DARBOUX TRANSFORMATIONS IN INTEGRABLE SYSTEMS
Theory and their Applications to Geometry

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Preface

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The soliton theory is an important branch of nonlinear science. On one hand, it describes various kinds of stable motions appearing in nature, such as solitary water wave, solitary signals in optical fibre etc., and has many applications in science and technology (like optical signal communication). On the other hand, it gives many effective methods of getting explicit solutions of nonlinear partial differential equations. Therefore, it has attracted much attention from physicists as well as mathematicians.

Nonlinear partial differential equations appear in many scientific problems. Getting explicit solutions is usually a difficult task. Only in certain special cases can the solutions be written down explicitly. However, for many soliton equations, people have found quite a few methods to get explicit solutions. The most famous ones are the inverse scattering method, Bäcklund transformation etc.. The inverse scattering method is based on the spectral theory of ordinary differential equations. The Cauchy problem of many soliton equations can be transformed to solving a system of linear integral equations. Explicit solutions can be derived when the kernel of the integral equation is degenerate. The Bäcklund transformation gives a new solution from a known solution by solving a system of completely integrable partial differential equations. Some complicated "nonlinear superposition formula" arise to substitute the superposition principle in linear science.

However, if the kernel of the integral equation is not degenerate, it is very difficult to get the explicit expressions of the solutions via the inverse scattering method. For the Bäcklund transformation, the nonlinear superposition formula is not easy to be obtained in general. In late 1970s, it was discovered by V. B. Matveev that a method given by G. Darboux a century ago for the spectral problem of second order ordinary differential equations can be extended to some important soliton equations. This method was called Darboux transformation. After that, it was found that this method is very effective for many partial differential equations. It is now playing an important role in mechanics, physics and differential geometry. V. B. Matveev and M. A. Salle published an important monograph [80] on this topic in 1991. Besides, an interesting monograph of C. Rogers and W. K. Schief [90] with many recent results was published in 2002.

The present monograph contains the Darboux transformations in matrix form and provides purely algebraic algorithms for constructing explicit solutions. Consequently, a basis of using symbolic calculations to obtain explicit exact solutions for many integrable systems is established. Moreover, the behavior of simple and multi-solutions, even in multi-dimensional cases, can be elucidated clearly. The method covers a series of important topics such as varies kinds of AKNS systems in $\mathbf{R}^{n+1}$, the construction of Bäcklund congruences and surfaces of constant Gauss curvature in $\mathbf{R}^3$ and $\mathbf{R}^{2,1}$, harmonic maps from two dimensional manifolds to the Lie group $U(n)$, self-dual Yang-Mills fields and the generalizations to higher dimensional case, Yang-Mills-Higgs fields in $2+1$ dimensional Minkowski and anti-de Sitter space, Laplace sequences of surfaces in projective spaces and two dimensional Toda equations. All these cases are stated in details. In geometric problems, the Lax pair is not only a tool, but also a geometric object to be studied. Many results in this monograph are obtained by the authors in recent years.

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Chapter 1

1+1 DIMENSIONAL INTEGRABLE SYSTEMS

Starting from the original Darboux transformation, we first discuss the classical form of the Darboux transformations for the KdV and the MKdV equation, then discuss the Darboux transformations for the AKNS system and more general systems. The coefficients in the evolution equations discussed here may depend on t. The Darboux matrices are constructed algebraically and the algorithm is purely algebraic and universal to whole hierarchies. The Darboux transformations for reduced systems are also concerned. We also present the relations between Darboux transformation and the inverse scattering theory, and show that the number of solitons (the number of eigenvalues) increases or decreases after the action of a Darboux transformation.

1.1 KdV equation, MKdV equation and their Darboux transformations

1.1.1 Original Darboux transformation

In 1882, G. Darboux [18] studied the eigenvalue problem of a linear partial differential equation of second order (now called the one-dimensional Schrödinger equation)

$$-\phi_{xx} - u(x)\phi = \lambda\phi. \tag{1.1}$$

Here $u(x)$ is a given function, called potential function; λ is a constant, called spectral parameter. He found out the following fact. If $u(x)$ and $\phi(x, \lambda)$ are two functions satisfying (1.1) and $f(x) = \phi(x, \lambda_0)$ is a solution of the equation (1.1) for $\lambda = \lambda_0$ where λ_0 is a fixed constant, then the functions u' and ϕ' defined by

$$u' = u + 2(\ln f)_{xx}, \quad \phi'(x,\lambda) = \phi_x(x,\lambda) - \frac{f_x}{f}\phi(x,\lambda) \tag{1.2}$$

satisfy

$$-\phi'_{xx} - u'\phi' = \lambda\phi', \tag{1.3}$$

which is of the same form as (1.1). Therefore, the transformation (1.2) transforms the functions (u, ϕ) to (u', ϕ') which satisfy the same equations. This transformation

$$(u, \phi) \longrightarrow (u', \phi'), \tag{1.4}$$

is the original Darboux transformation, which is valid for $f \neq 0$.

1.1.2 Darboux transformation for KdV equation

In 1885, the Netherlandish applied mathematicians Korteweg and de Vries introduced a nonlinear partial differential equation describing the motion of water wave, which is now called the Korteweg-de Vries equation (KdV equation)

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.5}$$

In the middle of 1960's, this equation was found out to be closely related to the Schrödinger equation mentioned above [87]. KdV equation (1.5) is the integrability condition of the linear system

$$\begin{aligned} &-\phi_{xx} - u\phi = \lambda\phi, \\ &\phi_t = -4\phi_{xxx} - 6u\phi_x - 3u_x\phi \end{aligned} \tag{1.6}$$

which is called the Lax pair of the KdV equation. Here u and ϕ are functions of x and t. (1.6) is the integrability condition of (1.5). In other words, (1.5) is the necessary and sufficient condition for $(\phi_{xx})_t = (\phi_t)_{xx}$ being an identity for all λ, where $(\phi_{xx})_t$ is computed from $\phi_{xx} = (-\lambda - u)\phi$ (the first equation of (1.6)) and $(\phi_t)_{xx}$ is given by the second equation of (1.6).

Since the first equation of the Lax pair of the KdV equation is just the Schrödinger equation, the Darboux transformation (1.2) can also be applied to the KdV equation, where the functions depend on t. Obviously the transformation keeps the first equation of (1.6) invariant, i.e., (u', ϕ') satisfies

$$-\phi'_{xx} - u'\phi' = \lambda\phi'. \tag{1.7}$$

Moreover, it is easily seen that (u', ϕ') satisfies the second equation of (1.6) as well. Therefore, u' satisfies the KdV equation, which is the integrability condition of (1.6). In summary, suppose one knows a solution u of the KdV equation, solving the linear equations (1.6) one gets $\phi(x,t,\lambda)$. Take λ to be a special value λ_0 and let $f(x,t)=\phi(x,t,\lambda_0)$, then $u'=u+2(\ln f)_{xx}$ gives a new solution of the KdV equation, and ϕ' given by (1.2) is a solution of the Lax pair corresponding to u'. This gives a way to obtain new solutions of the KdV equation.

This process can be done successively as follows. For a known solution u of (1.5), first solve a system of linear differential equations (1.6) and get ϕ. Then explicit calculation from (1.2) gives new special solutions of the KdV equation. Since ϕ' is known, it is not necessary to solve any linear differential equations again to obtain (u'',ϕ''). That is, we only need algebraic calculation to get (u'',ϕ'') etc.:

$$(u,\phi)\longrightarrow(u',\phi')\longrightarrow(u'',\phi'')\longrightarrow\cdots. \tag{1.8}$$

Therefore, we have extended the Darboux transformation for the Schrödinger equation to the KdV equation. The basic idea here is to get the new solutions of the nonlinear equation and the corresponding solutions of the Lax pair simultaneously from a known solution of the nonlinear equation and a solution of its Lax pair by using algebraic and differential computation. Note that the formula is valid only for $f\neq 0$. If $f=0$, the Darboux transformation will have singularities.

Remark 1 Let $\psi_1=\phi$, $\psi_2=\phi_x$, $\Psi=(\psi_1,\psi_2)^T$, then the Lax pair (1.6) can be written in matrix form as

$$\begin{aligned}\Psi_x&=\begin{pmatrix}0&1\\-\lambda-u&0\end{pmatrix}\Psi,\\ \Psi_t&=\begin{pmatrix}u_x&4\lambda-2u\\-4\lambda^2-2\lambda u+u_{xx}+2u^2&-u_x\end{pmatrix}\Psi.\end{aligned} \tag{1.9}$$

The transformation $\phi\to\phi'$ in (1.2) can also be rewritten as a transformation of Ψ, which can be realized via algebraic algorithm only. We shall discuss this Darboux transformation in matrix form later.

1.1.3 Darboux transformation for MKdV equation

The method of Darboux transformation can be applied to many other equations such as the MKdV equation, the sine-Gordon equation etc. [105]. We first take the MKdV equation as an example. General cases will be considered in the latter sections.

MKdV equation

$$u_t+6u^2u_x+u_{xxx}=0 \tag{1.10}$$

is the integrability condition of the over-determined linear system [2, 119]

$$\begin{aligned}
&\Phi_x = U\Phi = \begin{pmatrix} \lambda & u \\ -u & -\lambda \end{pmatrix}\Phi, \\
&\Phi_t = V\Phi \\
&\quad = \begin{pmatrix} -4\lambda^3 - 2u^2\lambda & -4u\lambda^2 - 2u_x\lambda - 2u^3 - u_{xx} \\ 4u\lambda^2 - 2u_x\lambda + 2u^3 + u_{xx} & 4\lambda^3 + 2u^2\lambda \end{pmatrix}\Phi,
\end{aligned} \tag{1.11}$$

that is, (1.10) is the necessary and sufficient condition for $\Phi_{xt} = \Phi_{tx}$ being an identity. The system (1.11) is called a Lax pair of (1.10) and λ a spectral parameter. Here Φ may be regarded as a column solution or a 2×2 matrix solution of (1.11).

There are several ways to derive the Darboux transformation for the MKdV equation. Here we use the Darboux matrix method.

For a given solution u of the MKdV equation, suppose we know a fundamental solution of (1.11)

$$\Phi(x,t,\lambda) = \begin{pmatrix} \Phi_{11}(x,t,\lambda) & \Phi_{12}(x,t,\lambda) \\ \Phi_{21}(x,t,\lambda) & \Phi_{22}(x,t,\lambda) \end{pmatrix} \tag{1.12}$$

which composes two linearly independent column solutions of (1.11).

Let λ_1, μ_1 be arbitrary real numbers and

$$\sigma = \frac{\Phi_{22}(x,t,\lambda_1) + \mu_1\Phi_{21}(x,t,\lambda_1)}{\Phi_{12}(x,t,\lambda_1) + \mu_1\Phi_{11}(x,t,\lambda_1)} \tag{1.13}$$

be the ratio of the two entries of a column solution of the Lax pair (1.11). Construct the matrix

$$D(x,t,\lambda) = \lambda I - \frac{\lambda_1}{1+\sigma^2}\begin{pmatrix} 1-\sigma^2 & 2\sigma \\ 2\sigma & \sigma^2 - 1 \end{pmatrix} \tag{1.14}$$

and let $\Phi'(x,t,\lambda) = D(x,t,\lambda)\Phi(x,t,\lambda)$. Then it is easily verified that $\Phi'(x,t,\lambda)$ satisfies

$$\Phi'_x = U'\Phi', \qquad \Phi'_t = V'\Phi', \tag{1.15}$$

where

$$U' = \begin{pmatrix} \lambda & u' \\ -u' & -\lambda \end{pmatrix},$$
$$V' = \begin{pmatrix} -4\lambda^3 - 2u'^2\lambda & -4u'\lambda^2 - 2u'_x\lambda - 2u'^3 - u'_{xx} \\ 4u'\lambda^2 - 2u'_x\lambda + 2u'^3 + u'_{xx} & 4\lambda^3 + 2u'^2\lambda \end{pmatrix} \tag{1.16}$$

with

$$u' = u + \frac{4\lambda_1\sigma}{1+\sigma^2}. \tag{1.17}$$

(1.15) and (1.16) are similar to (1.11). The only difference is that the u in (1.11) is replaced by u' defined by (1.17). For any solution Φ of (1.11), $D\Phi$ is a solution of (1.15), hence (1.15) is solvable for any given initial data (the value of Φ' at some point (x_0, t_0)). In other words, (1.15) is integrable. The integrability condition of (1.15) implies that u' is also a solution of the MKdV equation. Using this method, we obtain a new solution of the MKdV equation together with the corresponding fundamental solution of its Lax pair from a known one.

The above conclusions can be summarized as follows. Let u be a solution of the MKdV equation and Φ be a fundamental solution of its Lax pair. Take λ_1, μ_1 to be two arbitrary real constants, and let σ be defined by (1.13), then (1.17) gives a new solution u' of the MKdV equation, and the corresponding solution to the Lax pair can be taken as $D\Phi$. The transformation $(u,\Phi) \to (u',\Phi')$ is the Darboux transformation for the MKdV equation. This Darboux transformation in matrix form can be done successively and purely algebraically as

$$(u,\Phi) \longrightarrow (u',\Phi') \longrightarrow (u'',\Phi'') \longrightarrow \cdots. \tag{1.18}$$

Remark 2 Both (1.9) and (1.11) are of the form

$$\Phi_x = U\Phi, \qquad \Phi_t = V\Phi \tag{1.19}$$

where U and V are $N \times N$ matrices and independent of Φ. The integrability condition of (1.19) is

$$U_t - V_x + [U,V] = 0 \quad ([U,V] = UV - VU). \tag{1.20}$$

According to the elementary theory of linear partial differential equations, the solution of (1.19) exists uniquely for given initial data $\Phi(x_0,t_0) = \Phi_0$ if and only if (1.20) holds identically. Here Φ is an $N \times N$ matrix, or a *vector with N entries. In this case $\Phi(x,t)$ is determined by the ordinary differential equation*

$$d\Phi = (U\,\mathrm{d}x + V\,\mathrm{d}y)\Phi \tag{1.21}$$

along an arbitrary path connecting (x_0, t_0) and (x,t). (1.20) is also called a zero-curvature condition.

1.1.4 Examples: single and double soliton solutions

Starting with the trivial solution $u = 0$ of the MKdV equation, one can use the Darboux transformation to obtain the soliton solutions. For $u = 0$, the fundamental solution of the Lax pair can be obtained as

$$\Phi(x,t,\lambda) = \begin{pmatrix} \exp(\lambda x - 4\lambda^3 t) & 0 \\ 0 & \exp(-\lambda x + 4\lambda^3 t) \end{pmatrix} \tag{1.22}$$

by integrating (1.11). Take $\lambda_1 \neq 0$ and $\mu_1 = \exp(2\alpha_1) > 0$, then (1.13) gives

$$\sigma = \sigma_1 = \exp(-2\lambda_1 x + 8\lambda_1^3 t - 2\alpha_1), \tag{1.23}$$

hence

$$D = \lambda I - \frac{\lambda_1}{\cosh v_1}\begin{pmatrix} \sinh v_1 & 1 \\ 1 & -\sinh v_1 \end{pmatrix}, \tag{1.24}$$

where

$$v_1 = 2\lambda_1 x - 8\lambda_1^3 t + 2\alpha_1. \tag{1.25}$$

(1.17) gives the single soliton solution

$$u' = 2\lambda_1\ \mathrm{sech}(2\lambda_1 x - 8\lambda_1^3 t + 2\alpha_1), \tag{1.26}$$

of the MKdV equation.

(1.26) is called the single soliton solution because it has the following properties: (i) It is a travelling wave solution, i.e., it is in the form $u' = f(x - ct)$; (ii) For any t, $\lim_{x\to\pm\infty} u' = 0$. Speaking intuitively, u' is near 0 outside a small region, i.e., $|u| < 2|\lambda_1|\,\mathrm{sech}\,K$ when $|2\lambda_1 x - 8\lambda_1^3 t + 2\alpha_1| > K$.

The solution of the corresponding Lax pair is

$$\begin{aligned} \Phi'(x,t,\lambda) &= (\Phi'_{ij}(x,t,\lambda)) \\ &= D(x,t,\lambda)\begin{pmatrix} \exp(\lambda x - 4\lambda^3 t) & 0 \\ 0 & \exp(-\lambda x + 4\lambda^3 t) \end{pmatrix} \end{aligned} \tag{1.27}$$

where D is given by (1.24).

If we take u' as a seed solution, a new Darboux matrix can be constructed from Φ' and a series of new solutions of the MKdV equation can be obtained.

We write down the second Darboux transformation explicitly. Suppose u is a solution of the MKdV equation (1.10), Φ is a fundamental solution of the corresponding Lax pair (1.11). Construct the Darboux matrix $D = (D_{ij})$ according to (1.13) and (1.14) and let $\sigma = \sigma_1$. Moreover, take constants $\lambda_2 \neq 0$ ($\lambda_2 \neq \lambda_1$) and $\mu_2 = \exp(2\alpha_2)$. According to (1.13),

$$\sigma_2' = \frac{\Phi_{22}'(x,t,\lambda_2) + \mu_2 \Phi_{21}'(x,t,\lambda_2)}{\Phi_{12}'(x,t,\lambda_2) + \mu_2 \Phi_{11}'(x,t,\lambda_2)}. \tag{1.28}$$

Substituting $\Phi' = D\Phi$ into it, we have

$$\begin{aligned} \sigma_2' &= \left.\frac{D_{21}(\Phi_{12} + \mu_2\Phi_{11}) + D_{22}(\Phi_{22} + \mu_2\Phi_{21})}{D_{11}(\Phi_{12} + \mu_2\Phi_{11}) + D_{12}(\Phi_{22} + \mu_2\Phi_{21})}\right|_{\lambda=\lambda_2} \\ &= \frac{D_{21}(\lambda_2) + D_{22}(\lambda_2)\sigma_2}{D_{11}(\lambda_2) + D_{12}(\lambda_2)\sigma_2}, \end{aligned} \tag{1.29}$$

where

$$\sigma_2 = \frac{\Phi_{22}(x,t,\lambda_2) + \mu_2 \Phi_{21}(x,t,\lambda_2)}{\Phi_{12}(x,t,\lambda_2) + \mu_2 \Phi_{11}(x,t,\lambda_2)}. \tag{1.30}$$

Starting from $u = 0$, (1.26) and (1.27) are the single soliton solution and the corresponding fundamental solution of the Lax pair. Substituting (1.24), the expression of D, into (1.27), we have

$$\Phi'(x,t,\lambda) = \begin{pmatrix} (\lambda - \lambda_1 \tanh v_1)\mathrm{e}^{\lambda x - 4\lambda^3 t} & -\lambda_1 \operatorname{sech} v_1 \mathrm{e}^{-\lambda x + 4\lambda^3 t} \\ -\lambda_1 \operatorname{sech} v_1 \mathrm{e}^{\lambda x - 4\lambda^3 t} & (\lambda + \lambda_1 \tanh v_1)\mathrm{e}^{-\lambda x + 4\lambda^3 t} \end{pmatrix}, \tag{1.31}$$

hence

$$\sigma_2' = \frac{-\lambda_1 \operatorname{sech} v_1 + (\lambda_2 + \lambda_1 \tanh v_1)\exp(-v_2)}{(\lambda_2 - \lambda_1 \tanh v_1) - \lambda_1 \operatorname{sech} v_1 \exp(-v_2)}, \tag{1.32}$$

where

$$v_2 = 2\lambda_2 x - 8\lambda_2^3 t + 2\alpha_2, \quad (i = 1, 2). \tag{1.33}$$

According to (1.17),

$$\begin{aligned} u'' &= \frac{4\lambda_1\sigma_1}{1+\sigma_1^2} + \frac{4\lambda_2\sigma_2'}{1+\sigma_2'^2} \\ &= \frac{2(\lambda_2^2 - \lambda_1^2)(\lambda_2 \cosh v_1 - \lambda_1 \cosh v_2)}{(\lambda_1^2 + \lambda_2^2)\cosh v_1 \cosh v_2 - 2\lambda_1\lambda_2(1 + \sinh v_1 \sinh v_2)} \end{aligned} \tag{1.34}$$

is a new solution of the MKdV equation. This is called the double soliton solution of the MKdV equation. This name follows from the following asymptotic property of the solutions. We shall show that a double soliton solution is asymptotic to two single soliton solutions as $t \to \infty$.

Suppose $\lambda_2 > \lambda_1 > 0$, M is a fixed positive number. Let v_1 be bounded by $|v_1| \le M$, then $x \sim \infty$ as $t \sim \infty$. Since

$$v_2 = \frac{\lambda_2}{\lambda_1} v_1 - 8\lambda_2(\lambda_2^2 - \lambda_1^2)t + 2\alpha_2 - \frac{2\lambda_2\alpha_1}{\lambda_1}, \tag{1.35}$$

$v_2 \sim +\infty$ as $t \sim -\infty$, and

$$u'' \sim -2\lambda_1 \operatorname{sech}(v_1 - v_0) \tag{1.36}$$

as $t \to -\infty$ where

$$v_0 = \tanh^{-1} \frac{2\lambda_1\lambda_2}{\lambda_1^2 + \lambda_2^2}. \tag{1.37}$$

If $t \sim +\infty$, then $v_2 \sim -\infty$, and

$$u'' \sim -2\lambda_1 \operatorname{sech}(v_1 + v_0). \tag{1.38}$$

Hence, for fixed v_1 (i.e., the observer moves in the velocity $4\lambda_1^2$), the solution is asymptotic to one single soliton solution (corresponding to the parameter λ_1) as $t \sim -\infty$ or $t \sim +\infty$. However, there is a phase shift between the asymptotic solitons as $t \sim -\infty$ and $t \sim +\infty$. That is, the center of the soliton (the peak) moves from $v_1 = v_0$ to $v_1 = -v_0$.

Similarly, if $|v_2| \le M$, then

$$v_1 = \frac{\lambda_1}{\lambda_2} v_2 + 8\lambda_1(\lambda_2^2 - \lambda_1^2)t + 2\alpha_1 - \frac{2\lambda_1\alpha_2}{\lambda_2} \tag{1.39}$$

implies that $v_1 \sim \pm\infty$ as $t \sim \pm\infty$, and

$$\begin{aligned} u'' &\sim 2\lambda_2 \operatorname{sech}(v_2 + v_0), \quad t \sim -\infty, \\ u'' &\sim 2\lambda_2 \operatorname{sech}(v_2 - v_0), \quad t \sim +\infty. \end{aligned} \tag{1.40}$$

Finally, if $t \sim \pm\infty$ and both v_1, v_2 tend to $\pm\infty$ (i.e., the observer moves in the velocity $\neq 4\lambda_1^2, 4\lambda_2^2$), then $u'' \sim 0$. Therefore, whenever $t \sim +\infty$ or $t \sim -\infty$, u'' is asymptotic to two single soliton solutions (see Figures 1.1 – 1.3.

