpha^2-\beta^2)t),\quad \widetilde{v}=-4\alpha^2\mathrm{sech}^2(2\alpha x+\beta y).$$

If

$$\xi=ax+\frac{\mathrm{i}}{2}ay,\quad \eta=1$$

with a real, then the static solution

$$\widetilde{u}=\frac{\mathrm{i}a}{a^2x^2+\frac{1}{4}a^2y^2+1}$$

is obtained. If

$$\xi = a\Big(x + \frac{\mathrm{i}}{2}y\Big)^2 + \mathrm{i}at, \quad \eta = 1,$$

then we get the solution of solitary traveling wave

$$\widetilde{u} = \frac{-ay + 2ax\mathrm{i}}{a^2\Big(x^2 - \frac{1}{4}y^2\Big)^2 + a^2(xy+t)^2 + 1}.$$

Complicated solutions can be obtained by Darboux operator from this kind of simple solutions.

Next, we discuss briefly the KP hierarchy

$$\phi_y = \sigma\phi_{xx} + \sigma u\phi, \quad \phi_t = \sum_{j=0}^{n} v_{n-j}\partial^j\phi \tag{2-2-20}$$

(σ, v_0 are functions of t).

If this system is integrable,

$$\begin{aligned} 2\sigma_{j+1,x} &= v_{j,y} - \sigma v_{j,xx} + \sum_{k=0}^{j-1} C_{n-k}^{n-j} v_k \partial^{j-k} u - \sigma_t \delta_{j,n-2}, \\ \sigma u_t &= v_{n,y} - \omega v_{n,xx} + \sum_{k=0}^{n-1} \sigma v_k \partial^{n-k} u - \sigma_t u. \end{aligned} \tag{2-2-21}$$

This is a group of equations of unknowns u, v_j $(j = 1, \cdots, n)$.

Given any solution h of (2-2-20), $\partial - h_x h^{-1}$ is a Darboux operator transforming a solution of (2-2-21) to a solution of itself. To see this, we only need to verify that the Darboux transformation keeps the form of y-part of the Lax pair, i.e. $\widetilde{U}(\partial) = \sigma\partial^2 + \sigma\widetilde{u}$, which is easy to check. Moreover,

$$\begin{aligned} &\widetilde{u} = u + 2(h_x h^{-1})_x, \\ &\widetilde{v}_{j+1} = v_{j+1} + v_{j,x} - v_j h_x h^{-1} + \sum_{k=0}^{j-1} C_{n-k}^{n-j} \widetilde{v}_k \partial^{j-k}(h_x h^{-1}). \end{aligned} \tag{2-2-22}$$

If $n = 3$, (2-2-21) can be reduced to a differential equation of u. Especially, choose $\sigma^2 = \varepsilon = \pm 1$, $v_0 = 1$, $v_1 = 0$, $v_2 = 3u/2$, we get the KP equation

$$u_{tx} = \frac{1}{4}(u_{xxx} + 6uu_x)_x + \frac{3}{4}\varepsilon u_{yy}. \tag{2-2-23}$$

We can check directly that $\widetilde{u} = u + 2(h_x h^{-1})_x$ is a solution of (2-2-23) if u is a solution of it.

2.3 Application to 1+1 dimensional problem

In this section, we shall use Theorem 6 to get the corresponding conclusion for the Lax pair (0-6):

$$\lambda\Phi = U(x,t,\partial)\Phi, \quad \Phi_t = V(x,t,\partial)\Phi \tag{2-3-1}$$

where $(x,t) \in \Omega$ (simply connected domain in $\mathbf{R}^2$),

$$U(\partial) = \sum_{j=0}^{m} U_{m-j}(x,t)\partial^j, \quad V(\partial) = \sum_{j=0}^{n} V_{n-j}(x,t)\partial^j. \tag{2-3-2}$$

The integrability condition of (2-3-1) is

$$U_t(\partial) + [U(\partial), V(\partial)] = 0, \tag{2-3-3}$$

which is just (0-5).

$G(x,t,\partial) \in \mathcal{D}_N(\Omega)$ is a Darboux operator of (2-3-1) if there exist $\widetilde{U}(x,t,\partial)$, $\widetilde{V}(x,t,\partial) \in \mathcal{D}_N(\Omega)$ such that for any solution Φ of (2-3-1), $\widetilde{\Phi} = G(\partial)\Phi$ satisfies

$$\lambda\widetilde{\Phi} = \widetilde{U}(x,t,\partial)\widetilde{\Phi}, \quad \widetilde{\Phi}_t = \widetilde{V}(x,t,\partial)\widetilde{\Phi}. \tag{2-3-4}$$

For nondegenerate Darboux operator of first order $G(x,t,\partial) = \partial - S(x,t)$, we have

Theorem 8 *$\partial - S(x,t)$ is a Darboux operator of (2-3-1) if and only if $S = H_x H^{-1}$ where H is an $N \times N$ nondegenerate matrix solution of*

$$H\Lambda = U(\partial)H, \quad H_t = V(\partial)H, \tag{2-3-5}$$

and Λ is a constant triangular matrix.

Proof. Consider the equation

$$\Psi_y = U(\partial)\Psi, \quad \Psi_t = V(\partial)\Psi \tag{2-3-6}$$

for $(x,t) \in \Omega$, $y \in \mathbf{R}$. Suppose $\partial - S(x,t)$ is a Darboux operator of (2-3-1), then

$$\begin{aligned} 0 &= (\partial - S)U(\partial) - \widetilde{U}(\partial)(\partial - S), \\ S_t &= (\partial - S)V(\partial) - \widetilde{V}(\partial)(\partial - S) \end{aligned} \tag{2-3-7}$$

for certain $\widetilde{U}, \widetilde{V} \in \mathcal{D}_N(\Omega)$. Thus, $\partial - S$ is a Darboux operator of (2-3-6). According to Theorem 6, there exists a solution $H_0(x,y,t)$ of (2-3-6) such that $S = H_{0,x}H_0^{-1}$. Let $\Lambda_0 = H_0^{-1}H_{0,y}$, then

$$\begin{aligned} \Lambda_0 &= H_0^{-1}U(S)H_0, \\ \Lambda_{0,x} &= -H_0^{-1}H_{0,x}H_0^{-1}U(S)H_0 + H_0^{-1}(U(S))_x H_0 + H_0^{-1}U(S)H_{0,x} \\ &= H_0^{-1}\{(U(S))_x - [S, U(S)]\}H_0 = 0 \end{aligned}$$

by (2-1-13),

$$\begin{aligned}\Lambda_{0,y} &= -H_0^{-1}H_{0,y}H_0^{-1}U(S)H_0 + H_0^{-1}U(S)H_{0,y} = 0,\\ \Lambda_{0,t} &= -H_0^{-1}H_{0,y}H_0^{-1}U(S)H_0 + H_0^{-1}(U(S))_t H_0 + H_0^{-1}U(S)H_{0,t}\\ &= H_0^{-1}\{(U(S))_t[U(S),V(S)]\}H_0 = 0\end{aligned}$$

by (2-1-18). Hence, Λ_0 is a constant matrix, which is similar to an upper-triangular matrix: $\Lambda_0 = T\Lambda T^{-1}$, and

$$H_{0,y} = H_0 T\Lambda T^{-1},$$

or

$$H_0(x,y,t) = H(x,t)\exp(\Lambda y)T^{-1}$$

where H satisfies (2-3-5) and $S = H_x H^{-1}$.

Conversely, if H is a solution of (2-3-5), $S = H_x H^{-1}$, then it is easy to see that S satisfies (2-3-7), i.e. $\partial - S$ is a Darboux operator of (2-3-1). **QED.**

If $N = 1$, H satisfies the Lax pair itself.

For example, KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

possesses the Lax pair

$$\lambda\phi = -\phi_{xx} - u\phi, \quad \phi_t = 2(2\lambda - u)\phi_x + u_x\phi,$$

while Boussinesq equation

$$(u_{xxx} + 6uu_x)_x + 3\varepsilon u_{tt} = 0 \quad (\varepsilon = \pm 1)$$

possesses the Lax pair

$$\lambda\phi = \phi_{xxx} + \frac{3u}{2}\phi_x + w\phi, \quad \phi_t = \sigma\phi_{xx} + \sigma u\phi$$

($\sigma^2 = \varepsilon$, $w_x = \dfrac{3}{4\sigma}u_t + \dfrac{3}{4}u_{xx}$), their Darboux operators can be obtained form Theorem 8 directly (for the latter one, see [23]).

Using the same method as in the proof of Theorem 8, we can get Theorem 1 when U, V are polynomials of λ.

Bibliography

[1] Ablowitz M.J., BarYaacov D. and Fokas A.S., On the inverse scattering transform for the Kadomtsev-Petviashivili equation, *Stud. Appl. Math.* **69** (1983), 135.

[2] Beals R. and Coifman R.R., Multidimensional inverse scatterings and nonlinear partial differential equations, *Proc. Sym. Pure Math.* **43** (1985), 45.

[3] Beals R. and Coifman R.R., Scattering and Inverse scattering for first-order system II, *Inverse Problems* **3** (1987), 577.

[4] Boiti M., Konopelchenko B.G. and Pempinelli F., Bäcklund transformations via gauge transformations in 1+2 dimensions, *Inverse Problems* **1** (1985), 33.

[5] Boiti M. and Tu G.Z., Bäcklund transformations via gauge transformations, *Il Nuovo Cimento* **71B** (1982), 253.

[6] Davey A. and Stewartson K., On three-dimensional packets of surface waves, *Proc. Roy. Soc. London* **A338** (1974), 101.

[7] Fokas A.S. and Ablowitz M.J., On the inverse scattering transform of multidimensional nonlinear equations related to first-order system in the plane, *J. Math. Phys.* **25** (1984), 2494.