This fact means that: (i) a double soliton solution is asymptotic to two single soliton solutions as $t \to \pm\infty$; (ii) if two single solitons (the asymptotic behavior as $t \to -\infty$) interact, they will almost recover later

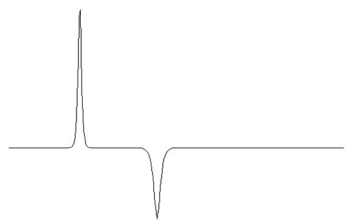


Figure 1.1. Double soliton solutions of the MKdV equation, $t = -1$

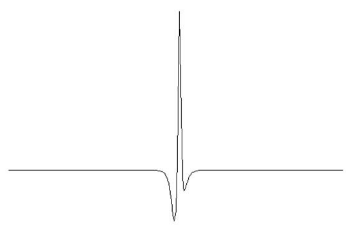


Figure 1.2. Double soliton solutions of the MKdV equation, $t = 0.1$

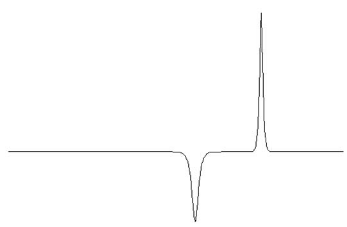


Figure 1.3. Double soliton solutions of the MKdV equation, $t = 1$

($t \to +\infty$). Both the shape and the velocity do not change. The only change is the phase shift. Physically speaking, there is elastic scattering between solitons. This is the most important character of solitons. The discovery of this property (first to the KdV equation) greatly promotes the progress of the soliton theory.

Remark 3 *Starting from the trivial solution $u = 0$, we can also obtain the single and double soliton solutions of the KdV equation by using the original Darboux transformation mentioned at the beginning of this section. The computation is simpler and is left for the reader.*

The Darboux transformation for the MKdV equation can be used to get not only the single and double soliton solutions, but also the multi-soliton solutions. Moreover, this method can be applied to many other nonlinear equations. We shall discuss the general problem in the next section.

1.1.5 Relation between Darboux transformations for KdV equation and MKdV equation

The Darboux transformation for the MKdV equation can also be derived from the "complexification" of the Schrödinger equation (1.1) directly. That is why the transformation given by the matrix D is also called a Darboux transformation.

Take a solution $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ of (1.11), then the first equation of (1.11) is

$$\begin{aligned} \phi_{1,x} &= \lambda\phi_1 + u\phi_2, \\ \phi_{2,x} &= -u\phi_1 - \lambda\phi_2. \end{aligned} \tag{1.41}$$

Let $\psi = \phi_1 + \mathrm{i}\phi_2$ and suppose λ is a real parameter, u is a real function, then ψ satisfies

$$\psi_{xx} = \lambda^2\psi - (\mathrm{i}u_x + u^2)\psi. \tag{1.42}$$

This is a complex Schrödinger equation with potential $(\mathrm{i}u_x + u^2)$. It can be checked directly that if u is a solution of the MKdV equation, then $w = \mathrm{i}u_x + u^2$ is a complex solution of the KdV equation $w_t + 6ww_x + w_{xxx} = 0$. The transformation from the solution u of the MKdV equation to the solution w of the KdV equation is called a Miura transformation.

Remark 4 Let $v = \mathrm{i}u$, then (1.10) is

$$v_t - 6v^2v_x + v_{xxx} = 0, \tag{1.10$'$}$$

and the Miura transformation becomes $w = v_x - v^2$. If v is a real solution of (1.10)$'$, then w is a real solution of the KdV equation.

Take a real number λ_0 and a solution $f = f_1 + \mathrm{i}f_2$ of the equation (1.42) for $\lambda = \lambda_0$, then $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is a solution of (1.41) for $\lambda = \lambda_0$. Using the conclusion to the KdV equation, we know that

$$\psi' = \psi_x - (f_x/f)\psi, \quad w' = w + 2(\ln f)_{xx} \tag{1.43}$$

satisfy (1.42) and w' is a solution of the KdV equation. Moreover, there is a corresponding u' satisfying the MKdV equation. Now we write down the explicit expression of u'.

Considering (1.41), the first equation of (1.43) can be rewritten in terms of the components as

$$\phi_1' + \mathrm{i}\phi_2' = \lambda\phi_1 - \mathrm{i}\lambda\phi_2 - \lambda_0\frac{\bar f}{f}(\phi_1 + \mathrm{i}\phi_2). \tag{1.44}$$

If λ and λ_0 are real numbers, then ϕ_1 and ϕ_2 can be chosen as real functions. (1.44) becomes

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \lambda - \lambda_0\dfrac{f_1^2 - f_2^2}{f_1^2 + f_2^2} & -\lambda_0\dfrac{2f_1f_2}{f_1^2 + f_2^2} \\ \lambda_0\dfrac{2f_1f_2}{f_1^2 + f_2^2} & -\lambda - \lambda_0\dfrac{f_1^2 - f_2^2}{f_1^2 + f_2^2} \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{1.45}$$

It should be noted that the matrix in the right hand side of (1.45) is the counterpart of the Darboux matrix defined by (1.24).

It can be checked that $\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix}$ satisfies

$$\begin{aligned} \phi_{1,x}' &= \lambda\phi_1' + u'\phi_2', \\ \phi_{2,x}' &= -u'\phi_1' - \lambda\phi_2' \end{aligned} \tag{1.46}$$

where

$$u' = -u - \frac{4\lambda_0 f_1 f_2}{f_1^2 + f_2^2}. \tag{1.47}$$

The integrability condition of (1.46) implies that u' is a solution of the MKdV equation.

Remark 5 The matrix given by (1.14) and that given by (1.45) are different by a left-multiplied factor $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *Notice that if* (u, ϕ_1, ϕ_2) *is a solution of (1.41), then* $(-u, \phi_1, -\phi_2)$ *is also a solution of (1.41). Therefore, the solution (1.17) given by (1.14) is the minus of the solution given by (1.47). Both u and* $-u$ *satisfy the MKdV equation. We can take any one transformation as the Darboux transformation.*

The matrix D is very important hereafter. It is called a Darboux matrix.

1.2 AKNS system

1.2.1 2 × 2 AKNS system

In order to generalize the Lax pair of the MKdV equation, V. E. Zakharov, A. B. Shabat and M. J. Ablowitz, D. J. Kaup, A. C. Newell,

H. Segur introduced independently a more general system [2, 119] which is now called the AKNS system. For simplicity, we first discuss the 2×2 AKNS system (i.e., the AKNS system of 2×2 matrices), and then the more general $N \times N$ AKNS system.

2×2 AKNS system is the linear system of differential equations

$$\begin{aligned}\Phi_x &= U\Phi = \lambda J\Phi + P\Phi,\\ \Phi_t &= V\Phi = \sum_{j=0}^{n} V_j \lambda^{n-j}\Phi,\end{aligned} \tag{1.48}$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}, \quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{1.49}$$

$$\begin{aligned}A &= \sum_{j=0}^{n} a_j(x,t)\lambda^{n-j},\\ B &= \sum_{j=0}^{n} b_j(x,t)\lambda^{n-j},\\ C &= \sum_{j=0}^{n} c_j(x,t)\lambda^{n-j},\end{aligned} \tag{1.50}$$

p, q, a_j, b_j, c_j are complex or real functions of x and t, λ is a real or complex parameter, called the spectral parameter. As mentioned in Remark 2, the integrability condition of (1.48)

$$U_t - V_x + [U, V] = 0 \tag{1.51}$$

should hold for all λ. In terms of the components, (1.51) becomes

$$\begin{aligned}A_x &= pC - qB,\\ B_x &= p_t + 2\lambda B - 2pA,\\ C_x &= q_t - 2\lambda C + 2qA.\end{aligned} \tag{1.52}$$

Both sides of the above equations are polynomials of λ. Expanding them in terms of the powers of λ, we have

$$\begin{aligned}&b_0 = c_0 = 0,\\ &a_{j,x} = pc_j - qb_j \quad (0 \le j \le n),\\ &b_{j+1} = \frac{1}{2}b_{j,x} + pa_j \quad (0 \le j \le n-1),\\ &c_{j+1} = -\frac{1}{2}c_{j,x} + qa_j \quad (0 \le j \le n-1),\end{aligned} \tag{1.53}$$

and the evolution equations

$$\begin{aligned} p_t &= b_{n,x} + 2pa_n, \\ q_t &= c_{n,x} - 2qa_n. \end{aligned} \tag{1.54}$$

(1.53) can be regarded as the equations to determine A, B, C, and (1.54) is a system of evolution equations of p and q. In (1.53), a_j, b_j, c_j can be derived through algebraic calculation, differentiation and integration. We can see later that they are actually polynomials of p, q and their derivatives with respect to x (without any integral expressions of p and q), the coefficients of which are arbitrary functions of t. After solving a_j, b_j, c_j from (1.53), we get the system of nonlinear evolution equations of p and q from (1.54).

For $j = 0, 1, 2, 3$,

$$\begin{aligned} a_0 &= \alpha_0(t), \quad b_0 = c_0 = 0, \\ a_1 &= \alpha_1(t), \quad b_1 = \alpha_0(t)p, \quad c_1 = \alpha_0(t)q, \\ a_2 &= -\frac{1}{2}\alpha_0(t)pq + \alpha_2(t), \\ b_2 &= \frac{1}{2}\alpha_0(t)p_x + \alpha_1(t)p, \\ c_2 &= -\frac{1}{2}\alpha_0(t)q_x + \alpha_1(t)q, \\ a_3 &= \frac{1}{4}\alpha_0(t)(pq_x - qp_x) - \frac{1}{2}\alpha_1(t)pq + \alpha_3(t), \\ b_3 &= \frac{1}{4}\alpha_0(t)(p_{xx} - 2p^2q) + \frac{1}{2}\alpha_1(t)p_x + \alpha_2(t)p, \\ c_3 &= \frac{1}{4}\alpha_0(t)(q_{xx} - 2pq^2) - \frac{1}{2}\alpha_1(t)q_x + \alpha_2(t)q. \end{aligned} \tag{1.55}$$

Here $\alpha_0(t)$, $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ are arbitrary functions of t, which are the integral constants in integrating a_0, a_1, a_2, a_3 from the second equation of (1.53).

Here are some simplest and most important examples.

EXAMPLE 1.1 *$n = 3$, $p = u$, $q = -1$, $\alpha_0 = -4$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$. In this case, $a_3 = -u_x$, $b_3 = -u_{xx} - 2u^2$, $c_3 = 2u$. (1.54) becomes the KdV equation*

$$u_t + u_{xxx} + 6uu_x = 0. \tag{1.56}$$

EXAMPLE 1.2 *$n = 3$, $p = u$, $q = -u$, $\alpha_0 = -4$, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then $a_3 = 0$, $b_3 = -u_{xx} - 2u^3$, $c_3 = u_{xx} + 2u^3$. The equation becomes the MKdV equation*

$$u_t + u_{xxx} + 6u^2u_x = 0. \tag{1.57}$$

EXAMPLE 1.3 *$n = 2$, $p = u$, $q = -\bar{u}$, $\alpha_0 = -2\mathrm{i}$, $\alpha_1 = \alpha_2 = 0$, $a_2 = -\mathrm{i}|u|^2$, $b_2 = -\mathrm{i}u_x$, $c_2 = -\mathrm{i}\bar{u}_x$. (1.54) is the nonlinear Schrödinger equation*

$$\mathrm{i}u_t = u_{xx} + 2|u|^2 u. \tag{1.58}$$

We have seen that for $j = 0, 1, 2, 3$, a_j, b_j, c_j are differential polynomials of p and q, i.e., polynomials of p, q and their derivatives with respect to x, whose coefficients are constants or arbitrary functions of t.

LEMMA 1.4 *a_j, b_j, c_j given by (1.53) are differential polynomials of p and q.*

Proof. Use induction. The conclusion is obviously true for $j = 0$.

Suppose a_j, b_j, c_j are differential polynomials of p and q for $j < l$, we will prove that a_l, b_l, c_l are also differential polynomials of p and q.

(1.53) implies that b_l, c_l are differential polynomials of p and q. Hence it is only necessary to prove that a_l is a differential polynomial of p and q. For $1 \le j \le l-1$,

$$\begin{aligned}
& b_j c_{l+1-j} - c_j b_{l+1-j} \\
= \ & b_j(q a_{l-j} - \frac{1}{2} c_{l-j,x}) - c_j(p a_{l-j} + \frac{1}{2} b_{l-j,x}) \\
= \ & (q b_j - p c_j) a_{l-j} - \frac{1}{2}(b_j c_{l-j,x} + c_j b_{l-j,x}) \\
= \ & -(a_j a_{l-j} + \frac{1}{2} b_j c_{l-j} + \frac{1}{2} c_j b_{l-j})_x + a_j a_{l-j,x} \\
& + \frac{1}{2}(b_{j,x} c_{l-j} + c_{j,x} b_{l-j}) \\
= \ & -(a_j a_{l-j} + \frac{1}{2} b_j c_{l-j} + \frac{1}{2} c_j b_{l-j})_x + (p a_j + \frac{1}{2} b_{j,x}) c_{l-j} \\
& -(q a_j - \frac{1}{2} c_{j,x}) b_{l-j} \\
= \ & -(a_j a_{l-j} + \frac{1}{2} b_j c_{l-j} + \frac{1}{2} c_j b_{l-j})_x + b_{j+1} c_{l-j} - c_{j+1} b_{l-j}.
\end{aligned} \tag{1.59}$$

Summarize for j from 1 to $l-1$, we have

$$b_1 c_l - c_1 b_l = -\sum_{j=1}^{l-1} (a_j a_{l-j} + \frac{1}{2} b_j c_{l-j} + \frac{1}{2} c_j b_{l-j})_x - (b_1 c_l - c_1 b_l), \tag{1.60}$$

i.e.,

$$p c_l - q b_l = -\sum_{j=1}^{l-1} \frac{1}{4a_0} (2 a_j a_{l-j} + b_j c_{l-j} + c_j b_{l-j})_x. \tag{1.61}$$

Hence

$$a_l = -\sum_{j=1}^{l-1} \frac{1}{4a_0} (2 a_j a_{l-j} + b_j c_{l-j} + c_j b_{l-j}) + \alpha_l(t) \tag{1.62}$$

is a differential polynomial of p and q. The lemma is proved.

Since $\{a_j, b_j, c_j\}$ are differential polynomials of p and q, we can define $\{a_j^0[p,q]\}$, $\{b_j^0[p,q]\}$, $\{c_j^0[p,q]\}$ recursively so that they satisfy the recursion relations (1.53) and the conditions $a_0^0[0,0] = 1$, $a_j^0[0,0] = 0$ $(1 \le j \le n)$. Clearly, these $\{a_j^0, b_j^0, c_j^0\}$ are uniquely determined as certain polynomials of p, q and their derivatives with respect to x. From (1.53), we have

LEMMA 1.5

$$\begin{aligned} &b_j^0[0,q] = 0, \quad c_j^0[p,0] = 0, \\ &a_j^0[p,0] = a_j^0[0,q] = 0 \quad (1 \le j \le n) \end{aligned} \tag{1.63}$$

for any p and q. Moreover, for any $\{a_j, b_j, c_j\}$ satisfying (1.53), there exist $\alpha_j(t)$ $(0 \le j \le n)$ such that

$$\begin{aligned} a_k[p,q] &= \sum_{j=0}^{k} \alpha_{k-j}(t) a_j^0[p,q], \\ b_k[p,q] &= \sum_{j=0}^{k} \alpha_{k-j}(t) b_j^0[p,q], \\ c_k[p,q] &= \sum_{j=0}^{k} \alpha_{k-j}(t) c_j^0[p,q]. \end{aligned} \tag{1.64}$$

Remark 6 For any positive integer n, the first equation of (1.48) (x-equation) is fixed, but the second one depends on the choice of $\alpha_0(t)$, $\cdots$, $\alpha_n(t)$. Therefore, the evolution equations (1.54) also depend on the choice of $\alpha_0(t)$, $\cdots$, $\alpha_n(t)$. This means that (1.54) is a series of equations, which is called the AKNS hierarchy. If $\alpha_0(t)$, $\cdots$, $\alpha_n(t)$ are all constants, then the evolution equations have the coefficients independent of t and form a series of infinite dimensional dynamical systems. Especially, if $\alpha_0 = \cdots = \alpha_{n-1} = 0$, $\alpha_n = 1$, then we obtain the normalized AKNS hierarchy, written as

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = K_n \begin{bmatrix} p \\ q \end{bmatrix}, \tag{1.65}$$

where K_n is a nonlinear differential operator defined by

$$K_n \begin{bmatrix} p \\ q \end{bmatrix} = \begin{pmatrix} b_{n,x}^0 + 2pa_n^0 \\ c_{n,x}^0 - 2qa_n^0 \end{pmatrix}. \tag{1.66}$$

From this definition and (1.53), we know that K_n *is derived from* K_{n-1} *by recursive algorithm.*

1.2.2 $N \times N$ AKNS system

In the last subsection, we introduced the 2×2 AKNS system. In order to obtain more nonlinear partial differential equations, the 2×2 Lax pair should be generalized naturally to the problems of $N \times N$ matrices. Therefore, we discuss the Lax pair

$$\begin{aligned} \Phi_x &= U\Phi = \lambda J\Phi + P(x,t)\Phi, \\ \Phi_t &= V\Phi = \sum_{j=0}^{n} V_j(x,t)\lambda^{n-j}\Phi \end{aligned} \tag{1.67}$$

where J is an $N \times N$ constant diagonal matrix, $P(x,t)$, $V_j(x,t)$ are $N \times N$ matrices and $P(x,t)$ is off-diagonal (i.e., its diagonal entries are all zero), λ is a spectral parameter. We assume that all the entries of J are distinct, though the assumption can be released with restrictions on P and V.

The integrability condition of (1.67) is still

$$U_t - V_x + [U,V] = 0. \tag{1.68}$$

Using the expressions of U and V in (1.67), we have

$$P_t - \sum_{j=0}^{n} V_{j,x}\lambda^{n-j} + \sum_{j=-1}^{n-1} [J, V_{j+1}]\lambda^{n-j} + \sum_{j=0}^{n} [P, V_j]\lambda^{n-j} = 0. \tag{1.69}$$

The coefficients of each power of λ on the left hand side should be zero. This leads to

$$\begin{aligned} &[J, V_0] = 0, \\ &[J, V_{j+1}] - V_{j,x} + [P, V_j] = 0 \qquad (0 \le j \le n-1), \\ &P_t - V_{n,x} + [P, V_n] = 0. \end{aligned} \tag{1.70}$$

For any $N \times N$ matrix M, we divide it as $M = M^{\text{diag}} + M^{\text{off}}$, where M^{diag} is the diagonal part of M and $M^{\text{off}} = M - M^{\text{diag}}$ (hence M^{off} is off-diagonal). Since J is diagonal with distinct diagonal entries and P

is off-diagonal, (1.70) is divided into

$$\begin{aligned} &V_0^{\text{off}} = 0, \\ &V_{j,x}^{\text{diag}} = [P, V_j^{\text{off}}]^{\text{diag}} \qquad (0 \le j \le n), \\ &[J, V_{j+1}^{\text{off}}] = V_{j,x}^{\text{off}} - [P, V_j]^{\text{off}} \qquad (0 \le j \le n-1), \end{aligned} \tag{1.71}$$

and

$$P_t = V_{n,x}^{\text{off}} - [P, V_n]^{\text{off}}. \tag{1.72}$$

We can solve V_j $(j = 0, \cdots, n)$ from (1.71) by differentiation and integration. In fact, similar to the 2×2 case, V_j can be obtained by differentiation and integration for $n = 0, 1, 2, 3$. They are differential polynomials of the entries of P. For general n, it can be proved by induction that each entry of V_j is a differential polynomial of the entries of P whose coefficients may depend on t. (The proof is omitted here. See [111]). Therefore, (1.72) gives a system of partial differential equations of the entries of P. We shall write $V_j[P]$ for the V_j to specify the dependence on P.

Example 1.6 *Let* $n = 1$, $J = A = \text{diag}(a_1, \cdots, a_N)$, $V_0 = B = \text{diag}(b_1, \cdots, b_N)$ *with* $a_i \ne a_j$ *and* $b_i \ne b_j$ $(i \ne j)$*. Take* $V_1 = Q(x,t)$ *whose diagonal entries are all* 0*, then, from (1.71),*

$$Q_{ij} = \frac{b_i - b_j}{a_i - a_j} P_{ij} \quad (i \ne j), \tag{1.73}$$

and the equation (1.72) becomes

$$P_t = Q_x - [P, Q]^{\text{off}}. \tag{1.74}$$

Written in terms of the components, it becomes

$$P_{ij,t} = c_{ij} P_{ij,x} + \sum_{k \ne i,j} (c_{ik} - c_{kj}) P_{ik} P_{kj} \tag{1.75}$$

where

$$c_{ij} = \frac{b_i - b_j}{a_i - a_j}. \tag{1.76}$$

This is a system of nonlinear partial differential equations of P_{ij} $(i \ne j)$*, called the* N *wave equation.*

Similar to the discussion in Lemma 1.5, let $V_j^0[P]$ be the solution of (1.71) satisfying $V_0[0] = I$, $V_l[0] = 0$ $(1 \le l \le n)$, then the following lemma holds.

Lemma 1.5′. The general solution of (1.71) can be expressed as

$$V_k[P] = \sum_{j=0}^{k} \alpha_j V^0_{k-j}[P]. \tag{1.77}$$

where $\alpha_0, \cdots, \alpha_n$ are the corresponding integral constants of $\{V_j[P]\}$, which are diagonal matrices independent of x but may depend on t.

1.3 Darboux transformation

1.3.1 Darboux transformation for AKNS system

Let

$$F(u, u_x, u_t, u_{xx}, \cdots) = 0, \tag{1.78}$$

be a system of partial differential equations where u is a function or a vector valued function. Consider the AKNS system

$$\begin{aligned} \Phi_x &= U\Phi = (\lambda J + P)\Phi, \\ \Phi_t &= V\Phi = \sum_{j=0}^{n} V_j \lambda^{n-j}\Phi, \end{aligned} \tag{1.79}$$

where J, P, V_j satisfy the condition in the last section and P is a differential polynomial of u. Suppose (1.78) is equivalent to (1.68), the integrability condition of (1.79), then (1.79) is called the Lax pair of (1.78). In this case, (1.78) is the evolution equation (1.72). The non-degenerate $N \times N$ matrix solution of (1.79) is called a fundamental solution of the Lax pair.

In this section, we suppose that the (off-diagonal) entries of P are independent and (1.78) is the system of differential equations (1.72) in which all off-diagonal entries of P are unknown functions. This system is called unreduced.

We first discuss the Darboux transformation for the unreduced AKNS system.

Definition 1.7 *Suppose $D(x, t, \lambda)$ is an $N \times N$ matrix. If for given P and any solution Φ of (1.79), $\Phi' = D\Phi$ satisfies a linear system*

$$\begin{aligned} \Phi'_x &= U'\Phi' = (\lambda J + P')\Phi', \\ \Phi'_t &= V'\Phi' = \sum_{j=0}^{n} V'_j \lambda^{n-j}\Phi', \end{aligned} \tag{1.80}$$

where P' is an off-diagonal $N \times N$ matrix function, then the transformation $(P, \Phi) \to (P', \Phi')$ is called a Darboux transformation for the unreduced AKNS system, $D(x, t, \lambda)$ is called a Darboux matrix. A Darboux matrix is of degree k if it is a polynomial of λ of degree k.

According to this definition, P' satisfies the equation

$$P_t'^{\text{off}} - V_{n,x}'^{\text{off}} + [P', V_n']^{\text{off}} = 0, \qquad (1.72)'$$

where the entries of V_n' are differential polynomials of P'. Later, we will see that $V_n' = V_n[P']$ when the Darboux matrix is a polynomial of λ. In this case $(1.72)'$ and (1.72) are the same partial differential equations.

Substituting $\Phi' = D\Phi$ into (1.80), we get

$$\begin{aligned} U' &= DUD^{-1} + D_x D^{-1}, \\ V' &= DVD^{-1} + D_t D^{-1}. \end{aligned} \qquad (1.81)$$

Proposition 1 *If D is a Darboux matrix for (1.79) and D' is a Darboux matrix for (1.80), then $D'D$ is a Darboux matrix for (1.79).*

Proof. Since D' is a Darboux matrix for (1.80), there exists $U'' = \lambda J + P''$ (P'' is off-diagonal) and $V'' = \sum_{j=0}^{n} V_j'' \lambda^{n-j}$ such that $\Phi'' = D'\Phi' = D'D\Phi$ satisfies

$$\Phi_x'' = U''\Phi'', \qquad \Phi_t'' = V''\Phi''. \qquad (1.82)$$

Hence, by definition, $D'D$ is a Darboux matrix for (1.79).

Remark 7 Any constant diagonal matrix K independent of λ is a Darboux matrix of degree 0*, since under its action according to (1.81),*

$$\begin{aligned} &\lambda J + P \to \lambda J + KPK^{-1}, \\ &\sum_{j=0}^{n} V_j \lambda^{n-j} \to \sum_{j=0}^{n} KV_j K^{-1} \lambda^{n-j}. \end{aligned} \qquad (1.83)$$

If we do not consider the relations among the entries of P, this kind of Darboux matrices are trivial.

Now we first consider the Darboux matrix of degree one, which is linear in λ. Suppose it has the form $\lambda I - S$ where S an $N \times N$ matrix function, I is the identity matrix. According to Proposition 1 and Remark 7, the discussion on the Darboux matrix $K(\lambda I - S)$ (K is a non-degenerate constant matrix which must be diagonal in order to get the first equation of (1.80)) can be reduced to the discussion on the Darboux matrix $\lambda I - S$. Therefore, to construct the Darboux matrix, it is only necessary to construct S.

The differential equations of S are derived as follows. From the first equation of (1.80),

$$(\lambda J+P')(\lambda I-S)\Phi=((\lambda I-S)\Phi)_x=(\lambda I-S)(\lambda J+P)\Phi-S_x\Phi. \quad (1.84)$$

It must hold for any solution of (1.79). Comparing the coefficients of the powers of λ, we have

$$P'=P+[J,S], \quad (1.85)$$

This is the expression of P'.

The term independent of λ in (1.84) gives

$$S_x=P'S-SP=PS-SP+JS^2-SJS, \quad (1.86)$$

i.e.,

$$S_x+[S,JS+P]=0. \quad (1.87)$$

This is the first equation which S satisfies.

The second equation of (1.80) leads to

$$\begin{aligned}\sum_{j=0}^{n}V_j'\lambda^{n-j}(\lambda I-S)\Phi &= ((\lambda I-S)\Phi)_t\\ = \quad & (\lambda I-S)\sum_{j=0}^{n}V_j\lambda^{n-j}\Phi-S_t\Phi. \end{aligned} \quad (1.88)$$

Comparing the coefficients of λ^{n+1}, λ^n, $\cdots$, λ, we can determine $\{V_j'\}$ recursively by

$$V_0'=V_0, \qquad V_{j+1}'=V_{j+1}+V_j'S-SV_j, \quad (1.89)$$

and get the second equation of S

$$S_t=V_n'S-SV_n. \quad (1.90)$$

From (1.89) V_j's can be expressed as

$$\begin{aligned} V_0' &= V_0,\\ V_j' &= V_j+\sum_{k=1}^{j}[V_{j-k},S]S^{k-1} \qquad (1\le j\le n), \end{aligned} \quad (1.91)$$

and (1.90) becomes

$$S_t+[S,\sum_{j=0}^{n}V_jS^{n-j}]=0. \quad (1.92)$$

THEOREM 1.8 *$\lambda I - S$ is a Darboux matrix for (1.79) if and only if S satisfies*

$$\begin{aligned} &S_x + [S, JS + P] = 0, \\ &S_t + [S, \sum_{j=0}^{n} V_j S^{n-j}] = 0. \end{aligned} \tag{1.93}$$

Moreover, under the action of the Darboux matrix $\lambda I - S$, $P' = P + [J, S]$.

Proof. Suppose $\lambda I - S$ is a Darboux matrix, then (1.93) is just (1.87) and (1.92) derived above. Conversely, if (1.87) and (1.92) hold, then for any solution Φ of (1.79), there are the relations (1.84) and (1.88). Hence (1.80) holds for the P' determined by (1.85) and the $\{V_j'\}$ determined by (1.89), which means that $\lambda I - S$ is a Darboux matrix.

This theorem implies that we need to solve S from the system of nonlinear partial differential equations (1.93) to get the Darboux matrix. Fortunately, most of the solutions of (1.93) can be constructed explicitly. The following theorem gives the explicit construction of the Darboux matrix of degree one.

Suppose P is a solution of (1.72). Take complex numbers $\lambda_1, \cdots, \lambda_N$ such that they are not all the same. Let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$. Let h_i be a column solution of (1.79) for $\lambda = \lambda_i$. $H = (h_1, \cdots, h_N)$. When $\det H \neq 0$, let

$$S = H \Lambda H^{-1}, \tag{1.94}$$

then we have the following theorem.

THEOREM 1.9 *The matrix $\lambda I - S$ defined by (1.94) is a Darboux matrix for (1.79).*

Proof. h_i is a solution of (1.79) for $\lambda = \lambda_i$, that is, it satisfies

$$\begin{aligned} &h_{i,x} = \lambda_i J h_i + P h_i, \\ &h_{i,t} = \sum_{j=0}^{n} V_j \lambda^{n-j} h_i. \end{aligned} \tag{1.95}$$

By taking the derivatives of $H = (h_1, h_2, \cdots, h_N)$ with respect to x and t, (1.95) is equivalent to

$$\begin{aligned} &H_x = J H \Lambda + P H, \\ &H_t = \sum_{j=0}^{n} V_j H \Lambda^{n-j}. \end{aligned} \tag{1.96}$$

Hence

$$\begin{aligned} S_x &= H_x \Lambda H^{-1} - H\Lambda H^{-1} H_x H^{-1} = [H_x H^{-1}, S] \\ &= [JS + P, S] \\ S_t &= H_t \Lambda H^{-1} - H\Lambda H^{-1} H_t H^{-1} = [H_t H^{-1}, S] \\ &= [\sum_{j=0}^{n} V_j S^{n-j}, S]. \end{aligned} \tag{1.97}$$

Therefore, the matrix S defined by (1.94) is a solution of (1.93). Theorem 1.8 implies that $\lambda I - S$ is a Darboux matrix for (1.79). The theorem is proved.

THEOREM 1.10 *For any given (x_0, t_0) and the matrix S_0, (1.93) has a solution satisfying $S(x_0, t_0) = S_0$. That is, the system is integrable.*

Proof. First suppose the Jordan form of S_0 is a diagonal matrix. Suppose its eigenvalues are $\lambda_1, \cdots, \lambda_N$ and the corresponding eigenvectors are h_{0i}. Let h_i be a solution of (1.95) satisfying $h_i(x_0, t_0) = h_{0i}$. Then these h_i are linearly independent in a neighborhood of (x_0, t_0). Theorem 1.9 implies that the Darboux matrix exists, i.e., (1.93) has a solution. If the Jordan form of S_0 is not diagonal, then there is a series of matrices $\{S_0^{(k)}\}$ such that the Jordan form of $S_0^{(k)}$ is diagonal and $S_0^{(k)} \to S_0$ as $k \to \infty$. Construct S according to (1.94) with initial value $S_0^{(k)}$, then $S^{(k)}$ solves (1.93). The smooth dependency of the solution of (1.93) to the initial value implies that $S^{(k)} \to S$ and $S_x^{(k)}$, $S_t^{(k)}$ converge in a neighborhood of (x_0, t_0). Therefore S is a solution of (1.93) with initial value S_0. Thus (1.93) is solvable for any given initial value, which means that it is integrable. The theorem is proved.

We can also prove this theorem by direct but tedious calculation.

This theorem implies that a Darboux matrix of degree one can be obtained either by (1.94) or the limit of such Darboux matrices.

Remark 8 h_i can be expressed as $h_i = \Phi(\lambda_i) l_i$ $(i = 1, 2, \cdots, N)$, where l_1, l_2, $\cdots$, l_N are N linearly independent constant column matrices. Hence H in (1.94) can be written as

$$H = (\Phi(\lambda_1) l_1, \Phi(\lambda_2) l_2, \cdots, \Phi(\lambda_N) l_N)\,. \tag{1.98}$$

This construction of Darboux matrix was given by [33, 94]. Theorem 1.9 and 1.10 implies that (1.94) contains all the matrices S which are similar to diagonal matrices and $\lambda I - S$ are Darboux matrices of degree one. A Darboux matrix expressed by (1.94) is called a diagonalizable Darboux matrix or Darboux matrix with explicit expressions. It is useful in constructing the solutions because it is expressed explicitly. Hereafter, we mostly use the diagonalizable Darboux matrices and the word "diagonalizable" is omitted.

The "single soliton solution" can be obtained by the Darboux transformation from the seed solution $P = 0$.

For $P = 0$, the fundamental solution of (1.67) is $\Phi = \mathrm{e}^{\lambda Jx+\Omega(\lambda,t)}$ where $\Omega(t) = \int \sum_{j=0}^{n} V_j[0](t)\lambda^{n-j}\,dt$ is a diagonal matrix. For any constants $\lambda_1, \cdots, \lambda_N$ and column matrices $l_1, \cdots, l_N$, let

$$H = \left(\mathrm{e}^{\lambda_1 Jx+\Omega(\lambda_1,t)}l_1, \cdots, \mathrm{e}^{\lambda_N Jx+\Omega(\lambda_N,t)}l_N\right), \tag{1.99}$$

then

$$P' = [J, H\Lambda H^{-1}] \tag{1.100}$$

is a solution of (1.72) and the fundamental solution of the Lax pair (1.80) is $\Phi' = (\lambda I - H\Lambda H^{-1})\Phi$.

Remark 9 If $V_j[0]$ depends on t, then $\Omega(\lambda_i, t)$ is not a linear function of t, hence the velocities of the solitons are not constants [45].

The double soliton solution can be obtained from P' by applying further Darboux transformation. Since Φ' is known in this process, P'' and Φ'' can be obtained by a purely algebraic algorithm. The multi-soliton solutions are obtained similarly.

For general AKNS system, $\det H \neq 0$ may not hold for all (x, t). Therefore, the solutions given by Darboux transformations may not be regular for all (x, t).

1.3.2 Invariance of equations under Darboux transformations

We have known that (P', V_j') and (P, V_j) satisfy the same recursion relations (1.71) and (1.72) holds true for the two sets of functions. V_j' is a differential polynomial of P' which is expressed by a similar equality as (1.77), i.e.,

$$V_k'[P'] = \sum_{j=0}^{k} \alpha_j'(t) V_{k-j}^0[P']. \tag{1.101}$$

Here we prove that actually the coefficients $\alpha_0'(t), \cdots, \alpha_n'(t)$ are the same as $\alpha_0(t), \cdots, \alpha_n(t)$ respectively. Therefore P' and P satisfy the same evolution equation (1.72).

THEOREM 1.11 *Suppose V_j's are differential polynomials of P satisfying (1.71). S is a matrix satisfying (1.87), V_i's are defined by (1.89) and $P' = P + [J, S]$. Then V_i's are differential polynomials of P' and*

$$V'_j[P'] = V_j[P'] \qquad (j = 1, 2, \cdots, n). \tag{1.102}$$

Proof. At first we see that the equation $P' = P + [J, S]$ is equivalent to $P = P' - [J, S]$ and the equation (1.87) is equivalent to

$$S_x = [P' + SJ, S]. \tag{1.103}$$

Moreover, for arbitrary x-function P'. This equation admits solutions in a neighborhood around any given point $x = x_0$. Thus we may consider P' as an arbitrary off-diagonal matrix-valued function of x.

From (1.89) we have

$$\begin{aligned}&[J, V'_{j+1}] - V'_{j,x} + [P', V'_j] \\ &= [J, V_{j+1}] - V_{j,x} + [P, V_j] + ([J, V'_j] - V'_{j-1,x} + [P', V'_{j-1}])S \\ &\quad -S([J, V_j] - V_{j-1,x} + [P, V_{j-1}]) \quad (j = 0, \cdots, n-1).\end{aligned} \tag{1.104}$$

Using induction, we know that

$$[J, V'_{j+1}] - V'_{j,x} + [P', V'_j] = 0 \quad (j = 0, \cdots, n-1) \tag{1.105}$$

from the equations (1.71) for $V_j[P]$. Moreover, we can prove

$$V'^{\text{diag}}_{n,x} - [P', V'_n]^{\text{diag}} = 0. \tag{1.106}$$

This means that V'_j and P' also satisfy (1.71). Therefore, as mentioned above, V'_j can be expressed as a differential polynomial of P': $V'_j = V'_j[P']$.

Let

$$\Delta_j[P'] = V'_j[P'] - V_j[P'], \tag{1.107}$$

(1.89) implies $\Delta_0 = 0$. Suppose $\Delta_k = 0$, then (1.71) leads to

$$[J, \Delta_{k+1}] = \Delta^{\text{off}}_{k,x} - [P', \Delta_k]^{\text{off}} = 0, \tag{1.108}$$

hence $\Delta^{\text{off}}_{k+1} = 0$. From (1.71),

$$\Delta^{\text{diag}}_{k+1,x} = [P', \Delta_{k+1}]^{\text{off}} = 0, \tag{1.109}$$

which means that $\Delta^{\text{diag}}_{k+1}[P']$ is independent of x. We should prove that $\Delta^{\text{diag}}_{k+1}[P']$ is independent of P'. Denote P'_{ij} be the entries of P', $P'^{(\alpha)}_{ij}$ be the αth derivative of P'_{ij} with respect to x. Suppose the order of the highest derivatives of P' in $\Delta_{k+1}^{\text{diag}}$ is r, then

$$0 = \frac{\partial \Delta_{k+1}^{\text{diag}}}{\partial x} = \sum_{i,j} \sum_{\alpha=0}^{r} \frac{\partial \Delta_{k+1}^{\text{diag}}}{\partial P_{ij}^{\prime(\alpha)}} P_{ij}^{\prime(\alpha+1)}. \tag{1.110}$$

In this equation, the coefficient of $P_{ij}^{\prime(r+1)}$ should be 0. Hence $\Delta_{k+1}^{\text{diag}}$ does not contain the rth derivative of P', which means that it is independent of P'. Especially, let $S = 0$, $P = P'$, then (1.89) implies $\Delta_{k+1}^{\text{diag}} = 0$. Thus (1.102) is proved

Theorem 1.11 implies that for the evolution equations (1.72) in the AKNS system, the Darboux transformation transforms a solution of an equation to a new solution of the same equation.

Note that the Darboux transformation

$$(P, \Phi) \longrightarrow (P', \Phi') \tag{1.111}$$

defined by

$$P' = P + [J, S], \quad \Phi' = (\lambda I - S)\Phi \tag{1.112}$$

can be taken successively in a purely algebraic algorithm and leads to an infinite series of solutions of the AKNS system:

$$(P, \Phi) \longrightarrow (P', \Phi') \longrightarrow (P'', \Phi'') \longrightarrow \cdots \tag{1.113}$$

1.3.3 Darboux transformations of higher degree and the theorem of permutability

The Darboux matrices discussed above are all of degree one. In this subsection, we construct Darboux matrices with explicit expressions which are the polynomials of λ of degree > 1. Then we derive the theorem of permutability from the Darboux matrices of degree two.

Clearly, the composition of r Darboux transformations of degree one gives a Darboux transformation of degree r. On the other hand, we can also construct the Darboux transformations of degree r directly.

As known above, a Darboux matrix of degree one is $D(x, t, \lambda) = \lambda I - S$ where S is given by (1.94) if it can be diagonalized (Theorem 1.9). Then, $SH = H\Lambda$ is equivalent to $D(x, t, \lambda_i)h_i = 0$ where h_i is a column solution of the Lax pair for $\lambda = \lambda_i$ such that $\det H = \det(h_1, \cdots, h_N) \neq 0$. This fact can be generalized to the Darboux matrix of degree r, that is, we can consider an $N \times N$ Darboux matrix in the form

$$D(x, t, \lambda) = \sum_{j=0}^{r} D_{r-j}(x, t)\lambda^j, \quad D_0 = I. \tag{1.114}$$

Take Nr complex numbers λ_1, λ_2, $\cdots$, λ_{Nr} and the column solution h_i of the Lax pair for $\lambda = \lambda_i$ $(i = 1, \cdots, Nr)$. Let

$$F_r = \begin{pmatrix} h_1 & h_2 & \cdots & h_{Nr} \\ \lambda_1 h_1 & \lambda_2 h_2 & \cdots & \lambda_{Nr} h_{Nr} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{r-1} h_1 & \lambda_2^{r-1} h_2 & \cdots & \lambda_{Nr}^{r-1} h_{Nr} \end{pmatrix} \tag{1.115}$$

which is an $Nr \times Nr$ matrix. The system $D(x,t,\lambda_i)h_i = 0$ is equivalent to

$$\sum_{j=0}^{r-1} D_{r-j}(x,t)\lambda_i^j h_i = -\lambda_i^r h_i \qquad (i = 1, \cdots, Nr) \tag{1.116}$$

and can be written as

$$(D_r, D_{r-1}, \cdots, D_1)F_r = -(\lambda_1^r h_1, \cdots, \lambda_{Nr}^r h_{Nr}). \tag{1.117}$$

This is a system of linear algebraic equations for $(D_r, D_{r-1}, \cdots, D_1)$. When $\det F_r \neq 0$, it has a unique solution $(D_r, D_{r-1}, \cdots, D_1)$. Therefore, when $\det F_r \neq 0$, there exists a unique $N \times N$ matrix $D(x,t,\lambda)$ satisfying $D(x,t,\lambda_i)h_i = 0$ $(i = 1, \cdots, Nr)$. We write it as $D(h_1, \cdots, h_{Nr}, \lambda)$ to indicate that D is constructed from $h_1, \cdots, h_{Nr}$.

The next theorem shows that it is a Darboux matrix and decomposable as a product of two Darboux matrices of lower degree [52, 74].

THEOREM 1.12 *Given Nr complex numbers $\lambda_1, \cdots, \lambda_{Nr}$. Let h_i be a column solution of the Lax pair (1.79) for $\lambda = \lambda_i$ $(i = 1, \cdots, Nr)$, F_r be defined by (1.115). Suppose* $\det F_r \neq 0$, *then the following conclusions hold.*

(1) There exists a unique matrix $D(h_1, \cdots, h_{Nr}, \lambda)$ in the form (1.114) such that

$$D(h_1, \cdots, h_{Nr}, \lambda_i)h_i = 0 \quad (i = 1, 2, \cdots, Nr). \tag{1.118}$$

In this case, $D(h_1, \cdots, h_{Nr}, \lambda)$ is a Darboux matrix of degree r for (1.79).

(2) If $\det F_{r-1} \neq 0$, *then the above Darboux matrix of degree r can be decomposed as*

$$\begin{aligned} & D(h_1, \cdots, h_{Nr}, \lambda) \\ = \; & D\Big(D(h_1, \cdots, h_{N(r-1)}, \lambda_{N(r-1)+1})h_{N(r-1)+1}, \cdots, \\ & D(h_1, \cdots, h_{N(r-1)}, \lambda_{Nr})h_{Nr}, \lambda\Big) \cdot \\ & \cdot D(h_1, \cdots, h_{N(r-1)}, \lambda). \end{aligned} \tag{1.119}$$

On the right hand side of this equality, the first term is a Darboux matrix of degree one and the second term is a Darboux matrix of degree $(r-1)$*.*

(3) The Darboux matrix $D(h_1,\cdots,h_{Nr},\lambda)$ *of degree* r *can be decomposed into the product of* r *Darboux matrices of degree one.*

(4) $\widetilde{P}=P-[J,D_1]$ *is a solution of (1.72).*

Proof. We first prove (2). Let

$$\Lambda_k = \operatorname{diag}(\lambda_{N(k-1)+1},\cdots,\lambda_{Nk}), \tag{1.120}$$

$$H_k = (h_{N(k-1)+1},\cdots,h_{Nk}), \tag{1.121}$$

then

$$F_r = \begin{pmatrix} H_1 & H_2 & \cdots & H_r \\ H_1\Lambda_1 & H_2\Lambda_2 & \cdots & H_r\Lambda_r \\ \vdots & \vdots & \ddots & \vdots \\ H_1\Lambda_1^{r-1} & H_2\Lambda_2^{r-1} & \cdots & H_r\Lambda_r^{r-1} \end{pmatrix}. \tag{1.122}$$

Since

$$F_{r-1} = \begin{pmatrix} H_1 & H_2 & \cdots & H_{r-1} \\ H_1\Lambda_1 & H_2\Lambda_2 & \cdots & H_{r-1}\Lambda_{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ H_1\Lambda_1^{r-2} & H_2\Lambda_2^{r-2} & \cdots & H_{r-1}\Lambda_{r-1}^{r-2} \end{pmatrix} \tag{1.123}$$

is non-degenerate, there is a matrix $D(h_1,\cdots,h_{N(r-1)},\lambda)$ of degree $(r-1)$ with respect to λ such that

$$D(h_1,\cdots,h_{N(r-1)},\lambda_i)h_i = 0 \qquad (i=1,2,\cdots,N(r-1)). \tag{1.124}$$

Let

$$h_i' = D(h_1,\cdots,h_{N(r-1)},\lambda_i)h_i \qquad (i=N(r-1)+1,\cdots,Nr). \tag{1.125}$$

Construct a Darboux matrix $D(h'_{N(r-1)+1},\cdots,h'_{Nr},\lambda)$ from h'_i and let

$$D'(\lambda) = D(h'_{N(r-1)+1},\cdots,h'_{Nr},\lambda)D(h_1,\cdots,h_{N(r-1)},\lambda), \tag{1.126}$$

then $D'(\lambda_i)h_i=0$ $(i=1,2,\cdots,N(r-1))$. Moreover, for $i=N(r-1)+1,\cdots,Nr$,

$$D'(\lambda_i)h_i = D(h'_{N(r-1)+1},\cdots,h'_{Nr},\lambda_i)h'_i = 0. \tag{1.127}$$

Hence $D'(\lambda)=D(\lambda)$.

Thus, we have decomposed the matrix of degree r determined by (1.116) to the product of a matrix of degree $(r-1)$ and a matrix of degree one, expressed by (1.119). This proves (2).

For $D(h_1, \cdots, h_{N(r-1)}, \lambda)$, if all the determinants of F_{r-2}, F_{r-3}, $\cdots$ are non-zero, then $D(h_1, \cdots, h_N, \lambda)$ can be decomposed to r matrices of degree one by repeating the above procedure. For $r = 1$, $D(h_1, \cdots, h_N, \lambda)$ is a Darboux matrix. Hence $D(h_1, \cdots, h_{Nr}, \lambda)$ is also a Darboux matrix and it can be decomposed to the product of r Darboux matrices of degree one:

$$D = (\lambda I - S_r) \cdots (\lambda I - S_1). \tag{1.128}$$

Since $\det F_r \neq 0$, we can always permute the subscripts of Λ_i and H_i so that all the determinants of F_{r-2}, F_{r-3}, $\cdots$ are non-zero. (3) is proved.