[8] Gu C.H., On the Bäcklund transformations for the generalized hierarchies of compound MKdV-SG equations, *Lett. Math. Phys. 12* (1986), 31.

[9] Gu C.H., On the Darboux form of Bäcklund transformations, *Proc. Nankai Symposium on Integrable Systems* (1987), to be published.

[10] Gu C.H. and Hu H.S., A unified explicit form of Bäcklund transformations for generalized hierarchies of KdV equations, *Lett. Math. Phys.* **11** (1986), 325.

[11] Gu C.H. and Zhou Z.X., On the Darboux matrices of Bäcklund transformations for the AKNS systems, *Lett. Math. Phys.* **13** (1987), 179.

[12] Gu C.H. and Zhou Z.X., Explicit form of Bäcklund transformations for $GL(N)$, $U(N)$ and $O(2N)$ principal chiral fields, preprint (1988).

[13] Gu X.S., L-degree Darboux transformation for the evolution equations which are associated with A.K.N.S. eigenvalue problem and its reductions, *Ann. Diff. Eqs.* **3** (1987), 13.

[14] Hirota R. and Satsuma J., A simple structure of superposition formula of the Bäcklund transformation, *J. Phys. Soc. Japan* **45** (1978), 1741.

[15] Kadomtsev B. and Petviashvili V.I., On the stability of solitary waves in a weakly dispersing media, *Sov. Phys. Dokl.* **15** (1970), 539.

[16] Kaup D.J. and Newell A.C., An exact solution for a derivative nonlinear Schrödinger equation, *J. Math. Phys.* **19** (1978), 798.

[17] Konno K. and Wadati M., Simple derivation of Bäcklund transformation from Riccati form of inverse method, *Prog. Theor. Phys.* **53** (1975), 1652.

[18] Konopelchenko B.G. and Dubrovsky V.G., Bäcklund-Cologero group and general form of integrable equations for the two-dimensional Gelfand-Dikij-Zakharov-Shabat problem, bilocal approach, *Physica* **16D** (1985), 79.

[19] Lax P.D., Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968), 467.

[20] Levi D., Toward a unification of the various techniques used to integrate non-linear partial differential equations: Bäcklund and Darboux transformations v.s. dressing method, *Rep. Math. Phys.* **23** (1986), 41.

[21] Levi D., Pilloni L. and Santini P.M., Bäcklund transformations for nonlinear evolution equations in 2+1 dimensions, *Phys. Lett.* **A81** (1981), 419.

[22] Levi D., Ragnisco O. and Sym A., Dressing method v.s. classical Darboux transformation, *Il Nuovo Cimento***83B** (1984), 34.

[23] Li Y.S. and Gu X.S., Generating solution of Boussinesq equation by Darboux transformation of three order eigenvalue differential equations, *Ann. Diff. Eqs.* **2** (1986), 419.

[24] Li Y.S., Gu X.S. and Zou M.R., Three kinds of Darboux transformations for the evolution equations which connect with A.K.N.S. eigenvalue problem, *Acta. Math. Sinica*, New Series **3** (1987), 143.

[25] Lin J.M., Evolution of the scattering data under classical Darboux transform for $su(2)$ soliton system, preprint (1987).

[26] Neugerbauer G. and Meinel R., General N-soliton solution of the AKNS class on arbitrary background, *Phys. Lett.* **100A** (1984), 467.

[27] Novikov S., Manakov S.V., Pitaevskii L.P. and Zakharov V.E., *Theory of Solitons*, Consultants Bureau (1984).

[28] Ogielski A.T., Prasad M.K., Sinha A. and Chau L.L., Bäcklund transformations and local conservation laws for principal chiral fields, *Phys. Lett.* **91B** (1980), 387.

[29] Rogers C. and Shadwick W.R., *Bäcklund Transformations and Their Applications*, Academic Press (1982).

[30] Sattinger D.H. and Zurkowski V.D., Gauge theory of Bäcklund transformations II, *Physica* **26D** (1987), 225.

[31] Tian C., Bäcklund transformation of nonlinear evolution equations, *Acta Math. Appl. Sinica* (1986).

[32] Tian C., Generalized KP equation and Miura transformation, preprint (1986).

[33] Wadati M., Sanuki H. and Konno K., Relationships among inverse method, Bäcklund transformations and infinite number of conservation laws, *Prog. Theor. Phys.* **53** (1975), 418.

[34] Zakharov V.E. and Shabat A.B., Integration of the nonlinear equations of mathematical physics by the method of the inverse scattering problem, *Func. Anal. Appl.* **13** (1980), 166.

[35] Zhang Y.J., A correspondence between the Kaup-Newell hierarchy and the AKNS hierarchy and the Darboux transformation for the Kaup-Newell eigenvalue problem, *J. China Univ. Sci. Tech.* (1987).

[36] Zhou Z.X., On the Darboux transformation for 1+2 dimensional equations, *Lett. Math. Phys.* **16** (1988), 9.

[37] Zhou Z.X., General form of Darboux transformations for Lax pairs, preprint IC/88/30, ICTP (1988).