Since D is the product of r Darboux matrices of degree one, D itself is a Darboux matrix. Hence (1) holds.

Note that

$$D_1 = -(S_1 + \cdots + S_r). \tag{1.129}$$

After the transformation of $\lambda I - S_1$, $P \to P' = P + [J, S_1]$. Then after the transformation of $\lambda I - S_2$, $P' \to P'' = P' + [J, S_2]$, and so on. Hence, after the transformation of D,

$$P \to P + [J, S_1 + \cdots + S_r] = P - [J, D_1]. \tag{1.130}$$

Therefore, $P - [J, D_1]$ is a solution of (1.72). (4) is proved. This proves the lemma.

Darboux transformation has an important property — the theorem of permutability. This theorem originated from the Bäcklund transformation of the sine-Gordon equation and there are a lot of generalizations and various proofs. The proof here is given by [52] (2×2 case) and [33] ($N \times N$ case). This proof does not depend on any boundary conditions and the permutation of the parameters is expressed definitely.

From the solution $(P, \Phi(\lambda))$, we can construct the Darboux transformation with parameters $\lambda_1^{(1)}$, $\cdots$, $\lambda_N^{(1)}$ and the solutions $h_i^{(1)} = \Phi(\lambda_i) l_i^{(1)}$ of the Lax pair. Then the solution $(P^{(1)}, \Phi^{(1)}(\lambda))$ is obtained. Here $l_i^{(1)}$'s are N constant vectors. Next, construct a Darboux matrix for $(P^{(1)}$, $\Phi^{(1)}(\lambda))$ with parameters $\lambda_1^{(2)}$, $\cdots$, $\lambda_N^{(2)}$ and $l_i^{(2)}$ to get $(P^{(1,2)}, \Phi^{(1,2)}(\lambda))$. On the other hand, construct the Darboux transformation for $(P, \Phi(\lambda))$ with parameters $\lambda_i^{(2)}$ and $l_i^{(2)}$ to get $(P^{(2)}, \Phi^{(2)}(\lambda))$. Then construct the Darboux transformation with parameters $\lambda_i^{(1)}$ and $l_i^{(1)}$ to get $(P^{(2,1)}$, $\Phi^{(2,1)}(\lambda))$. The following theorem holds.

THEOREM 1.13 *(Theorem of permutability) Suppose*

$$\det \begin{pmatrix} H_1 & H_2 \\ H_1\Lambda_1 & H_2\Lambda_2 \end{pmatrix} \neq 0, \tag{1.131}$$

then

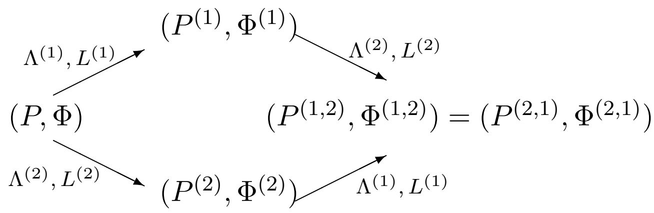
$$(P^{(1,2)}, \Phi^{(1,2)}(\lambda)) = (P^{(2,1)}, \Phi^{(2,1)}(\lambda)). \tag{1.132}$$

Proof. Theorem 1.12 implies that $\Phi^{(1,2)}(\lambda)$ and $\Phi^{(2,1)}(\lambda)$ are both obtained from $\Phi(\lambda)$ by the action of the Darboux transformation of degree two, and are expressed by

$$\begin{aligned} \Phi^{(1,2)}(\lambda) &= D(h_1^{(1)}, \cdots, h_N^{(1)}, h_1^{(2)}, \cdots, h_N^{(2)}, \lambda), \\ \phi^{(2,1)}(\lambda) &= D(h_1^{(2)}, \cdots, h_N^{(2)}, h_1^{(1)}, \cdots, h_N^{(1)}, \lambda). \end{aligned} \tag{1.133}$$

From (1) of Theorem 1.12, we know that the right hand side of the above equations are equal. Hence the theorem of permutability holds.

The theorem of permutability can be expressed by the following Bianchi diagram:



(1.134)

Here $L^{(1)}$ and $L^{(2)}$ denote the sets $\{l_i^{(1)}\}$ and $\{l_i^{(2)}\}$ respectively.

Remark 10 The Darboux transformation of higher degree is much more complicated than the Darboux transformation of degree one. The theorem of decomposition implies that Darboux transformations of degree one can generate Darboux transformations of higher degree. Therefore, we can use Darboux transformations of degree one successively instead of a Darboux transformation of higher degree so as to avoid the calculation of the determinant of a matrix of very high order (of order Nr). Since the algorithm for the Darboux transformation of degree one is purely algebraic and independent of the seed solution P, it is quite convenient to calculate the solutions using symbolic calculation with computer. However, some solutions, e.g., multi-solitons can be expressed by an explicit formulae by using Darboux transformations of higher degree [80].

Remark 11 The proof for Theorem 1.13 is for the Darboux transformations with explicit expressions. Since any Darboux transformation without explicit expression is a limit of Darboux transformations with explicit expressions, the theorem of permutability also holds for the Darboux transformations without explicit expressions.

Now we compute the more explicit expression of the Darboux matrix of degree two. Suppose it is constructed from (Λ_1, H_1) and (Λ_2, H_2) which satisfy (1.131). Let $S_j = H_j \Lambda_j H_j^{-1}$ and denote

$$\Lambda_\alpha = \mathrm{diag}(\lambda_1^{(\alpha)}, \cdots, \lambda_N^{(\alpha)}), \quad H_\alpha = (h_1^{(\alpha)}, \cdots, h_N^{(\alpha)}).$$

After the action of $\lambda I - S_1$, $h_j^{(2)}$ is transformed to $(\lambda_j^{(2)} I - S_1) h_j^{(2)}$. Hence H_2 is transformed to $\widetilde{H}_2 = H_2 \Lambda_2 - S_1 H_2 = (S_2 - S_1) H_2$. The second Darboux matrix of degree one is $\lambda I - \widetilde{S}_2$ where

$$\widetilde{S}_2 = \widetilde{H}_2 \Lambda_2 \widetilde{H}_2^{-1} = (S_2 - S_1) S_2 (S_2 - S_1)^{-1}. \tag{1.135}$$

According to (1.131), $S_2 - S_1$ is non-degenerate. The Darboux matrix of degree two is

$$\begin{aligned} D(\lambda) &= (\lambda I - \widetilde{S}_2)(\lambda I - S_1) \\ &= \lambda^2 I - \lambda (S_2^2 - S_1^2)(S_2 - S_1)^{-1} + (S_2 - S_1) S_2 (S_2 - S_1)^{-1} S_1. \end{aligned} \tag{1.136}$$

It is easy to check that $D(\lambda)$ is symmetric to S_1 and S_2. Therefore, we can also obtain the theorem of permutability by this symmetry.

1.3.4 More results on the Darboux matrices of degree one

In this subsection, we show that the Darboux matrix method in Theorem 1.9 can be applied not only to the AKNS system, but also to many other evolution equations, especially to the Lax pairs whose U and V are polynomials of λ. On the other hand, we also show that those Darboux matrices include all the diagonalizable Darboux matrices of the form $\lambda I - S$, and any non-diagonalizable Darboux matrix can be obtained from the limit of diagonalizable Darboux matrices.

We generalize the Lax pair (1.79) to

$$\begin{aligned} \Phi_x &= U \Phi, \\ \Phi_t &= V \Phi, \end{aligned} \tag{1.137}$$

where U and V are polynomials of the spectral parameter λ:

$$\begin{aligned} U(x,t,\lambda) &= \sum_{j=0}^{m} U_j(x,t)\lambda^{m-j}, \\ V(x,t,\lambda) &= \sum_{j=0}^{n} V_j(x,t)\lambda^{n-j}, \end{aligned} \tag{1.138}$$

U_j's and V_j's are $N \times N$ matrices.

Clearly, the integrability condition of (1.137) is

$$U_t - V_x + [U, V] = 0. \tag{1.139}$$

In this subsection, we still discuss the Darboux matrices for the Lax pairs without reductions. That is, we suppose that all the entries of U_j's and V_j's are independent except for the partial differential equations (1.139). This is to say that apart from the integrability condition (1.139), there is no other constraint. Therefore, the nonlinear partial differential equation to be studied is just (1.139), i.e., the equations given by the coefficients of each power of λ in (1.139) and the unknowns are the $N \times N$ matrices U_j and V_j $(j = 0, 1, \cdots, n)$. Compared with Subsection 1.3.1, $D = \lambda I - S$ is a Darboux matrix if and only if there exist

$$\begin{aligned} U'(x,t,\lambda) &= \sum_{j=0}^{m} U'_j(x,t)\lambda^{m-j}, \\ V'(x,t,\lambda) &= \sum_{j=0}^{n} V'_j(x,t)\lambda^{n-j} \end{aligned} \tag{1.140}$$

such that $\Phi' = (\lambda I - S)\Phi$ satisfies

$$\Phi'_x = U'\Phi', \qquad \Phi'_t = V'\Phi' \tag{1.141}$$

where Φ is a fundamental solution of (1.137).

Clearly, U' and V' have the expressions

$$\begin{aligned} U' &= DUD^{-1} + D_x D^{-1}, \\ V' &= DVD^{-1} + D_t D^{-1} \end{aligned} \tag{1.142}$$

and they satisfy

$$U'_t - V'_x + [U', V'] = 0. \tag{1.143}$$

The remaining problem is to obtain S so that (1.141) holds. If S is obtained, we have the Darboux transformation

$$(U, V, \Phi) \to (U', V', \Phi'). \tag{1.144}$$

Comparing to Theorem 1.8, 1.9 and 1.10, we have

Theorem 1.8′. $\lambda I - S$ is a Darboux matrix of degree one for (1.137) if and only if S satisfies

$$S_x + [S, U(S)] = 0, \qquad S_t + [S, V(S)] = 0. \tag{1.145}$$

Here

$$U(S) = \sum_{j=0}^{m} U_j S^{m-j}, \qquad V(S) = \sum_{j=0}^{n} V_j S^{n-j}. \tag{1.146}$$

Now suppose (U, V) satisfies the integrability condition (1.139). For given constant diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, let h_i be a column solution of (1.137) for $\lambda = \lambda_i$, $H = (h_1, \cdots, h_N)$. If $\det H \neq 0$, let $S = H\Lambda H^{-1}$, then the following theorems holds.

Theorem 1.9′. The matrix $\lambda I - S$ is a Darboux matrix for (1.137).

Theorem 1.10′. The system (1.145) is integrable.

The proofs are omitted since they are similar to the proofs for the corresponding theorems above.

Note that for the AKNS system, we can solve $V_i[P]$'s from a system of differential equations by choosing "integral constants" and these $V_i[P]$'s are differential polynomials of P. The remaining equation is only the equation (1.72) for P. In the present case, all the entries of U_i and V_i are regarded as independent unknowns satisfying the partial differential equations (1.139).

The inverse of Theorem 1.9′ also holds.

THEOREM 1.14 *(1) If $\lambda I - S$ is a Darboux matrix for (1.137) and S is diagonalized at one point, then there exists a constant diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ and column solutions h_i's of the Lax pair (1.137) for $\lambda = \lambda_i$ $(i = 1, 2, \cdots, N)$ such that $H = (h_1, \cdots, h_N)$ and $S = H\Lambda H^{-1}$.*

(2) If $\lambda I - S$ is a Darboux matrix for (1.137) but it can not be diagonalized at any points, then there exist a series of Darboux matrices $\lambda I - S_k$ such that S_k's and their derivatives with respect to x and t converge to S and its derivatives respectively.

The proof is similar to that for Theorem 1.10.

EXAMPLE 1.15 *An example of a Darboux matrix which is not diagonalizable everywhere.*

Consider the Lax pair

$$\begin{aligned}\Phi_x &= \begin{pmatrix} \lambda & p \\ q & -\lambda \end{pmatrix} \Phi, \\ \Phi_t &= \begin{pmatrix} -2\mathrm{i}\lambda^2 + \mathrm{i}pq & -2\mathrm{i}\lambda p - \mathrm{i}p_x \\ -2\mathrm{i}\lambda q + \mathrm{i}q_x & 2\mathrm{i}\lambda^2 - \mathrm{i}pq \end{pmatrix} \Phi\end{aligned} \tag{1.147}$$

whose integrability condition leads to the nonlinear evolution equations

$$\mathrm{i}p_t = p_{xx} - 2p^2 q, \qquad -\mathrm{i}q_t = q_{xx} - 2pq^2. \tag{1.148}$$

This system of equations has a solution

$$p = \alpha\ \mathrm{sech}(\alpha x) e^{-\mathrm{i}\alpha^2 t}, \qquad q = -\alpha\ \mathrm{sech}(\alpha x) e^{\mathrm{i}\alpha^2 t}, \tag{1.149}$$

which is derived from the trivial solution $p = q = 0$ by the Darboux matrix $D = \lambda I - H\Lambda H^{-1}$ with

$$\begin{aligned}\Lambda &= \frac{\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ H &= \begin{pmatrix} e^{-\mathrm{i}\alpha^2 t/2} & 0 \\ 0 & e^{\mathrm{i}\alpha^2 t/2} \end{pmatrix} \begin{pmatrix} e^{\alpha x/2} & -e^{-\alpha x/2} \\ e^{-\alpha x/2} & e^{\alpha x/2} \end{pmatrix}.\end{aligned} \tag{1.150}$$

Now we take (1.149) as a seed solution, whose corresponding fundamental solution of the Lax pair (1.147) is

$$(\lambda I - H\Lambda H^{-1}) \begin{pmatrix} e^{\lambda x - 2\mathrm{i}\lambda^2 t} & 0 \\ 0 & e^{-\lambda x + 2\mathrm{i}\lambda^2 t} \end{pmatrix}. \tag{1.151}$$

Take

$$\Lambda^{(\epsilon)} = \begin{pmatrix} \epsilon & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.152}$$

then we can choose

$$H^{(\epsilon)} = \begin{pmatrix} h_{11}^{(\epsilon)} & h_{12}^{(\epsilon)} \\ h_{21}^{(\epsilon)} & h_{22}^{(\epsilon)} \end{pmatrix}, \tag{1.153}$$

where

$$\begin{aligned} h_{11}^{(\epsilon)} &= \Big(\frac{2\epsilon^2}{\alpha} - \epsilon\tanh(\alpha x)\Big)e^{\theta} - \frac{\alpha}{2}\,\mathrm{sech}(\alpha x)e^{-\mathrm{i}\alpha^2 t-\theta},\\ h_{12}^{(\epsilon)} &= -\epsilon\tanh(\alpha x) + \mathrm{sech}(\alpha x)e^{-\mathrm{i}\alpha^2 t},\\ h_{21}^{(\epsilon)} &= -\epsilon\,\mathrm{sech}(\alpha x)e^{\mathrm{i}\alpha^2 t+\theta} + \Big(\epsilon + \frac{\alpha}{2}\tanh(\alpha x)\Big)e^{-\theta},\\ h_{22}^{(\epsilon)} &= -\epsilon\,\mathrm{sech}(\alpha x)e^{\mathrm{i}\alpha^2 t} - \tanh(\alpha x),\\ \theta &= \epsilon x - 2\mathrm{i}\epsilon^2 t. \end{aligned} \tag{1.154}$$

When $\epsilon \to 0$,

$$H^{(\epsilon)}\Lambda^{(\epsilon)}(H^{(\epsilon)})^{-1} \to S, \tag{1.155}$$

$$S = \frac{\alpha}{\Delta}\begin{pmatrix} \sinh(\alpha x)e^{-\mathrm{i}\alpha^2 t} & e^{-2\mathrm{i}\alpha^2 t}\\ -\sinh^2(\alpha x) & -\sinh(\alpha x)e^{-\mathrm{i}\alpha^2 t}\end{pmatrix}, \tag{1.156}$$

$$p' = \frac{2\alpha e^{-2\mathrm{i}\alpha^2 t}}{\Delta}, \qquad q' = \frac{2\alpha\sinh^2(\alpha x)}{\Delta} \tag{1.157}$$

where

$$\Delta = (\alpha + 2 - 2\,\mathrm{sech}(\alpha x)e^{-\mathrm{i}\alpha^2 t})\cosh^2(\alpha x). \tag{1.158}$$

Note that both eigenvalues of S are zero, but $S \neq 0$. Hence S is not diagonalizable. However, from the construction of S, we know that $\lambda I - S$ is a Darboux matrix, i.e., it satisfies (1.145).

Finally, the conclusions for the Darboux transformations of higher degree and the theorem of permutability in Subsection 1.3.3 also hold for the general Lax pair (1.137). Moreover, when U and V in (1.137) are generalized to rational functions of λ, similar conclusions hold [121].

1.4 KdV hierarchy, MKdV-SG hierarchy, NLS hierarchy and AKNS system with $u(N)$ reduction

In the last section, we discussed the Darboux transformations for the AKNS system and more general systems. In those cases, we supposed that there were no reductions. In particular, there were no restrictions among the off-diagonal entries of P. However, in many cases, there are constraints on P and the Darboux transformation should keep those constraints. This problem is solved in many cases. Nevertheless, it should be very interesting to establish a systematic method to treat with reduced problems.

In this section, we first discuss some equations when $N = 2$ and there is certain relation between p and q. They are the important special cases of the 2×2 AKNS system: (1) KdV hierarchy: p is real and $q = -1$; (2) MKdV-SG hierarchy: $q = -p$ is real; (3) Nonlinear Schrödinger hierarchy: $q = -\bar{p}$. These special cases were studied widely (e.g. [82, 88, 91, 105, 117, 118]). Here we use a unified method to deal with the whole hierarchy, and the coefficients may depend on t [32, 45]. At the end of this section, we discuss the general AKNS system with $u(N)$ reduction. This is a generalization of the nonlinear Schrödinger hierarchy and has many applications to other problems.

1.4.1 KdV hierarchy

Consider the Lax pair [45]

$$\Phi_x = U\Phi, \qquad \Phi_t = V\Phi \tag{1.159}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ \zeta - u & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{1.160}$$

A, B and C are polynomials of the spectral parameter ζ.

Compared with Section 1.2, the integrability condition (1.68) leads to

$$\begin{aligned} &-A_x + C - B(\zeta - u) = 0, \qquad B_x + 2A = 0, \\ &-u_t - C_x + 2A(\zeta - u) = 0. \end{aligned} \tag{1.161}$$

The first two equations imply

$$\begin{aligned} A &= -\frac{1}{2}B_x, \\ C &= \zeta B - uB - \tfrac{1}{2}B_{xx}. \end{aligned} \tag{1.162}$$

Substituting (1.162) into (1.161) we get

$$u_t = -2(\zeta - u)B_x + u_x B + \frac{1}{2}B_{xxx}. \tag{1.163}$$

Let

$$B = \sum_{j=0}^{n} b_j(x,t)\zeta^{n-j}, \tag{1.164}$$

then (1.163) leads to

$$\begin{aligned} &b_{0,x} = 0, \\ &b_{j+1,x} = ub_{j,x} + \frac{1}{2}u_x b_j + \tfrac{1}{4}b_{j,xxx} \quad (0 \le j \le n-1), \end{aligned} \tag{1.165}$$

$$u_t = 2ub_{n,x} + u_x b_n + \frac{1}{2} b_{n,xxx}. \tag{1.166}$$

(1.166) is the equation of u. When $n \geq 2$, it is called a KdV equation of higher order.

Similar to Lemma 1.5, (1.165) leads to

$$b_k = \sum_{j=0}^{k} \alpha_{k-j}(t) b_j^0[u], \tag{1.167}$$

where $b_j^0[u]$'s satisfy the recursion relations (1.165) and $b_0^0[0] = 1$, $b_j^0[0] = 0$ $(j \geq 1)$. Clearly $b_j^0[u]$'s are determined by (1.165) uniquely.

The first few b_j^0's are

$$\begin{aligned} &b_0^0 = 1, \quad b_1^0 = \frac{1}{2}u, \\ &b_2^0 = \frac{1}{8}u_{xx} + \frac{3}{8}u^2, \cdots . \end{aligned} \tag{1.168}$$

The corresponding equations are

$n = 0$: Linear equation

$$u_t = \alpha_0(t) u_x. \tag{1.169}$$

$n = 1$:

$$u_t = \alpha_0(t)\Big(\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x\Big) + \alpha_1(t) u_x. \tag{1.170}$$

If $\alpha_0 =$ constant, $\alpha_1 = 0$, (1.170) is the standard KdV equation.

$n = 2$:

$$\begin{aligned} u_t = \ & \alpha_0(t)\Big(\frac{1}{16}u_{xxxxx} + \frac{5}{8}uu_{xxx} + \frac{5}{4}u_x u_{xx} + \frac{15}{8}u^2 u_x\Big) \\ & + \alpha_1(t)\Big(\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x\Big) + \alpha_2(t) u_x, \end{aligned} \tag{1.171}$$

which is called the KdV equation of 5th-order.

Next we discuss the Darboux transformation for the KdV hierarchy by using the general results for the AKNS system. It seems that the calculation is tedious. However, we can see the application of the general results more clearly. The method here is valid to the whole hierarchy comparing to the special method in Section 1.1.

The U and V given by (1.160) are different from those of the AKNS system. However, the Lax pair can be transformed to a Lax pair in the AKNS system by a similar transformation given by a constant matrix depending on ζ.

Let

$$
\begin{aligned}
&R = \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix} \qquad (\lambda^2 = \zeta), \\
&\Psi = R\Phi, \\
&\widetilde{U} = RUR^{-1} = \begin{pmatrix} \lambda & u \\ -1 & -\lambda \end{pmatrix}, \\
&\widetilde{V} = RVR^{-1} = \begin{pmatrix} \lambda B - A & \lambda^2 B - 2\lambda A - C \\ -B & A - \lambda B \end{pmatrix},
\end{aligned} \tag{1.172}
$$

then Ψ satisfies

$$
\Psi_x = \widetilde{U}\Psi, \qquad \Psi_t = \widetilde{V}\Psi. \tag{1.173}
$$

This is the Lax pair for the KdV equation in the AKNS form. The Darboux transformation can be constructed based on the discussion in Section 1.3. Take two constants λ_1, λ_2 and column solutions h_1, h_2 of the Lax pair when $\lambda = \lambda_1, \lambda_2$ respectively. Moreover, we want that the matrices given by the Darboux transformation are still of the form of (1.172). That is,

$$
\begin{aligned}
&\widetilde{U}' = \begin{pmatrix} \lambda & u' \\ -1 & -\lambda \end{pmatrix}, \\
&\widetilde{V}' = \begin{pmatrix} \lambda B[u'] - A[u'] & \lambda^2 B[u'] - 2\lambda A[u'] - C[u'] \\ -B[u'] & A[u'] - \lambda B[u'] \end{pmatrix}.
\end{aligned} \tag{1.174}
$$

This condition (especially that the $(2,1)$ entry of $\widetilde{U}'$ is -1) holds only when λ_2, λ_1, h_2 and h_1 are specified.

Suppose $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution of the Lax pair (1.173) for $\lambda = \lambda_0$, then $\begin{pmatrix} \alpha + 2\lambda_0\beta \\ \beta \end{pmatrix}$ is a solution of (1.173) for $\lambda = -\lambda_0$. Thus we choose

$$
\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix}, \quad H = \begin{pmatrix} \alpha & \alpha + 2\lambda_0\beta \\ \beta & \beta \end{pmatrix}. \tag{1.175}
$$

Let

$$
S = H\Lambda H^{-1} = \begin{pmatrix} -\lambda_0 - \dfrac{1}{\tau} & \dfrac{1}{\tau^2} + \dfrac{2\lambda_0}{\tau} \\ -1 & \dfrac{1}{\tau} + \lambda_0 \end{pmatrix} \tag{1.176}
$$

where $\tau = \beta/\alpha$, and

$$\widetilde{D} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\lambda I - S), \tag{1.177}$$

then after the action of the Darboux transformation given by the Darboux matrix $\widetilde{D}$,

$$\widetilde{U}' = \widetilde{D}\widetilde{U}\widetilde{D}^{-1} + \widetilde{D}_x\widetilde{D}^{-1} = \begin{pmatrix} \lambda & u' \\ -1 & -\lambda \end{pmatrix} \tag{1.178}$$

where

$$u' = -u - 2\left(\frac{1}{\tau^2} + \frac{2\lambda_0}{\tau}\right). \tag{1.179}$$

According to the general discussion to the AKNS system, V' is given by the second equation of (1.174). Therefore, the Darboux transformation given by $\widetilde{D}$ in (1.177) is a Darboux transformation from any equation in the KdV hierarchy to the same equation.

Next we compare the results here with those in Section 1.1. If $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution of (1.173) for $\lambda = \lambda_0$, then the corresponding solution of (1.159) is

$$R^{-1}(\lambda_0) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\beta \\ \alpha + \lambda_0\beta \end{pmatrix}. \tag{1.180}$$

Let σ be the ratio of the second and the first components, i.e.,

$$\sigma = \frac{\alpha + \lambda_0\beta}{-\beta} = -\frac{1}{\tau} - \lambda_0, \tag{1.181}$$

then σ satisfies

$$\sigma_x = \lambda_0^2 - u - \sigma^2, \tag{1.182}$$

and

$$S = \begin{pmatrix} \sigma & \sigma^2 - \lambda_0^2 \\ -1 & -\sigma \end{pmatrix}. \tag{1.183}$$

In order to get the Darboux matrix for the Lax pair in the form (1.159), let

$$\begin{aligned} D &= R^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\lambda I - S)R = \begin{pmatrix} -\sigma & 1 \\ \lambda^2-\lambda_0^2+\sigma^2 & -\sigma \end{pmatrix} \\ &= \begin{pmatrix} -\sigma & 1 \\ \zeta-\zeta_0+\sigma^2 & -\sigma \end{pmatrix} \qquad (\zeta_0=\lambda_0^2). \end{aligned} \tag{1.184}$$

Then

$$\begin{aligned} U' &= DUD^{-1} + D_x D^{-1} \\ &= \begin{pmatrix} 0 & 1 \\ \zeta - 2\zeta_0 + u + 2\sigma^2 & 0 \end{pmatrix}. \end{aligned} \tag{1.185}$$

Hence the action of D keeps the x part of the Lax pair invariant, and transforms u to

$$u' = 2\zeta_0 - u - 2\sigma^2 \tag{1.186}$$

(the same as (1.179)). Theorem 1.11 implies that $V'[u] = V[u']$, i.e., the Darboux transformation keeps the t part invariant. Therefore, we have

Theorem 1.16 *Suppose u is a solution of (1.166), ζ_0 is a non-zero real constant, $\begin{pmatrix} a \\ b \end{pmatrix}$ is a solution of the Lax pair (1.159) for $\zeta = \zeta_0$, $\sigma = b/a$, then*

$$D = \begin{pmatrix} -\sigma & 1 \\ \zeta-\zeta_0+\sigma^2 & -\sigma \end{pmatrix} \tag{1.187}$$

is a Darboux matrix for (1.159). It transforms a solution u of (1.166) to a new solution

$$u' = 2\zeta_0 - u - 2\sigma^2 \tag{1.188}$$

of the same equation.

Remark 12 In order to let $\widetilde{U}'$ and $\widetilde{U}$ have the same $(2,1)$-entry -1, the Darboux matrix (1.184) is chosen as $D = R^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\lambda I - S)R$, not $R^{-1}(\lambda I - S)R$. This guarantees that the transformation transforms a solution of (1.166) to a solution of the same equation (1.166).

Remark 13 Let $\Phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$*, then* ϕ *satisfies*

$$\begin{aligned} \phi_{xx} &= (\zeta - u)\phi, \\ \phi_t &= A\phi + B\phi_x. \end{aligned} \tag{1.189}$$

It is similar to the system in Section 1.1. However, here A *and* B *can be polynomials of* ζ *of arbitrary degrees, whose coefficients are differential polynomials of* u*. The problem discussed in Section 1.1 was a special case.*

In Theorem 1.16, $b = a_x$*, hence* $\sigma = a_x/a$*. The transformation* D *in (1.184) gives*

$$\phi \to \phi' = \phi_x - \sigma\phi = \phi_x - \frac{a_x}{a}\phi, \tag{1.190}$$

and (1.182), (1.186) give the original Darboux transformation

$$u' = u + 2(\ln a)_{xx}. \tag{1.191}$$

Remark 14 From (1.165), we can get b_0*,* b_1*,* $\cdots$ *recursively, whose integral constants can be functions of* t*. Therefore, the coefficients of the nonlinear equations can be functions of* t*, as in the examples (1.170) and (1.171). The solutions of the equations whose coefficients depending on* t *differ a lot from the solutions of the equations whose coefficients independent of* t*. In the latter case, each soliton moves in a fixed velocity and the soliton with larger amplitude moves faster. However, in the former case, each soliton can have varying velocity (e.g. oscillates), and the soliton with larger amplitude may move slower.*

1.4.2 MKdV-SG hierarchy

Consider the Lax pair [32]

$$\begin{aligned} \Phi_x = U\Phi &= \begin{pmatrix} \lambda & p \\ -p & -\lambda \end{pmatrix} \Phi, \\ \Phi_t = V\Phi &= \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi, \end{aligned} \tag{1.192}$$

where A, B and C are polynomials of λ and λ^{-1} satisfying

$$A(-\lambda) = -A(\lambda), \qquad B(-\lambda) = -C(\lambda). \tag{1.193}$$

Moreover, suppose

$$A = \sum_{j=0}^{n+m} a_j \lambda^{2n-2j+1} \tag{1.194}$$

($m \geq 0$, $n \geq 0$). Unlike the AKNS system, here A, B and C are not restricted to the polynomials of λ, but the negative powers of λ are allowed. The term with lowest power of λ in (1.194) is $a_{n+m}\lambda^{-2m+1}$.

The integrability condition $U_t - V_x + [U, V] = 0$ leads to

$$\begin{aligned} &A_x = p(B+C), \\ &p_t - B_x - 2pA + 2\lambda B = 0, \\ &p_t + C_x + 2pA + 2\lambda C = 0. \end{aligned} \tag{1.195}$$

Hence

$$\begin{aligned} B+C &= \frac{A_x}{p} = \sum_{j=0}^{n+m} \frac{a_{j,x}}{p}\lambda^{2n-2j+1}, \\ B-C &= \frac{(B+C)_x + 4pA}{2\lambda} \\ &= \sum_{j=0}^{n+m} \left(\frac{1}{2}\left(\frac{a_{j,x}}{p}\right)_x + 2pa_j\right)\lambda^{2n-2j}, \end{aligned} \tag{1.196}$$

(thus $B(-\lambda) = -C(\lambda)$ holds automatically) and

$$p_t = \frac{1}{2}(B-C)_x - \lambda(B+C). \tag{1.197}$$

Comparing the coefficients of λ in (1.197), we can obtain the recursion relations among a_j's. They include two parts. The first part

$$\begin{aligned} &a_{0,x} = 0, \\ &a_{j+1,x} = \frac{1}{4}p\left(\left(\frac{a_{j,x}}{p}\right)_x + 4a_j p\right)_x \qquad (j = 0, 1, \cdots, n-1) \end{aligned} \tag{1.198}$$

are obtained from the coefficients of positive powers of λ and the second part

$$\begin{aligned} &\left(\left(\frac{a_{n+m,x}}{p}\right)_x + 4a_{n+m}p\right)_x = 0, \\ &\left(\left(\frac{a_{j,x}}{p}\right)_x + 4a_j p\right)_x = 4\frac{a_{j+1,x}}{p} \\ &(j = n+m-1, \cdots, n+1) \end{aligned} \tag{1.199}$$

are obtained from the coefficients of negative powers of λ. Moreover, the term without λ leads to the equation

$$p_t - \frac{1}{4}\left(\left(\frac{a_{n,x}}{p}\right)_x + 4a_n p\right)_x + \frac{a_{n+1,x}}{p} = 0. \tag{1.200}$$

The first few a_j's $(0 \le j \le n)$ are

$$\begin{aligned} a_0 &= \alpha_0(t), \\ a_1 &= \frac{1}{2}\alpha_0(t)p^2 + \alpha_1(t), \\ a_2 &= \alpha_0(t)\left(\frac{1}{4}pp_{xx} - \frac{1}{8}p_x^2 + \frac{3}{8}p^4\right) + \frac{1}{2}\alpha_1(t)p^2 + \alpha_2(t), \\ &\cdots . \end{aligned} \tag{1.201}$$

If V does not contain negative powers of λ, i.e., $m = 0$, then from the general conclusion to the AKNS system, all a_j's are differential polynomials of p. The equation (1.200) becomes

$$p_t - \frac{1}{4}\left(\left(\frac{a_{n,x}}{p}\right)_x + 4a_n p\right)_x = 0. \tag{1.202}$$

This is called the MKdV hierarchy. By using the notion a_j^0 in Section 1.2, these equations can be written as

$$p_t + \sum_{j=0}^{n} \alpha_j(t) M_{n-j}[p] = 0, \tag{1.203}$$

where

$$M_l[p] = -\frac{1}{4}\left(\left(\frac{a_{l,x}^0}{p}\right)_x + 4a_l^0 p\right)_x \quad (l = 0, 1, \cdots, n). \tag{1.204}$$

Especially, if $n = 1$ and $\alpha_0 = -4$, $\alpha_1 = 0$, then (1.202) becomes the MKdV equation

$$p_t + p_{xxx} + 6p^2 p_x = 0. \tag{1.205}$$

Next, we consider the negative powers of λ in V. Take $p = -u_x/2$ and suppose it satisfies the boundary condition: $u - k\pi$ and its derivatives tend to 0 fast enough as $x \to -\infty$ (k is an integer).

The first equation of (1.199) gives

$$\left(\left(\frac{a_{n+m,x}}{u_x}\right)_x + a_{n+m}u_x\right)_x = 0. \tag{1.206}$$

Write a_{n+m} as a function of u, then the above equation becomes

$$((a_{n+m,uu} + a_{n+m})u_x)_x = 0. \tag{1.207}$$

The boundary condition as $x \to -\infty$ gives $a_{n+m,uu} + a_{n+m} = 0$. Hence

$$a_{n+m} = \alpha\cos(u + \beta) \tag{1.208}$$

where α and β are constants.

Now take a special a_{n+m}: $a^0_{n+m} = \frac{1}{4}\cos u$. a_{n+j} $(j = 1, 2, \cdots, m-1)$ can be determined as follows. Let $g_{n+j} = a^0_{n+j,x}/p$, then $g_{n+m} = \frac{1}{2}\sin u$, and

$$a^0_{j-1} = \int_{-\infty}^{x} p g_{j-1}\, dx + a^-_{j-1} \quad (n+2 \le j \le n+m) \tag{1.209}$$

where a^-_{j-1} is the limit of a^0_{j-1} as $x \to -\infty$. From the boundary condition $\lim\limits_{x\to-\infty}(g_{j-1})_x = 0$, the recursion relations (1.199) become

$$\frac{1}{4}(g_{j-1})_x + p\left(a^-_{j-1} + \int_{-\infty}^{x} p g_{j-1}\, dx\right) = \int_{-\infty}^{x} g_j\, dx. \tag{1.210}$$

Moreover, suppose

$$\lim_{x\to-\infty} g_{j-1} = 0, \tag{1.211}$$

then

$$\begin{aligned} &g_{j-1} + 4\int_{-\infty}^{x} p(\xi)\left(a^-_{j-1} + \int_{-\infty}^{\xi} p(\zeta) g_{j-1}(\zeta)\, d\zeta\right) d\xi \\ &= 4\int_{-\infty}^{x}\int_{-\infty}^{\xi} g_j(\zeta)\, d\zeta\, d\xi. \end{aligned} \tag{1.212}$$

This is an integral equation of Volterra type. It has a unique solution in the class of functions which tend to zero fast enough together with its derivatives as $x \to -\infty$.

Take $a^-_{j-1} = 0$ and write the solution of (1.212) as

$$g_{j-1} = Q(g_j) = Q^2(g_{j+1}) = \cdots = \frac{1}{2}Q^{n+m-j+1}[\sin u]. \tag{1.213}$$

Here Q is the operator to determine g_{j-1} from g_j defined by (1.212). g_{j-1} is not a differential polynomial of g_j.

If $n = 0$, $\alpha_0 = 0$, then we obtain the SG hierarchy

$$p_t + \frac{1}{2}\sum_{j=0}^{m-1} \beta_j(t) Q^{m-j-1}[\sin u] = 0 \tag{1.214}$$

where $\beta_j(t)$'s are arbitrary functions of t.

Generally, we have the compound MKdV-SG hierarchy

$$\begin{aligned} &p_t + \sum_{j=0}^{n} \alpha_j(t) M_{n-j}[p] + \frac{1}{2}\sum_{j=0}^{m-1} \beta_j(t) Q^{m-j-1}[\sin u] = 0, \\ &(p = -\tfrac{u_x}{2}). \end{aligned} \tag{1.215}$$

EXAMPLE 1.17 *$n = 0$, $m = 2$, $\beta_0 = 0$, $\beta_1 = 1$, then, $g_2 = \frac{1}{2}\sin u$, and the equation becomes the sine-Gordon equation*

$$u_{xt} = \sin u. \tag{1.216}$$

EXAMPLE 1.18 *$n = 1$, $m = 2$, $\alpha_0 = -4$, $\alpha_1 = 0$, $\beta_0 = 0$, $\beta_1 = 1$, then the equation becomes the equation describing one-dimensional nonlinear lattice of atoms [70]*

$$u_{xt} + \frac{3}{2}u_x^2 u_{xx} + u_{xxxx} - \sin u = 0. \tag{1.217}$$

Now we consider the Darboux transformation. If $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution of (1.192) for $\lambda = \lambda_0$, then $\begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$ is a solution of (1.192) for $\lambda = -\lambda_0$. Therefore, we can choose

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix}, \qquad H = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \tag{1.218}$$

where $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution of (1.192) for $\lambda = \lambda_0$. Let $\sigma = \beta/\alpha$,

$$S = H\Lambda H^{-1} = \frac{\lambda_0}{1+\sigma^2}\begin{pmatrix} 1-\sigma^2 & 2\sigma \\ 2\sigma & \sigma^2 - 1 \end{pmatrix} \tag{1.219}$$

and denote $\tan\dfrac{\theta}{2} = \sigma$, then

$$S = \lambda_0 \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{1.220}$$

From $\sigma_x = -p(1+\sigma^2) - 2\lambda_0\sigma$, we have

$$\theta_x = -2p - 2\lambda_0 \sin\theta. \tag{1.221}$$

By direct calculation,

$$(\lambda I - S)U(\lambda I - S)^{-1} - S_x(\lambda I - S)^{-1} = \begin{pmatrix} \lambda & p' \\ -p' & -\lambda \end{pmatrix} \tag{1.222}$$

where

$$p' = p + 2\lambda_0 \sin\theta = -p - \theta_x, \tag{1.223}$$

or equivalently,

$$u' = -u + 2\theta \tag{1.224}$$

for suitable choice of the integral constant.

It remains to prove that the Darboux matrix $\lambda I - S$ keeps the reduction of MKdV-SG hierarchy. This includes (1) the transformed A', B' and C' still satisfy $A'(-\lambda) = -A'(\lambda)$ and $B'(-\lambda) = -C(\lambda)$; (2) the coefficients $\alpha_j(t)$'s keeps invariant.

Since $V^T(-\lambda) = -V(\lambda)$, $S^T = S$ and $(\lambda I + S)^T(\lambda I - S) = \lambda^2 I - S^2 = (\lambda^2 - \lambda_0^2)I$, it can be verified by direct calculation that $V'^T(-\lambda) = -V'(\lambda)$ holds. This proves (1).

(2) is proved as follows. For a_j $(j \leq n)$, this has been proved for the AKNS system; for a_j $(j \geq n+1)$, the conclusion follows from the boundary condition at infinity.

Therefore, the following theorem holds.

THEOREM 1.19 *Suppose u is a solution of (1.200), λ_0 is a non-zero real number, $\begin{pmatrix} a \\ b \end{pmatrix}$ is a solution of the Lax pair (1.192) for $\lambda = \lambda_0$. Let $\theta = 2\tan^{-1}(b/a)$,*

$$S = \lambda_0 \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \tag{1.225}$$

then $\lambda I - S$ is a Darboux matrix for (1.192). It transforms a solution p of (1.200) to the solution $p' = p + 2\lambda_0 \sin\theta$ of the same equation. Moreover, $u' = -u + 2\theta$ with suitable boundary condition, where $p = -u_x/2$, $p' = -u'_x/2$.

Remark 15 For the sine-Gordon equation, the Bäcklund transformation is a kind of method to get explicit solutions, which was known in the nineteenth century. In that method, to obtain a new solution from a known solution, there is an integrable system of differential equations to be solved (moreover, one can obtain explicit expression by using the theorem of permutability and the nonlinear superposition formula). Using Darboux transformation, that explicit expression can be obtained directly. This will be discussed in Chapter 4 together with the related geometric problems.

1.4.3 NLS hierarchy

The Lax pair for the nonlinear Schrödinger hierarchy (NLS hierarchy) is

$$\begin{aligned}\Phi_x = U\Phi &= \begin{pmatrix} \lambda & p \\ -\bar{p} & -\lambda \end{pmatrix}\Phi,\\ \Phi_t = V\Phi &= \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\Phi,\end{aligned} \tag{1.226}$$

where A, B and C are polynomials of λ (λ, p, A, B and C are complex-valued) satisfying

$$A(-\bar{\lambda}) = -\overline{A(\lambda)}, \qquad B(-\bar{\lambda}) = -\overline{C(\lambda)} \tag{1.227}$$

(i.e., $V^*(-\bar{\lambda}) = -V(\lambda)$ where $*$ refers to the complex conjugate transpose of a matrix). This is also a special case of the AKNS system. We shall construct a Darboux matrix keeping this reduction.

The integrability condition $U_t - V_x + [U, V] = 0$ is

$$\begin{aligned}A_x &= pC + \bar{p}B,\\ B_x &= p_t + 2\lambda B - 2pA,\\ C_x &= -\bar{p}_t - 2\lambda C - 2\bar{p}A.\end{aligned} \tag{1.228}$$

We can use (1.55) to write down the coefficients of the powers of λ in A, B and C. They depend on $p, p_x, \cdots$ and the integral constants $\alpha_j(t)$. Moreover, there is a nonlinear evolution equation

$$p_t = b_{n,x} + 2pa_n. \tag{1.229}$$

Especially, for $n = 2$, $\alpha_0 = -2\mathrm{i}$, $\alpha_1 = \alpha_2 = 0$, the equation is the nonlinear Schrödinger equation

$$\mathrm{i}p_t = p_{xx} + 2|p|^2 p. \tag{1.230}$$

The Darboux transformation for the nonlinear Schrödinger hierarchy is also constructed from the choice of Λ and H. Suppose $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution of (1.226) for $\lambda = \lambda_0$, then $\begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}$ is a solution of (1.226) for

$\lambda = -\bar{\lambda}_0$. Hence, we can choose

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\bar{\lambda}_0 \end{pmatrix}, \qquad H = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \tag{1.231}$$

$$S = H\Lambda H^{-1} = \frac{1}{1+|\sigma|^2} \begin{pmatrix} \lambda_0 - \bar{\lambda}_0|\sigma|^2 & (\lambda_0 + \bar{\lambda}_0)\bar{\sigma} \\ (\lambda_0 + \bar{\lambda}_0)\sigma & -\bar{\lambda}_0 + \lambda_0|\sigma|^2 \end{pmatrix}, \tag{1.232}$$

where $\sigma = \beta/\alpha$. Since $\det H \neq 0$, S can be defined globally. It can be checked that H satisfies

$$H^*H = |\alpha|^2 + |\beta|^2, \tag{1.233}$$

hence S satisfies

$$\begin{aligned} &S^*S = |\lambda_0|^2, \\ &S - S^* = \lambda_0 - \bar{\lambda}_0. \end{aligned} \tag{1.234}$$

Therefore, under the action of the Darboux matrix $\lambda I - S$, U is transformed to

$$U' = (\lambda I - S)U(\lambda I - S)^{-1} - S_x(\lambda I - S)^{-1} = \begin{pmatrix} \lambda & p' \\ -\bar{p}' & -\lambda \end{pmatrix} \tag{1.235}$$

where

$$p' = p + 2S_{12} = p + \frac{2(\lambda_0 + \bar{\lambda}_0)\bar{\sigma}}{1+|\sigma|^2}. \tag{1.236}$$

From the discussion on the AKNS system (Theorem 1.11), we know that $V' = (\lambda I - S)V(\lambda I - S)^{-1} - S_t(\lambda I - S)^{-1}$ is also a polynomial of λ and $V'^*(-\bar{\lambda}) = -V'(\lambda)$ holds. Moreover, $\lambda I - S$ gives a Darboux transformation from an equation in the nonlinear Schrödinger hierarchy to the same equation. This leads to the following theorem.

Theorem 1.20 *Suppose p is a solution of (1.229), λ_0 is a non-real complex number, $\begin{pmatrix} a \\ b \end{pmatrix}$ is a solution of the Lax pair (1.226) for $\lambda = \lambda_0$. Let $\sigma = b/a$,*

$$S = \frac{1}{1+|\sigma|^2} \begin{pmatrix} \lambda_0 - \bar{\lambda}_0|\sigma|^2 & (\lambda_0 + \bar{\lambda}_0)\bar{\sigma} \\ (\lambda_0 + \bar{\lambda}_0)\sigma & -\bar{\lambda}_0 + \lambda_0|\sigma|^2 \end{pmatrix}, \tag{1.237}$$

then $\lambda I - S$ is a Darboux matrix for (1.226). It transforms a solution p of (1.229) to a solution

$$p' = p + \frac{2(\lambda_0 + \bar{\lambda}_0)\bar{\sigma}}{1 + |\sigma|^2} \tag{1.238}$$

of the same equation.

1.4.4 AKNS system with $u(N)$ reduction

For the nonlinear Schrödinger hierarchy, (1.226) and (1.227) imply

$$U(-\bar{\lambda}) = -U(\lambda)^*, \quad V(-\bar{\lambda}) = -V(\lambda)^*. \tag{1.239}$$

Here we generalize it to the AKNS system.

For the AKNS system (1.226), if U and V satisfy (1.239), then we say that (1.226) has $u(N)$ reduction, because $U(\lambda)$ and $V(\lambda)$ are in the Lie algebra $u(N)$ when λ is purely imaginary. This is a very popular reduction.

We want to construct Darboux matrix which keeps $u(N)$ reduction. That is, after the action of the Darboux matrix, the derived potentials $U'(\lambda)$ and $V'(\lambda)$ must satisfy

$$U'(-\bar{\lambda}) = -U'(\lambda)^*, \quad V'(-\bar{\lambda}) = -V'(\lambda)^*. \tag{1.240}$$

With this additional condition, Λ and H in (1.94) can not be arbitrary. They should satisfy the following two conditions:

(1) $\lambda_1, \cdots, \lambda_N$ can only be μ or $-\bar{\mu}$ where μ is a complex number (μ is not real).

(2) If $\lambda_j \neq \lambda_k$, then

$$h_j^* h_k = 0 \tag{1.241}$$

holds at one point (x_0, t_0).

In fact, if (1.241) holds at one point, then it holds everywhere. This is proved as follows.

When $\lambda_j \neq \lambda_k$, $\lambda_k = -\bar{\lambda}_j$, hence

$$\begin{aligned} &h_{k,x} = U(\lambda_k)h_k, \quad h_{k,t} = V(\lambda_k)h_k, \\ &h_{j,x}^* = h_j^* U(\lambda_j)^* = -h_j^* U(-\bar{\lambda}_j) = -h_j^* U(\lambda_k), \\ &h_{j,t}^* = h_j^* V(\lambda_j)^* = -h_j^* V(-\bar{\lambda}_j) = -h_j^* V(\lambda_k). \end{aligned} \tag{1.242}$$

This implies that

$$(h_j^* h_k)_x = 0, \quad (h_j^* h_k)_t = 0. \tag{1.243}$$

Therefore, $h_j^* h_k = 0$ holds everywhere if it holds at one point.

THEOREM 1.21 *If λ_j's, h_j's satisfy the above conditions (1) and (2), $H = (h_1, \cdots, h_N)$, then* $\det H \neq 0$ *holds everywhere if it holds at one point. Moreover, U' and V' given by (1.81) satisfy*

$$U'(-\bar{\lambda}) = -U'(\lambda)^*, \quad V'(-\bar{\lambda}) = -V'(\lambda)^*. \tag{1.244}$$

Proof. Let (x_0, t_0) be a fixed point. Then by the property of linear differential equation, all $\{h_\alpha\}$ with $\lambda_\alpha = \mu$ are linearly independent if they are linearly independent at (x_0, t_0). Likewise, all $\{h_\alpha\}$ with $\lambda_\alpha = \bar{\mu}$ are also linearly independent if they are linearly independent at (x_0, t_0). Moreover, (1.241) implies that all $\{h_1, \cdots, h_N\}$ are linearly independent if they are linearly independent at (x_0, t_0). Therefore, $\det H \neq 0$ and $S = H\Lambda H^{-1}$ is globally defined.

According to the definition of S,

$$Sh_j = \lambda_j h_j, \quad h_k^* S^* = h_k^* \bar{\lambda}_k. \tag{1.245}$$

Hence

$$h_k^*(S - S^*)h_j = (\lambda_j - \bar{\lambda}_k)h_k^* h_j. \tag{1.246}$$

If $\lambda_j = \mu$, $\lambda_k = -\bar{\mu}$, then

$$h_k^*(S - S^*)h_j = 0. \tag{1.247}$$

If $\lambda_j = \lambda_k = \mu$ (or $\lambda_j = \lambda_k = -\bar{\mu}$), then

$$h_k^*(S - S^*)h_j = (\mu - \bar{\mu})h_k^* h_j. \tag{1.248}$$

Hence

$$S - S^* = (\mu - \bar{\mu})I. \tag{1.249}$$

On the other hand, from (1.245), we have

$$h_k^* S^* S h_j = \lambda_j \bar{\lambda}_k h_k^* h_j. \tag{1.250}$$

If $\lambda_j = \mu$, $\lambda_k = -\bar{\mu}$, then

$$h_k^* S^* S h_j = 0. \tag{1.251}$$

If $\lambda_j = \lambda_k = \mu$ (or $\lambda_j = \lambda_k = -\bar{\mu}$), then

$$h_k^* S^* S h_j = |\mu|^2 h_k^* h_j, \tag{1.252}$$

Therefore,

$$S^* S = |\mu|^2 I. \tag{1.253}$$

From (1.249) and (1.253), we obtain

$$(\bar{\lambda} I + S)^*(\lambda I - S) = (\lambda - \mu)(\lambda + \bar{\mu})I. \tag{1.254}$$

According to the action of the Darboux transformation on V_j,

$$\begin{aligned}&\sum_{j=0}^{m} V_j'\lambda^{m-j}\\&=(\lambda I-S)\sum_{j=0}^{m} V_j\lambda^{m-j}(\lambda I-S)^{-1}+(\lambda I-S)_t(\lambda I-S)^{-1},\end{aligned}\tag{1.255}$$

$$\begin{aligned}&(\sum_{j=0}^{m} V_j'\lambda^{m-j})^*\\&=(\lambda I-S)^{*-1}\sum_{j=0}^{m} V_j^*\bar{\lambda}^{m-j}(\lambda I-S)^*+(\lambda I-S)^{*-1}(\lambda I-S)_t^*\\&=-(\bar{\lambda}I+S)\sum_{j=0}^{m} V_j(-\bar{\lambda})^{m-j}(\bar{\lambda}I+S)^{-1}-(\bar{\lambda}I+S)_t(\bar{\lambda}I+S)^{-1}\\&=-\sum_{j=0}^{m} V_j'(-\bar{\lambda})^{m-j}.\end{aligned}\tag{1.256}$$

Hence $V'(-\bar{\lambda})=-V'(\lambda)^*$. Likewise, $U'(-\bar{\lambda})=-U'(\lambda)^*$. The theorem is proved.

As in Section 1.3, a Darboux transformation of higher degree can be derived by the composition of Darboux transformations of degree one. However, with the $u(N)$ reduction, we have also the following special and more direct construction [117, 17].

Suppose we take l times of Darboux transformations of degree one. Each Darboux transformation is constructed from Λ_α, H_α ($\alpha=1,\cdots,l$). In each $\Lambda_\alpha=\mathrm{diag}(\lambda_1^{(\alpha)},\cdots,\lambda_k^{(\alpha}})$, suppose $\lambda_1^{(\alpha)}=\cdots=\lambda_k^{(\alpha)}=\mu_\alpha$, $\lambda_{k+1}^{(\alpha)}=\cdots=\lambda_N^{(\alpha)}=-\bar{\mu}_\alpha$. Here k is the same for all α. For each $\lambda_j^{(\alpha)}$, solve the Lax pair and get a solution $h_j^{(\alpha)}$ satisfying the orthogonal relations (1.241).

Denote $H_\alpha=(h_1^{(\alpha)},\cdots,h_N^{(\alpha)})$, $\mathring{H}_\alpha=(h_1^{(\alpha)},\cdots,h_k^{(\alpha)})$. Let

$$\Gamma_{\alpha\beta}=\frac{\mathring{H}_\alpha^*\mathring{H}_\beta}{\mu_\beta+\bar{\mu}_\alpha},\tag{1.257}$$

$$D(\lambda)=\prod_{\gamma=1}^{l}(\lambda+\bar{\mu}_\gamma)\left(1-\sum_{\alpha,\beta=1}^{l}\frac{\mathring{H}_\alpha(\Gamma^{-1})_{\alpha\beta}\mathring{H}_\beta^*}{\lambda+\bar{\lambda}_\beta}\right).\tag{1.258}$$

Now we prove that $D(\lambda)$ is a Darboux matrix.

Let $\hat{H}_\alpha = (h_{k+1}^{(\alpha)}, \cdots, h_N^{(\alpha)})$. Then $H_\alpha = (\overset{\circ}{H}_\alpha, \hat{H}_\alpha)$ and $\overset{\circ}{H}{}^*_\alpha \hat{H}_\alpha = 0$ for all $\alpha = 1, \cdots, l$. Hence

$$D(\mu_\alpha)\overset{\circ}{H}_\alpha = 0, \quad D(-\bar{\mu}_\alpha)\hat{H}_\alpha = 0. \tag{1.259}$$

According to Theorem 1.12, $D(\lambda)$ is a Darboux matrix.

Moreover, the inverse of $D(\lambda)$ can be written out explicitly as

$$D(\lambda)^{-1} = \prod_{\gamma=1}^{l}(\lambda + \bar{\mu}_\gamma)^{-1}\left(1 + \sum_{\alpha,\beta=1}^{l} \frac{\overset{\circ}{H}_\alpha(\Gamma^{-1})_{\alpha\beta}\overset{\circ}{H}{}^*_\beta}{\lambda - \lambda_\alpha}\right). \tag{1.260}$$

(1.258) gives a compact form of Darboux matrix of higher degree. Although it is special, it is very useful.

1.5 Darboux transformation and scattering, inverse scattering theory

The scattering and inverse scattering theory is an important part of the soliton theory. It transforms the problem of solving the Cauchy problem of a nonlinear partial differential equation to the problem of describing the spectrum and eigenfunctions of the Lax pair. Here we consider the 2×2 AKNS system as an example to show the outline of the scattering and inverse scattering theory (see [23] for details). Moreover, we discuss the change of the scattering data under Darboux transformation for $su(2)$ reduction. For the KdV equation, the problem can be solved similarly, but the scattering and inverse scattering theory is simpler.

1.5.1 Outline of the scattering and inverse scattering theory for the 2×2 AKNS system

First, we give the definition of the scattering data for the 2×2 complex AKNS system. In order to coincide with the usual scattering theory, let $\lambda = -\mathrm{i}\zeta$, then the first equation of the 2×2 AKNS system (1.48) becomes

$$\Phi_x = \begin{pmatrix} -\mathrm{i}\zeta & p \\ q & \mathrm{i}\zeta \end{pmatrix}\Phi. \tag{1.261}$$

Suppose p, q and their derivatives with respect to x decay fast enough at infinity. Let $\mathbf{C}$ be the complex plane and $\mathbf{R}$ be the real line. Besides $\mathbf{C}_+$ and $\mathbf{C}_-$ are the upper and lower half plane of $\mathbf{C}$ respectively, i.e., $\mathbf{C}_+ = \{z \in \mathbf{C} \mid \operatorname{Im}\zeta > 0\}$, $\mathbf{C}_- = \{z \in \mathbf{C} \mid \operatorname{Im}\zeta < 0\}$.

Property 1. For each one of the following boundary conditions, the equation (1.261) has a unique column solution

$$(1)\qquad \psi_r(x,\zeta)=R(x,\zeta)\mathrm{e}^{-\mathrm{i}\zeta x},\quad \lim_{x\to-\infty}R(x,\zeta)=\begin{pmatrix}1\\0\end{pmatrix},\qquad (\operatorname{Im}\zeta\geq 0),\tag{1.262}$$

$$(2)\qquad \widetilde{\psi}_r(x,\zeta)=\widetilde{R}(x,\zeta)\mathrm{e}^{\mathrm{i}\zeta x},\quad \lim_{x\to-\infty}\widetilde{R}(x,\zeta)=\begin{pmatrix}0\\1\end{pmatrix},\qquad (\operatorname{Im}\zeta\leq 0),\tag{1.263}$$

$$(3)\qquad \psi_l(x,\zeta)=L(x,\zeta)\mathrm{e}^{\mathrm{i}\zeta x},\quad \lim_{x\to+\infty}L(x,\zeta)=\begin{pmatrix}0\\1\end{pmatrix},\qquad (\operatorname{Im}\zeta\geq 0),\tag{1.264}$$

$$(4)\qquad \widetilde{\psi}_l(x,\zeta)=\widetilde{L}(x,\zeta)\mathrm{e}^{-\mathrm{i}\zeta x},\quad \lim_{x\to+\infty}\widetilde{L}(x,\zeta)=\begin{pmatrix}1\\0\end{pmatrix},\qquad (\operatorname{Im}\zeta\leq 0).\tag{1.265}$$

Moreover, ψ_r, ψ_l (resp. $\widetilde{\psi}_r$, $\widetilde{\psi}_l$) are continuous for $\zeta\in\mathbf{C}_+\cup\mathbf{R}$ (resp. $z\in\mathbf{C}_-\cup\mathbf{R}$), and holomorphic with respect to ζ in $\mathbf{C}_+$ (resp. $\mathbf{C}_-$). These solutions are called Jost solutions.

If $\zeta\in\mathbf{R}$, then ψ_l and $\widetilde{\psi}_l$ are linearly independent. Hence, there exist functions $r_+(\zeta)$, $r_-(\zeta)$, $\widetilde{r}_+(\zeta)$ and $\widetilde{r}_-(\zeta)$ such that

$$\begin{aligned}\psi_r&=r_+\psi_l+r_-\widetilde{\psi}_l,\\ \widetilde{\psi}_r&=\widetilde{r}_+\psi_l+\widetilde{r}_-\widetilde{\psi}_l.\end{aligned}\tag{1.266}$$

Considering the Wronskian determinant between ψ_r, ψ_l and the Wronskian determinant between $\widetilde{\psi}_r$, $\widetilde{\psi}_l$, we have

Property 2. For $\zeta\in\mathbf{R}$,

$$\begin{aligned}r_-(\zeta)&=R_1(x,\zeta)L_2(x,\zeta)-R_2(x,\zeta)L_1(x,\zeta),\\ \widetilde{r}_+(\zeta)&=\widetilde{R}_2(x,\zeta)\widetilde{L}_1(x,\zeta)-\widetilde{R}_1(x,\zeta)\widetilde{L}_2(x,\zeta).\end{aligned}\tag{1.267}$$

$r_-(\zeta)$ can be holomorphically extended to $\mathbf{C}_+\cup\mathbf{R}$, and $\widetilde{r}_+(\zeta)$ can be holomorphically extended to $\mathbf{C}_-\cup\mathbf{R}$. Here R_1 and R_2 are two components of the vector R, i.e., $R=(R_1,R_2)^T$. L_1, L_2, $\widetilde{R}_1$, $\widetilde{R}_2$, $\widetilde{L}_1$, $\widetilde{L}_2$ have the similar meanings.

The asymptotic properties of the four Jost solutions in Property 1 as $x \to \pm\infty$ are listed in the next property.

Property 3. (1) The following limits hold uniformly for ζ:

$$\begin{aligned}
&\lim_{x\to-\infty} R(x,\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_+ \cup \mathbf{R},\\
&\lim_{x\to-\infty} \widetilde{R}(x,\zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_- \cup \mathbf{R},\\
&\lim_{x\to+\infty} L(x,\zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_+ \cup \mathbf{R},\\
&\lim_{x\to+\infty} \widetilde{L}(x,\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_- \cup \mathbf{R}.
\end{aligned} \tag{1.268}$$

(2) The following limits hold uniformly for ζ in a compact subset:

$$\begin{aligned}
&\lim_{x\to+\infty} R(x,\zeta) = \begin{pmatrix} r_-(\zeta) \\ 0 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_+,\\
&\lim_{x\to+\infty} \widetilde{R}(x,\zeta) = \begin{pmatrix} 0 \\ \widetilde{r}_+(\zeta) \end{pmatrix}, \qquad \zeta \in \mathbf{C}_-,\\
&\lim_{x\to-\infty} L(x,\zeta) = \begin{pmatrix} 0 \\ r_-(\zeta) \end{pmatrix}, \qquad \zeta \in \mathbf{C}_+,\\
&\lim_{x\to-\infty} \widetilde{L}(x,\zeta) = \begin{pmatrix} \widetilde{r}_+(\zeta) \\ 0 \end{pmatrix}, \qquad \zeta \in \mathbf{C}_-.
\end{aligned} \tag{1.269}$$

(3) The following limits hold uniformly for real $\zeta \in \mathbf{R}$:

$$\lim_{x\to+\infty} \left| R(x,\zeta) - \begin{pmatrix} r_-(\zeta) \\ r_+(\zeta)\mathrm{e}^{2\mathrm{i}\zeta x} \end{pmatrix} \right| = 0, \qquad \zeta \in \mathbf{R},$$

$$\lim_{x\to+\infty} \left| \widetilde{R}(x,\zeta) - \begin{pmatrix} \widetilde{r}_-(\zeta)\mathrm{e}^{-2\mathrm{i}\zeta x} \\ \widetilde{r}_+(\zeta) \end{pmatrix} \right| = 0, \qquad \zeta \in \mathbf{R},$$

$$\lim_{x\to-\infty} \left| L(x,\zeta) - \begin{pmatrix} -\widetilde{r}_-(\zeta)\mathrm{e}^{-2\mathrm{i}\zeta x} \\ r_-(\zeta) \end{pmatrix} \right| = 0, \qquad \zeta \in \mathbf{R},$$

$$\lim_{x\to-\infty}\left|\widetilde{L}(x,\zeta)-\begin{pmatrix}\widetilde{r}_+(\zeta)\\ -r_+(\zeta)\mathrm{e}^{2\mathrm{i}\zeta x}\end{pmatrix}\right|=0,\qquad \zeta\in\mathbf{R}.$$

Rewrite (1.261) as

$$\mathcal{L}\Phi=\zeta\Phi,\tag{1.270}$$

where

$$\mathcal{L}=\begin{pmatrix}\mathrm{i}&0\\0&-\mathrm{i}\end{pmatrix}\left(\frac{\mathrm{d}}{\mathrm{d}x}-\begin{pmatrix}0&p\\q&0\end{pmatrix}\right),\tag{1.271}$$

then (1.261) becomes a spectral problem of a linear ordinary differential operator. We consider its spectrum in $L^2(\mathbf{R})\times L^2(\mathbf{R})$.

If $\zeta\in\mathbf{C}_+$ and $r_-(\zeta)=0$, then (1.267) implies that ψ_r and ψ_l are linearly dependent. Hence $\psi_r\to 0$ as $x\to\pm\infty$. Similarly, if $\zeta\in\mathbf{C}_-$ and $\widetilde{r}_+(\zeta)=0$, then $\widetilde{\psi}_r$ and $\widetilde{\psi}_l$ are linearly dependent. Hence $\widetilde{\psi}_r\to 0$ as $x\to\pm\infty$. Since $r_-(\zeta)$ and $\widetilde{r}_+(\zeta)$ are holomorphic in $\mathbf{C}_+$ and $\mathbf{C}_-$ respectively, their zeros are discrete. These zeros are the eigenvalues of $\mathcal{L}$. The set of all eigenvalues of $\mathcal{L}$ is denoted by $IP\sigma(\mathcal{L})$. If $\zeta\in\mathbf{R}$, then it can be proved that (1.270) has a nontrivial bounded solution. $\sigma(\mathcal{L})=\mathbf{R}\cup IP\sigma(\mathcal{L})$ is called the spectrum of the operator $\mathcal{L}$. Its compliment $\mathbf{C}-\sigma(\mathcal{L})$ is called the regular set of $\mathcal{L}$.

Property 4. If $r_-(\zeta)\neq 0$ and $\widetilde{r}_+(\zeta)\neq 0$ hold for $\zeta\in\mathbf{R}$, then $IP\sigma(\mathcal{L})$ is a finite set.

Hereafter, we always suppose $r_-(\zeta)\neq 0$ and $r_+(\zeta)\neq 0$ when $\zeta\in\mathbf{R}$. First we consider the eigenvalues.

If $\zeta\in IP\sigma(\mathcal{L})$, then ψ_r and ψ_l are linearly dependent. Suppose

$$\begin{aligned}\psi_r(x,\zeta)&=\alpha(\zeta)\psi_l(x,\zeta)\quad(\zeta\in\mathbf{C}_+\cap IP\sigma(\mathcal{L})),\\ \widetilde{\psi}_r(x,\zeta)&=\widetilde{\alpha}(\zeta)\widetilde{\psi}_l(x,\zeta)\quad(\zeta\in\mathbf{C}_-\cap IP\sigma(\mathcal{L})).\end{aligned}\tag{1.272}$$

Denote $IP\sigma(\mathcal{L})\cap\mathbf{C}_+=\{\zeta_1,\cdots\zeta_d\}$ and $IP\sigma(\mathcal{L})\cap\mathbf{C}_-=\{\widetilde{\zeta}_1,\cdots\widetilde{\zeta}_{\widetilde{d}}\}$ to be the set of eigenvalues in $\mathbf{C}_+$ and $\mathbf{C}_-$ respectively. Moreover, suppose $\zeta_1,\cdots,\widetilde{\zeta}_{\widetilde{d}}$ are all simple zeros. Corresponding to each eigenvalue, there is a constant

$$\begin{aligned}C_k&=\alpha(\zeta_k)\Big/\frac{\mathrm{d}r_-(\zeta_k)}{\mathrm{d}\zeta}\quad(k=1,\cdots,d),\\ \widetilde{C}_k&=\widetilde{\alpha}(\zeta_k)\Big/\frac{\mathrm{d}\widetilde{r}_+(\zeta_k)}{\mathrm{d}\zeta}\quad(k=1,\cdots,\widetilde{d}).\end{aligned}\tag{1.273}$$

Using these data, we define the functions

$$B_d(y) = -\mathrm{i}\sum_{k=1}^{d} C_k \mathrm{e}^{\mathrm{i}\zeta_k y}, \quad \widetilde{B}_d(y) = \mathrm{i}\sum_{k=1}^{\widetilde{d}} \widetilde{C}_k \mathrm{e}^{-\mathrm{i}\zeta_k y}. \tag{1.274}$$

Next, we consider the continuous spectrum $\zeta \in \mathbf{R}$. As is known, the Fourier transformation of a Schwarz function ϕ is

$$F(\phi)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(s)\mathrm{e}^{-\mathrm{i}ks}\, ds. \tag{1.275}$$

It can be extended to $L^2(\mathbf{R})$ and becomes a bounded map from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$.

Property 5.

$$L(x,\cdot) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in L^2(\mathbf{R}), \quad \widetilde{L}(x,\cdot) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in L^2(\mathbf{R}). \tag{1.276}$$

Denote

$$\begin{aligned} N(x,s) &= \frac{1}{\sqrt{2\pi}} F\left(L(x,\cdot) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)(s), \\ \widetilde{N}(x,s) &= \tfrac{1}{\sqrt{2\pi}} F\left(\widetilde{L}(x,\cdot) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)(s), \quad (s \geq 0), \end{aligned} \tag{1.277}$$

then

$$\begin{aligned} L(x,\zeta) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^{+\infty} N(x,s)\mathrm{e}^{\mathrm{i}\zeta s}\, ds, \quad \forall \zeta \in \mathbf{C}_+ \cup \mathbf{R}, \\ \widetilde{L}(x,\zeta) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \textstyle\int_0^{+\infty} \widetilde{N}(x,s)\mathrm{e}^{-\mathrm{i}\zeta s}\, ds, \quad \forall \zeta \in \mathbf{C}_- \cup \mathbf{R}, \end{aligned} \tag{1.278}$$

and the above integrals converge absolutely. Moreover,

$$p(x) = -2N_1(x,0), \qquad q(x) = -2\widetilde{N}_2(x,0), \tag{1.279}$$

where the subscripts refer to the components.

For $\zeta \in \mathbf{R}$, denote

$$b(\zeta) = \frac{r_+(\zeta)}{r_-(\zeta)}, \quad \widetilde{b}(\zeta) = \frac{\widetilde{r}_-(\zeta)}{\widetilde{r}_+(\zeta)}. \tag{1.280}$$

It can be proved that

$$b, \widetilde{b} \in L^2(\mathbf{R}) \cap L^1(\mathbf{R}) \cap C^0(\mathbf{R}). \tag{1.281}$$

Hence, we can define

$$\begin{aligned} B_c(y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(\zeta)\mathrm{e}^{\mathrm{i}\zeta y}\,d\zeta, \\ \widetilde{B}_c(y) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widetilde{b}(\zeta)\mathrm{e}^{-\mathrm{i}\zeta y}\,d\zeta. \end{aligned} \tag{1.282}$$

The data

$$\{\zeta_k, C_k\,(k=1,\cdots,d),\ \widetilde{\zeta}_k, \widetilde{C}_k\,(k=1,\cdots,\widetilde{d}),\ b(\zeta), \widetilde{b}(\zeta)\,(\zeta \in \mathbf{R})\} \tag{1.283}$$

are called the scattering data corresponding to (p,q), denoted by $\Sigma(p,q)$. We can also call

$$\{r_-(\zeta)\ (\zeta \in \mathbf{C}_+ \cup \mathbf{R}),\ \widetilde{r}_+(\zeta)\ (\zeta \in \mathbf{C}_- \cup \mathbf{R})\} \tag{1.284}$$

the scattering data, since the data in (1.283) can be obtained from the data in (1.284).

Define $B = B_c + B_d$ and $\widetilde{B} = \widetilde{B}_c + \widetilde{B}_d$ according to (1.274) and (1.282).

Property 6. N and $\widetilde{N}$ satisfy the follow system of linear integral equations (Gelfand-Levitan-Marchenko equations)

$$\begin{aligned} &N(x,s) + \widetilde{B}(2x+s)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^{+\infty} \widetilde{N}(x,\sigma)\widetilde{B}(2x+s+\sigma)\,d\sigma = 0, \\ &\widetilde{N}(x,s) + B(2x+s)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^{+\infty} N(x,\sigma)B(2x+s+\sigma)\,d\sigma = 0. \end{aligned} \tag{1.285}$$

If the scattering data $\{\zeta_k, C_k, \widetilde{\zeta}_k, \widetilde{C}_k, b(\zeta), \widetilde{b}(\zeta)\}$ are known, N and $\widetilde{N}$ are solved from the above integral equations and (1.279) gives (p,q).

The process to get scattering data from (p,q) is called the scattering process. It needs to solve the spectral problem of ordinary differential equations. The process to get (p,q) from the scattering data is called the inverse scattering process. It needs to solve linear integral equations.

Now we consider the evolution of the scattering data. In the AKNS system, p and q are functions of (x,t). Therefore, we should consider the full AKNS system (with time t)

$$\Phi_x = \begin{pmatrix} -\mathrm{i}\zeta & p \\ q & \mathrm{i}\zeta \end{pmatrix}\Phi, \quad \Phi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\Phi, \tag{1.286}$$

where A, B and C are polynomials of ζ,

$$\begin{aligned} A &= \sum_{j=0}^{n} a_j(-\mathrm{i}\zeta)^{n-j}, \\ B &= \sum_{j=0}^{n} b_j(-\mathrm{i}\zeta)^{n-j}, \\ C &= \sum_{j=0}^{n} c_j(-\mathrm{i}\zeta)^{n-j}. \end{aligned} \tag{1.287}$$

We also suppose

$$A|_{p=q=0} = \mathrm{i}\omega(\zeta, t). \tag{1.288}$$

Lemma 1.5 implies $B|_{p=q=0} = C|_{p=q=0} = 0$.

Property 7. Suppose (p, q) satisfies the equations

$$p_t = b_{n,x} + 2pa_n, \qquad q_t = c_{n,x} - 2qa_n \tag{1.289}$$

given by the integrability condition, then the evolution of the corresponding scattering data is given by

$$\begin{aligned} &r_-(\zeta, t) = r_-(\zeta, 0) \qquad \zeta \in \mathbf{C}_+ \cup \mathbf{R}, \\ &\widetilde{r}_+(\zeta, t) = \widetilde{r}_+(\zeta, 0) \qquad \zeta \in \mathbf{C}_- \cup \mathbf{R}, \\ &r_+(\zeta, t) = r_+(\zeta, 0) \exp(-2\mathrm{i} \int_0^t \omega(\zeta, \tau)\, d\tau) \qquad \zeta \in \mathbf{R}, \\ &\widetilde{r}_-(\zeta, t) = \widetilde{r}_-(\zeta, 0) \exp(2\mathrm{i} \int_0^t \omega(\zeta, \tau)\, d\tau) \qquad \zeta \in \mathbf{R}, \end{aligned} \tag{1.290}$$

and

$$\begin{aligned} &\zeta_k(t) = \zeta_k(0), \\ &\widetilde{\zeta}_k(t) = \widetilde{\zeta}_k(0), \\ &C_k(t) = C_k(0) \exp(-2\mathrm{i} \int_0^t \omega(\zeta, \tau)\, d\tau), \\ &\widetilde{C}_k(t) = \widetilde{C}_k(0) \exp(2\mathrm{i} \int_0^t \omega(\zeta, \tau)\, d\tau), \\ &b(\zeta, t) = b(\zeta, 0) \exp(-2\mathrm{i} \int_0^t \omega(\zeta, \tau)\, d\tau), \\ &\widetilde{b}(\zeta, t) = \widetilde{b}(\zeta, 0) \exp(2\mathrm{i} \int_0^t \omega(\zeta, \tau)\, d\tau). \end{aligned} \tag{1.291}$$

(1.290) or (1.291) gives the evolution of the scattering data explicitly.

In summary, the process of solving the initial value problem of nonlinear evolution equations (1.289) of (p, q) is as follows. Here the initial condition is $t = 0 : p = p_0,\ q = q_0$.

For given (p_0, q_0), first solve the x-part of the Lax pair (1.286) for $p = p_0$, $q = q_0$ and get the scattering data corresponding to p_0 and q_0. Then, using the evolution of the scattering data (1.291), the scattering data corresponding to $(p(t), q(t))$ are obtained. Finally, solve the integral equations (1.285) to get $(p(t), q(t))$. Therefore, the inverse scattering method changes the initial value problem of nonlinear partial differential equations to the problem of solving systems of linear integral equations. This gives an effective way to solve the initial value problem. Especially, when $b_r = \widetilde{b}_r = 0$, $B_c = \widetilde{B}_c = 0$, (1.285) has a degenerate kernel. Hence it can be solved algebraically and the soliton solutions can be obtained. Please see [23] for details.

Remark 16 Denote $\Sigma(p, q)$ to be the scattering data corresponding to $(p(x, t), q(x, t))$, $p_0(x)$ and $q_0(x)$ to be the initial values of p and q at $t = 0$, then the procedure of inverse scattering method can be shown in the following diagram:

$$\begin{array}{lcc} t = 0 : (p_0, q_0) & \xrightarrow{\text{scattering}} & \Sigma(p_0, q_0) \\ & & \big\downarrow \\ t = t : \quad (p, q) & \xleftarrow{\text{inverse scattering}} & \Sigma(p, q) \end{array} \tag{1.292}$$

For a linear equation, if "scattering" is changed to "Fourier transformation" and "inverse scattering" is changed to "inverse Fourier transformation" in the above diagram, then it becomes the diagram for solving the initial value problem by Fourier transformations which has been used extensively for linear problems. Therefore, the scattering and inverse scattering method can be regarded as a kind of Fourier method for nonlinear problems.

1.5.2 Change of scattering data under Darboux transformations for $su(2)$ AKNS system

For the AKNS system, the scattering data include

$$\{\zeta_k, C_k, \widetilde{\zeta}_k, \widetilde{C}_k, b(\zeta), \widetilde{b}(\zeta)\}. \tag{1.293}$$

The $su(2)$ AKNS system means that $U, V \in su(2)$ for $\zeta \in \mathbf{R}$, i.e., $q = -\bar{p}$, $\bar{A} = -A$, $C = -\bar{B}$. Therefore, it is just the nonlinear Schrödinger hierarchy. The Lax pair is

$$
\begin{aligned}
\Phi_x &= \begin{pmatrix} -\mathrm{i}\zeta & p \\ -\bar{p} & \mathrm{i}\zeta \end{pmatrix} \Phi, \\
\Phi_t &= \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi
\end{aligned}
\tag{1.294}
$$

where $\overline{A(\bar{\zeta})} = -A(\zeta)$, $\overline{B(\bar{\zeta})} = -C(\zeta)$. Here we consider the $su(2)$ AKNS system instead of general 2×2 AKNS system because the Darboux transformation may exist globally in this case.

If $\begin{pmatrix} \alpha(\zeta) \\ \beta(\zeta) \end{pmatrix}$ is a solution of (1.294), then $\begin{pmatrix} -\bar{\beta}(\bar{\zeta}) \\ \bar{\alpha}(\bar{\zeta}) \end{pmatrix}$ is also its solution. This leads to the following property.

Property 8. For the Lax pair (1.294), if $\zeta \in \mathbf{R}$, then there are following relations among the Jost solutions and the scattering data:

$$
\begin{aligned}
\widetilde{R}_1 &= -\bar{R}_2, \quad \widetilde{R}_2 = \bar{R}_1, \\
\widetilde{L}_1 &= \bar{L}_2, \quad \widetilde{L}_2 = -\bar{L}_1,
\end{aligned}
\tag{1.295}
$$

$$
\widetilde{r}_+(\zeta) = \bar{r}_-(\zeta), \qquad \widetilde{r}_-(\zeta) = \bar{r}_+(\zeta). \tag{1.296}
$$

By reordering the eigenvalues,

$$
\widetilde{d} = d, \quad \widetilde{\zeta}_k = \bar{\zeta}_k, \quad \widetilde{C}_k = -\bar{C}_k, \quad \widetilde{b}(\zeta) = \bar{b}(\zeta) \quad (\zeta \in \mathbf{R}). \tag{1.297}
$$

Therefore, for the $su(2)$ AKNS system, the scattering data can be reduced to $\zeta_k \in \mathbf{C}_+$, C_k $(k = 1, 2, \cdots, d)$ and $b(\zeta)$ $(\zeta \in \mathbf{R})$.

Now we consider the change of the scattering data under Darboux transformations.

From the discussion on the nonlinear Schrödinger hierarchy, we know that if p is defined globally on $(-\infty, +\infty)$, so is the Darboux matrix. In order to use the scattering theory, we want that p and its derivatives tend to 0 fast enough at infinity.

Take a constant μ and a column solution of the Lax pair

$$
\psi_r(\zeta_0) - \mu\psi_l(\zeta_0) = \begin{pmatrix} R_1(\zeta_0)\mathrm{e}^{-\mathrm{i}\zeta_0 x} - \mu L_1(\zeta_0)\mathrm{e}^{\mathrm{i}\zeta_0 x} \\ R_2(\zeta_0)\mathrm{e}^{-\mathrm{i}\zeta_0 x} - \mu L_2(\zeta_0)\mathrm{e}^{\mathrm{i}\zeta_0 x} \end{pmatrix} \tag{1.298}
$$

$(\zeta_0 \in \mathbf{C}_+)$.

Let

$$
\sigma = \frac{R_2(\zeta_0) - \mu L_2(\zeta_0)\mathrm{e}^{2\mathrm{i}\zeta_0 x}}{R_1(\zeta_0) - \mu L_1(\zeta_0)\mathrm{e}^{2\mathrm{i}\zeta_0 x}} \tag{1.299}
$$

be the ratio of the second and the first components. Then the Darboux matrix is

$$\begin{aligned}&-\mathrm{i}\zeta I-S\\&=-\mathrm{i}\zeta I-\frac{1}{1+|\sigma|^2}\begin{pmatrix}-\mathrm{i}\zeta_0-\mathrm{i}\bar{\zeta}_0|\sigma|^2 & (-\mathrm{i}\zeta_0+\mathrm{i}\bar{\zeta}_0)\bar{\sigma}\\(-\mathrm{i}\zeta_0+\mathrm{i}\bar{\zeta}_0)\sigma & -\mathrm{i}\bar{\zeta}_0-\mathrm{i}\zeta_0|\sigma|^2\end{pmatrix},\end{aligned}\tag{1.300}$$

and the solution is transformed by

$$p'=p+2\mathrm{i}\frac{(\bar{\zeta}_0-\zeta_0)\bar{\sigma}}{1+|\sigma|^2}.\tag{1.301}$$

The change of the scattering data under Darboux transformation is given by the following theorem [75].

THEOREM 1.22 *If the scattering data for (1.294) are $r_-(\zeta)$ ($\zeta\in\mathbf{C}_+\cup\mathbf{R}$), $r_+(\zeta)$ ($\zeta\in\mathbf{R}$) and $\alpha(\zeta_k)$ ($k=1,\cdots,d$), then, under the action of the Darboux matrix (1.300) ($\mu\neq 0$, $\zeta_0\in\mathbf{C}_+$), the scattering data are changed as follows:*

(1) If ζ_0 is not an eigenvalue, then, after the action of the Darboux transformation, the number of eigenvalues increase one. All the original eigenvalues are not changed, and ζ_0 is a unique additional eigenvalue. Moreover,

$$\begin{aligned}&r'_-(\zeta)=\frac{\zeta-\zeta_0}{\zeta-\bar{\zeta}_0}r_-(\zeta)\quad(\zeta\in\mathbf{C}_+\cup\mathbf{R}),\\&r'_+(\zeta)=r_+(\zeta)\quad(\zeta\in\mathbf{R}),\\&\alpha'(\zeta_k)=\alpha(\zeta_k)\quad(k=1,\cdots,d),\\&\alpha'(\zeta_0)=1/\mu,\end{aligned}\tag{1.302}$$

hence

$$\begin{aligned}&b'(\zeta)=\frac{\zeta-\bar{\zeta}_0}{\zeta-\zeta_0}b(\zeta)\quad(\zeta\in\mathbf{R}),H\\&C'_k=\frac{\zeta_k-\bar{\zeta}_0}{\zeta_k-\zeta_0}C_k\quad(k=1,\cdots,d),H\\&C'_0=\frac{\zeta_0-\bar{\zeta}_0}{\mu r_-(\zeta_0)}.\end{aligned}\tag{1.303}$$

(2) If ζ_0 is an eigenvalue: $\zeta_0=\zeta_j$, and $\mu\neq\alpha(\zeta_j)$, then, after the action of the Darboux transformation, ζ_0 is no longer an eigenvalue.

Moreover,

$$\begin{aligned} r'_-(\zeta) &= \frac{\zeta - \bar{\zeta}_0}{\zeta - \zeta_0} r_-(\zeta) \quad (\zeta \in \mathbf{C}_+ \cup \mathbf{R}), \\ r'_+(\zeta) &= r_+(\zeta) \quad (\zeta \in \mathbf{R}), \\ \alpha'(\zeta_k) &= \alpha(\zeta_k) \quad (k = 1, \cdots, d,\ k \neq j), \end{aligned} \tag{1.304}$$

and

$$\begin{aligned} b'(\zeta) &= \frac{\zeta - \zeta_0}{\zeta - \bar{\zeta}_0} b(\zeta) \quad (\zeta \in \mathbf{R}), H \\ C'_k &= \frac{\zeta_k - \zeta_0}{\zeta_k - \bar{\zeta}_0} C_k \quad (k = 1, \cdots, d,\ k \neq j). \end{aligned} \tag{1.305}$$

Proof. (1) $\zeta_0 \notin IP\sigma(L)$.

Then, both the numerator and denominator of (1.299) are not 0. Property 3 implies

$$\lim_{x \to -\infty} \sigma = \infty, \qquad \lim_{x \to +\infty} \sigma = 0. \tag{1.306}$$

Hence

$$\begin{aligned} \lim_{x \to -\infty} (-\mathrm{i}\zeta I - S) &= \begin{pmatrix} -\mathrm{i}\zeta + \mathrm{i}\bar{\zeta}_0 & 0 \\ 0 & -\mathrm{i}\zeta + \mathrm{i}\zeta_0 \end{pmatrix}, \\ \lim_{x \to +\infty} (-\mathrm{i}\zeta I - S) &= \begin{pmatrix} -\mathrm{i}\zeta + \mathrm{i}\zeta_0 & 0 \\ 0 & -\mathrm{i}\zeta + \mathrm{i}\bar{\zeta}_0 \end{pmatrix}. \end{aligned} \tag{1.307}$$

Under the action of the Darboux transformation, the Jost solutions are changed to

$$\begin{aligned} \psi'_r(x,t,\zeta) &= \frac{1}{-\mathrm{i}\zeta + \mathrm{i}\bar{\zeta}_0} (-\mathrm{i}\zeta I - S) \psi_r(x,t,\zeta), \\ \psi'_l(x,t,\zeta) &= \frac{1}{-\mathrm{i}\zeta + \mathrm{i}\bar{\zeta}_0} (-\mathrm{i}\zeta I - S) \psi_l(x,t,\zeta). \end{aligned} \tag{1.308}$$

Hence

$$R' = \frac{1}{-\mathrm{i}\zeta + \mathrm{i}\bar{\zeta}_0} (-\mathrm{i}\zeta I - S) R. \tag{1.309}$$

If $\zeta \in \mathbf{C}_+$,

$$\lim_{x \to +\infty} R' = \begin{pmatrix} \dfrac{\zeta - \zeta_0}{\zeta - \bar{\zeta}_0} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_-(\zeta) \\ 0 \end{pmatrix}. \tag{1.310}$$

Thus

$$r'_-(\zeta) = \frac{\zeta - \zeta_0}{\zeta - \bar{\zeta}_0} r_-(\zeta), \tag{1.311}$$

$r'_-(\zeta)$ has an additional zero ζ_0 than $r_-(\zeta)$. This means that ζ_0 is a new eigenvalue. For $\zeta \in \mathbf{R}$,

$$R' \sim \begin{pmatrix} \dfrac{\zeta-\zeta_0}{\zeta-\bar{\zeta}_0} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_-(\zeta) \\ r_+(\zeta)\mathrm{e}^{2\mathrm{i}\zeta x} \end{pmatrix}. \tag{1.312}$$

Hence $r'_+(\zeta) = r_+(\zeta)$, and

$$b'(\zeta) = \frac{\zeta-\bar{\zeta}_0}{\zeta-\zeta_0} b(\zeta). \tag{1.313}$$

If ζ_k is a zero of $r_-(\zeta)$, then (1.308) implies $\alpha'(\zeta_k) = \alpha(\zeta_k)$, and

$$C'_k = \alpha'(\zeta_k) \Bigg/ \frac{\mathrm{d}r'_-(\zeta_k)}{\mathrm{d}\zeta} = \frac{\zeta_k-\bar{\zeta}_0}{\zeta_k-\zeta_0} C_k. \tag{1.314}$$

When $\zeta = \zeta_0$,

$$\begin{aligned} \psi'_r(x,t,\zeta_0) &= \frac{1}{1+|\sigma|^2} \begin{pmatrix} |\sigma|^2 & -\bar{\sigma} \\ -\sigma & 1 \end{pmatrix} \psi_r(x,t,\zeta_0), \\ \psi'_l(x,t,\zeta_0) &\frac{1}{1+|\sigma|^2} \begin{pmatrix} |\sigma|^2 & -\bar{\sigma} \\ -\sigma & 1 \end{pmatrix} \psi_l(x,t,\zeta_0), \end{aligned} \tag{1.315}$$

$$\alpha'(\zeta_0) = \frac{\sigma L_1 \exp(\mathrm{i}\zeta_0 x) - L_2 \exp(\mathrm{i}\zeta_0 x)}{\sigma R_1 \exp(-\mathrm{i}\zeta_0 x) - R_2 \exp(-\mathrm{i}\zeta_0 x)} = \frac{1}{\mu}, \tag{1.316}$$

$$C'_0 = \alpha'(\zeta_0) / \frac{\mathrm{d}r'_-(\zeta_0)}{\mathrm{d}\zeta} = \frac{\zeta_0-\bar{\zeta}_0}{\mu r_-(\zeta_0)}. \tag{1.317}$$

(1) is proved.

(2) $\zeta_0 = \zeta_j \in IP\sigma(L)$, $\mu \neq \alpha(\zeta_j)$.

Now

$$\sigma = \frac{R_2(\zeta_j)}{R_1(\zeta_j)} = \frac{L_2(\zeta_j)}{L_1(\zeta_j)}, \tag{1.318}$$

hence

$$\lim_{x\to-\infty} \sigma = 0, \quad \lim_{x\to+\infty} \sigma = \infty, \tag{1.319}$$

$$\begin{aligned} \lim_{x\to-\infty} (-\mathrm{i}\zeta I - S) &= \begin{pmatrix} -\mathrm{i}\zeta + \mathrm{i}\zeta_0 & 0 \\ 0 & -\mathrm{i}\zeta + \mathrm{i}\bar{\zeta}_0 \end{pmatrix}, \\ \lim_{x\to+\infty} (-\mathrm{i}\zeta I - S) &= \begin{pmatrix} -\mathrm{i}\zeta + \mathrm{i}\bar{\zeta}_0 & 0 \\ 0 & -\mathrm{i}\zeta + \mathrm{i}\zeta_0 \end{pmatrix}. \end{aligned} \tag{1.320}$$

Under the action of the Darboux transformation, the Jost solutions become

$$\begin{aligned}\psi_r'(x,t,\zeta) &= \frac{1}{-\mathrm{i}\zeta+\mathrm{i}\zeta_0}(-\mathrm{i}\zeta I - S)\psi_r(x,t,\zeta),\\ \psi_l'(x,t,\zeta) &= \frac{1}{-\mathrm{i}\zeta+\mathrm{i}\zeta_0}(-\mathrm{i}\zeta I - S)\psi_l(x,t,\zeta).\end{aligned} \tag{1.321}$$

For $\zeta \in \mathbf{C}_+$,

$$\lim_{x\to+\infty} R' = \begin{pmatrix} \dfrac{\zeta-\bar\zeta_0}{\zeta-\zeta_0} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_-(\zeta) \\ 0 \end{pmatrix}, \tag{1.322}$$

and for $\zeta \in \mathbf{R}$,

$$R' \sim \begin{pmatrix} \dfrac{\zeta-\bar\zeta_0}{\zeta-\zeta_0} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_-(\zeta) \\ r_+(\zeta)\mathrm{e}^{2\mathrm{i}\zeta x} \end{pmatrix}. \tag{1.323}$$

Hence

$$\begin{aligned} r_-'(\zeta) &= \frac{\zeta-\bar\zeta_0}{\zeta-\zeta_0} r_-(\zeta) \quad (\zeta \in \mathbf{C}_+ \cup \mathbf{R}),\\ r_+'(\zeta) &= r_+(\zeta) \quad (\zeta \in \mathbf{R}),\end{aligned} \tag{1.324}$$

and

$$b'(\zeta) = \frac{\zeta-\zeta_0}{\zeta-\bar\zeta_0} b(\zeta). \tag{1.325}$$

From (1.324) we know that the Darboux transformation removes the eigenvalue ζ_0 $(=\zeta_j)$.

If $\zeta = \zeta_k$ $(k \neq j)$, then $\psi_r' = \alpha(\zeta_k)\psi_l'$, hence

$$\begin{aligned} \alpha'(\zeta_k) &= \alpha(\zeta_k),\\ C_k' &= \alpha'(\zeta_k)\Big/\frac{\mathrm{d}r_-'(\zeta_k)}{\mathrm{d}\zeta} = \frac{\zeta_k-\zeta_0}{\zeta_k-\bar\zeta_0} C_k. \end{aligned} \tag{1.326}$$

The theorem is proved.

We have given the formulae for the change of the scattering data under Darboux transformation in the $su(2)$ case. A Darboux transformation increase or decrease the number of eigenvalues (number of solitons). However, it does not affect the scattering data related to the continuous spectrum. Thus we can use the Darboux transformation to change a general inverse scattering problem to an inverse scattering problem without eigenvalues.

Remark 17 *For the KdV equation, $q = 1$ (or -1) in the Lax pair (1.286). Since q does not tend to zero at infinity, the above conclusions can not be applied directly. However, the inverse scattering theory for the KdV equation is actually simpler than the AKNS system (see [23]). The conclusions similar to Theorem 1.22 for the KdV equation holds true as well [29].*

Chapter 2

2+1 DIMENSIONAL INTEGRABLE SYSTEMS

This chapter is devoted to the Darboux transformations of 2+1 dimensional integrable systems. Starting from the KP equation, we discuss the Darboux transformation for 2+1 dimensional AKNS system and more general systems. Unlike the Darboux matrices in 1+1 dimensions, the Darboux transformations here are given by differential operators (called Darboux operators). The construction of the Darboux operators is uniform to all the equations in the system, as in the 1+1 dimensional case. The binary Darboux transformation, which is a kind of Darboux transformation in integral form, is introduced briefly. Explicit solutions of the DSI equation can be obtained by the combination of Darboux transformation and binary Darboux transformation. Moreover, the nonlinear constraint method is used to separate the differentials in the 2+1 dimensional AKNS system so that the Darboux transformation in 1+1 dimensions can be used to get the localized soliton solutions.

2.1 KP equation and its Darboux transformation

A 2+1 dimensional integrable system has three independent variables (x, y, t) where x and y usually refer to space variables and t refers to time variable. A typical 2+1 dimensional integrable partial differential equation is the Kadomtsev-Petviashvili equation (KP equation) [68]

$$u_{xt} = (u_{xxx} + 6uu_x)_x + 3\alpha^2 u_{yy}, \tag{2.1}$$

where $\alpha = \pm 1$ or $\pm \mathrm{i}$. (2.1) is called the KPI equation if $\alpha = \pm 1$, and the KPII equation if $\alpha = \pm \mathrm{i}$. The KP equation is the natural generalization of the KdV equation, which describes the motion of two dimensional water wave. (2.1) can also be written as

$$v_{xt} = v_{xxxx} + 6v_x v_{xx} + 3\alpha^2 v_{yy} \tag{2.2}$$

where v satisfies $v_x = u$. The KP equation has a Lax pair

$$\phi_y = \alpha^{-1}\phi_{xx} + \alpha^{-1}u\phi, \tag{2.3}$$

$$\phi_t = 4\phi_{xxx} + 6u\phi_x + 3(\alpha v_y + u_x)\phi. \tag{2.4}$$

We first derive ϕ_{yt} by differentiating (2.3) with respect to t and inserting the expression of ϕ_t. Also, we can derive ϕ_{ty} by differentiating (2.4) with respect to y and inserting the expression of ϕ_y. The equality $\phi_{yt} = \phi_{ty}$ is equivalent to (2.1) when $\phi \neq 0$. The proof of this fact is direct, which is left for the reader. Therefore, (2.1) is the integrability condition of the overdetermined system (2.3) and (2.4).

The Darboux transformation for the KP equation is similar with that for the KdV equation. It can be constructed as follows. Let h be a solution of the Lax pair (2.3) and (2.4). For any solution ϕ of (2.3) and (2.4), define

$$\phi' = \phi_x - (h_x/h)\phi, \tag{2.5}$$

then ϕ' is a solution of

$$\begin{aligned} &\phi'_y = \alpha^{-1}\phi'_{xx} + \alpha^{-1}u'\phi', \\ &\phi'_t = 4\phi'_{xxx} + 6u'\phi'_x + 3(\alpha v'_y + u'_x)\phi' \end{aligned} \tag{2.6}$$

where

$$u' = u + 2(h_x/h)_x, \qquad v' = v + 2h_x/h. \tag{2.7}$$

Comparing (2.6) with (2.3) and (2.4), the only difference is that (u, ϕ) is changed to (u', ϕ'). Hence (2.7) gives a new solution u' of the KP equation [77].

Similar to 1+1 dimensions, if the seed solution u is simple enough, we can solve the Lax pair (2.3) and (2.4) to get h, then (2.5) gives a more complicated solution of the KP equation. Especially, if $u = v = 0$, then (2.3) and (2.4) becomes

$$\begin{aligned} &\phi_y = \alpha^{-1}\phi_{xx} \\ &\phi_t = 4\phi_{xxx}. \end{aligned} \tag{2.8}$$

Therefore, for any solution h of (2.8) with $h \neq 0$, $u' = 2(h_x/h)_x$ gives a solution of the KP equation.

Example 2.1 *For $\alpha = 1$, h can be chosen as*

$$h = e^{\lambda x+\lambda^2 y+4\lambda^3 t} + 1, \tag{2.9}$$

where λ is a real constant, then

$$u' = \frac{\lambda^2}{2} \operatorname{sech}^2 \Big(\frac{1}{2}(\lambda x + \lambda^2 y + 4\lambda^3 t)\Big) \tag{2.10}$$

is a solution of the KPI equation.

Example 2.2 *For $\alpha = -\mathrm{i}$, let*

$$h = e^{\lambda x+\mathrm{i}\lambda^2 y+4\lambda^3 t} + e^{-\bar\lambda x+\mathrm{i}\bar\lambda^2 y-4\bar\lambda^3 t}, \tag{2.11}$$

where $\lambda = a + b\mathrm{i}$ is a complex constant, then we obtain a solution of the KPII equation:

$$u' = 2a^2 \operatorname{sech}^2(ax - 2aby + 4(a^3 - 3ab^2)t). \tag{2.12}$$

These two solutions are both travelling waves, i.e., they are of form $u' = f(t + a_1 x + a_2 y)$ and u' is invariant along the line $t + a_1 x + a_2 y =$ constant on the (x, y) plane. For fixed t, u' is a non-zero constant along certain lines (for KPI, they are $\lambda x+\lambda^2 y =$ constant, while for KPII, they are $ax - 2aby =$ constant), and u' tends to zero exponentially at infinity along other lines. Hence the region where u' is far from zero forms a band on the (x, y) plane. This kind of solutions are call "line-solitons". This does not happen in 1+1 dimensions.

Suppose we have known a solution u of the KP equation and a set of solutions $\{\phi\}$ of the corresponding Lax pair. Let h be a special ϕ, then $u' = u + 2(\ln h)_{xx}$ is a solution of the KP equation. Moreover, $\phi' = \phi_x - (h_x/h)\phi$ gives the set of solutions of the Lax pair for u'. Now we take a special ϕ' as h', then we can obtain another solution $u'' = u' + 2(\ln h')_{xx}$ of the KP equation and the solution $\phi'' = \phi'_x - (h'_x/h')\phi'$ of the corresponding Lax pair by constructing Darboux transformation with h'. Continuing this procedure, we obtain a series of solutions of the KP equation without solving differential equations.

Except the first step, this algorithm can be realized by algebraic computation and differentiations. Therefore, it can be done by symbolic calculation. The solutions are global for all (x, y, t) if $h, h', h'' \cdots$ do not equal zero. This process can be expressed as

$$(u, \phi) \longrightarrow (u', \phi') \longrightarrow (u'', \phi'') \longrightarrow \cdots. \tag{2.13}$$

The differential operator of order three on the right hand side of (2.4) can be changed to differential operators of arbitrary order, then we get the KP hierarchy

$$\begin{aligned}\phi_y &= \alpha^{-1}\phi_{xx} + \alpha^{-1}u\phi,\\ \phi_t &= \sum_{j=0}^{n} v_{n-j}\partial^j\phi,\end{aligned} \tag{2.14}$$

$(\partial = \partial/\partial x)$. Computing the integrability condition of (2.14) and letting all the coefficients of the derivatives of ϕ with respect to x be zero, we have

$$2v_{j+1,x} = \alpha v_{j,y} - v_{j,xx} + \sum_{k=0}^{j-1} C_{n-k}^{n-j} v_k \partial^{j-k} u, \tag{2.15}$$

$$u_t = \alpha v_{n,y} - v_{n,xx} + \sum_{k=0}^{n-1} v_k \partial^{n-k} u. \tag{2.16}$$

In (2.15), v_{j+1} can be solved by integration. Unlike the 1+1 dimensional systems such as the KdV hierarchy, here, in general, v_j's can not be expressed as differential polynomials of u. Therefore, $(u, v_1, \cdots, v_n)$ are regarded as a set of unknowns of (2.15) – (2.16). The Darboux transformation is still valid for this system. In practical problems, some additional relations among $(v_1, \cdots, v_n, u)$ should be satisfied. This is called a reduction of the original one. In that case, we should choose proper h so that the relations among $v_1, \cdots, v_n, u$ keeps after the Darboux transformation. Usually this is a difficult problem and some special cases can be solved by certain techniques.

2.2 2+1 dimensional AKNS system and DS equation

2+1 dimensional AKNS system is

$$\Phi_y = J\Phi_x + P\Phi, \qquad \Phi_t = \sum_{j=0}^{n} V_{n-j}\partial^j\Phi, \tag{2.17}$$

where J is an $N \times N$ constant diagonal matrix, $P(x, y, t)$ is an off-diagonal $N \times N$ matrix, $V_j(x, y, t)$'s are also $N \times N$ matrices, $\partial = \partial/\partial x$. For simplicity, we assume that the diagonal entries of J are distinct. Moreover, we consider the non-degenerate Φ only.

The integrability condition of (2.17) leads to

$$[J, V_{j+1}^{\text{off}}] = V_{j,y}^{\text{off}} - JV_{j,x}^{\text{off}} - [P, V_j]^{\text{off}} + \sum_{k=0}^{j-1} C_{n-k}^{n-j} (V_k\partial^{j-k}P)^{\text{off}}, \tag{2.18}$$

$$V_{j,y}^{\text{diag}} - JV_{j,x}^{\text{diag}} = [P, V_j^{\text{off}}]^{\text{diag}} - \sum_{k=0}^{j-1} C_{n-k}^{n-j} (V_k \partial^{j-k} P)^{\text{diag}}, \tag{2.19}$$

$$P_t = V_{n,y}^{\text{off}} - JV_{n,x}^{\text{off}} - [P, V_n]^{\text{off}} + \sum_{k=0}^{n-1} (V_k \partial^{n-k} P)^{\text{off}}. \tag{2.20}$$

Here the superscripts "diag" and "off" refer to the diagonal and off-diagonal part of a matrix respectively.

Usually V_j's are not differential polynomials of P. But they can be generated from P by differentiation and integration with respect to x. (2.19) and (2.20) are regarded as a system of partial differential equations for P and V_j^{diag}'s $(j = 0, 1, \cdots, n)$ where V_j^{off}'s $(j = 1, \cdots, n)$ are determined by (2.18). (2.17) is the Lax pair of this system of equations.

A typical equation in 2+1 dimensional AKNS system is the Davey-Stewartson equation (DS equation), which is the natural generalization of the nonlinear Schrödinger equation in 2+1 dimensions.

Take $N = 2$, $n = 2$ in (2.17) and let

$$\begin{aligned}
&J = \alpha^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & u \\ -\epsilon \bar{u} & 0 \end{pmatrix}, \\
&\alpha = \pm 1 \text{ or } \pm \mathrm{i}, \qquad \epsilon = \pm 1, \\
&V_0 = 2\mathrm{i}\alpha J, \qquad V_1 = 2\mathrm{i}\alpha P, \\
&V_2 = \mathrm{i}\alpha \begin{pmatrix} w_1 & u_x + \alpha u_y \\ -\epsilon \bar{u}_x + \alpha \epsilon \bar{u}_y & w_2 \end{pmatrix}.
\end{aligned} \tag{2.21}$$

where u, w_1, w_2 are complex-valued functions, $\bar{u}$ is the complex conjugate of u. Then, (2.17) becomes

$$\begin{aligned}
\Phi_y = {} & \alpha^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_x + \begin{pmatrix} 0 & u \\ -\epsilon \bar{u} & 0 \end{pmatrix} \Phi, \\
\Phi_t = {} & 2\mathrm{i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_{xx} + 2\mathrm{i}\alpha \begin{pmatrix} 0 & u \\ -\epsilon \bar{u} & 0 \end{pmatrix} \Phi_x \\
& + \mathrm{i}\alpha \begin{pmatrix} w_1 & u_x + \alpha u_y \\ -\epsilon \bar{u}_x + \alpha \epsilon \bar{u}_y & w_2 \end{pmatrix} \Phi.
\end{aligned} \tag{2.22}$$

(2.19) and (2.20) lead to

$$\begin{aligned} &\mathrm{i}u_t = -u_{xx} - \alpha^2 u_{yy} - \alpha u(w_1 - w_2), \\ &w_{1,y} - \alpha^{-1} w_{1,x} = \epsilon(|u|^2)_x + \alpha\epsilon(|u|^2)_y, \\ &w_{2,y} + \alpha^{-1} w_{2,x} = \epsilon(|u|^2)_x - \alpha\epsilon(|u|^2)_y, \end{aligned} \tag{2.23}$$

and $\overline{w_2 - w_1} = \alpha^{-2}(w_2 - w_1)$. Denote

$$v = -\epsilon|u|^2 + \frac{1}{2\alpha}(w_1 - w_2), \tag{2.24}$$

then (2.23) becomes

$$\begin{aligned} &\mathrm{i}u_t = -u_{xx} - \alpha^2 u_{yy} - 2\epsilon\alpha^2|u|^2 u - 2\alpha^2 uv, \\ &v_{xx} - \alpha^2 v_{yy} + 2\epsilon(|u|^2)_{xx} = 0. \end{aligned} \tag{2.25}$$

(2.25) is called the DSI equation if $\epsilon = 1$, $\alpha = \pm 1$, and DSII equation if $\epsilon = 1$, $\alpha = \pm\mathrm{i}$. They describe the motion of long wave and short wave in the water of finite depth [20].

2.3 Darboux transformation

2.3.1 General Lax pair

Similar to the KP equation, we also want to construct the Darboux transformation for the AKNS system. Here we first discuss the Darboux transformation for the following more general Lax pair without any reductions.

Consider Lax pair

$$\Phi_y = U(x, y, t, \partial)\Phi, \qquad \Phi_t = V(x, y, t, \partial)\Phi, \tag{2.26}$$

where

$$\begin{aligned} U(x, y, t, \partial) &= \sum_{j=0}^{m} U_{m-j}(x, y, t)\partial^j, \\ V(x, y, t, \partial) &= \sum_{j=0}^{n} V_{n-j}(x, y, t)\partial^j \end{aligned} \tag{2.27}$$

are differential operators with respect to x whose coefficients U_j's and V_j's are $N \times N$ matrices. For simplicity, we write $U(x, y, t, \partial) = U(\partial)$, $V(x, y, t, \partial) = V(\partial)$.

$\Phi_{yt} = \Phi_{ty}$ can be obtained by differentiating the first equation of (2.26) with respect to t or by differentiating the second equation with respect to x. Let these two equal, we get

$$U_t(\partial) - V_y(\partial) + [U(\partial), V(\partial)] = 0. \tag{2.28}$$

(2.26) is called integrable if (2.28) holds. (2.28) is the generalization of the zero-curvature equations in 1+1 dimensions. It gives a system of partial differential equations by equating all the coefficients of ∂ to be zero.

Remark 18 *The existence and uniqueness of the solutions of a system of partial differential equations are very difficult problems. The local solvability of a system of linear partial differential equations have been studied by many authors. In the present case, even (2.28) holds, local solution of (2.26) near $t = t_0$, $y = y_0$ with initial data $\Phi(t_0, x, y_0) = \Phi_0(x)$ may not exist. However, if each set of equations in (2.26) is locally solvable and the solutions are smooth enough with respect to the parameters y and t, $U(\partial)$ and $V(\partial)$ satisfy (2.28), then (2.26) is locally solvable. This follows from the following consideration. Suppose that the initial data $(x_0, y_0, t_0, \Phi_0(x))$ are given. First solve the first set of equations of (2.26) at $t = t_0$ with initial value $\Phi_1(x, y_0) = \Phi_0(x)$ and get the solution $\Phi_1(x, y)$. Using $\Phi_1(x, y)$ as the initial value, solve the second set of equations of (2.26) for fixed y and get the solution $\Phi(x, y, t)$. Using (2.28) and the second equation of (2.26), we have*

$$(\Phi_y - U(\partial)\Phi)_t = V(\partial)(\Phi_y - U(\partial)\Phi). \tag{2.29}$$

Therefore, $\Phi_y = U(\partial)\Phi$ holds identically near (x_0, y_0, t_0) by the uniqueness of the solution.

No matter whether the existence and uniqueness hold, (2.28) is called the integrability condition of (2.26). It gives a system of nonlinear partial differential equations of $U(\partial)$ and $V(\partial)$. (2.26) is called the Lax pair of this system of nonlinear partial differential equations. It is interesting to see that we can apply Darboux transformation as well provided that the set of solutions of (2.26) is not empty.

2.3.2 Darboux transformation of degree one

Similar to 1+1 dimensional case, we can define Darboux operator for the integrable nonlinear partial differential equations (2.28) and there Lax pair (2.26).

Definition 2.3 *A differential operator $D(x, y, t, \partial)$ with respect to x is called a Darboux operator for (2.26) if there exist differential operators $U'(\partial)$ and $V'(\partial)$ with respect to x such that for any solution Φ of (2.26), $\Phi' = D(\partial)\Phi$ satisfies*

$$\Phi'_y = U'(\partial)\Phi', \qquad \Phi'_t = V'(\partial)\Phi'. \tag{2.30}$$

The transformation $(\Phi, U(\partial), V(\partial)) \to (\Phi', U'(\partial), V'(\partial))$ *given by* $D(\partial)$ *is called a Darboux transformation.*

Substituting $\Phi' = D\Phi$ into (2.30), we have

$$\begin{aligned} D_y(\partial) &= U'(\partial)D(\partial) - D(\partial)U(\partial), \\ D_t(\partial) &= V'(\partial)D(\partial) - D(\partial)V(\partial), \end{aligned} \tag{2.31}$$

and

$$U'_t(\partial) - V'_y(\partial) + [U'(\partial), V'(\partial)] = 0. \tag{2.32}$$

(2.31) is the necessary and sufficient condition for $D(\partial)$ being a Darboux operator. Hence, if $(U(\partial), V(\partial))$ is a solution of (2.28), so is $(U'(\partial), V'(\partial))$. This means that the Darboux transformation gives a new solution of (2.28). Our main task is to construct the solution D of (2.31).

We first discuss the most fundamental Darboux operator, the Darboux operator of degree one. This is the Darboux operator in the form $D(x, y, t, \partial) = \partial - S(x, y, t)$. The Darboux operator of higher degree will be discussed later. In order to get the general construction of S, we first derive the equations that S should satisfy.

For a matrix $M(x)$, we define a sequence of matrices $M^{(j)}$ by $M^{(0)} = I$ and

$$M^{(j+1)} = M_x^{(j)} + M^{(j)}M, \tag{2.33}$$

then, for any solution Φ of the equation $\Phi_x = M\Phi$, $\partial^j\Phi = M^{(j)}\Phi$ holds.

For any differential operator

$$U(\partial) = \sum_{j=0}^{k} U_{k-j}\partial^j, \qquad V(\partial) = \sum_{j=0}^{k} V_{k-j}\partial^j \tag{2.34}$$

and an $N \times N$ matrix S, we define

$$U(S) = \sum_{j=0}^{k} U_{k-j}S^{(j)}, \qquad V(S) = \sum_{j=0}^{k} V_{k-j}S^{(j)}. \tag{2.35}$$

Suppose that Φ satisfies $\Phi_x = S\Phi$, then $U(\partial)\Phi = U(S)\Phi$, $V(\partial)\Phi = V(S)\Phi$. Notice that $U(S)$ and $V(S)$ are not given by replacing ∂ in $U(\partial)$ and $V(\partial)$ with S. Actually they are obtained by replacing ∂^j with $S^{(j)}$.

THEOREM 2.4 *$\partial - S$ is a Darboux operator for (2.26) if and only if S satisfies*

$$\begin{aligned} S_y + [S, U(S)] &= (U(S))_x, \\ S_t + [S, V(S)] &= (V(S))_x. \end{aligned} \tag{2.36}$$

Proof. Suppose $\partial - S$ is a Darboux operator for (2.26), then the first equation of (2.31) is

$$S_y - (\partial - S)U(\partial) + U'(\partial)(\partial - S) = 0. \tag{2.37}$$

Let Ψ be the fundamental solution of $\Psi_x = S\Psi$, then

$$S_y\Psi = (\partial - S)U(S)\Psi = (U(S))_x\Psi - [S, U(S)]\Psi. \tag{2.38}$$

This gives the first equation of (2.36). The second equation is derived similarly. The necessity of (2.36) is proved.

Conversely, suppose S is a solution of (2.36). Define

$$U'(\partial) = \sum_{j=0}^{m} U'_{m-j}\partial^j, \tag{2.39}$$

where U'_j's are determined recursively by

$$\begin{aligned} &U'_0 = U_0, \\ &U'_{j+1} = U_{j+1} + U_{j,x} - SU_j + \sum_{k=0}^{j} C_{m-k}^{m-j} U'_k \partial^{j-k} S. \end{aligned} \tag{2.40}$$

Then

$$S_y - (\partial - S)U(\partial) + U'(\partial)(\partial - S) \tag{2.41}$$

does not contain any terms with ∂, i.e., it is a matrix-valued function of x, y and t. On the other hand, for any fundamental solution Φ of $\Psi_x = S\Psi$, (2.36) leads to

$$(S_y - (\partial - S)U(\partial) + U'(\partial)(\partial - S))\Psi = 0. \tag{2.42}$$

Hence, as a matrix,

$$S_y - (\partial - S)U(\partial) + U'(\partial)(\partial - S) = 0. \tag{2.43}$$

This shows that $\partial - S$ satisfies the first equation of (2.31). The second one can be proved similarly. Therefore, $\partial - S$ is a Darboux operator for (2.26). The theorem is proved.

THEOREM 2.5 *$\partial - S$ is a Darboux operator for (2.26) if and only if there exists an $N \times N$ non-degenerate matrix solution H of (2.26) such that $S = H_x H^{-1}$ [122].*

Proof. First we prove the sufficiency, i.e., to show the S satisfies (2.36). From (2.26),

$$\begin{aligned} S_y &= H_{xy}H^{-1} - SH_yH^{-1} = (U(S)H)_xH^{-1} - SU(S) \\ &= [U(S), S] + (U(S))_x, \end{aligned} \tag{2.44}$$

which is the first equation of (2.36). The second one is derived in the same way.

Now suppose S satisfies (2.36). We shall show that the system of equations

$$H_x = SH, \qquad H_y = U(\partial)H, \qquad H_t = V(\partial)H, \tag{2.45}$$

has a solution. Clearly (2.45) is equivalent to

$$H_x = SH, \qquad H_y = U(S)H, \qquad H_t = V(S)H. \tag{2.46}$$

Hence, we only need to verify the integrability conditions of (2.46).

Let Ψ be a fundamental solution of $\Psi_x = S\Psi$. (2.37) implies

$$(\Psi_y - U(\partial)\Psi)_x = (S\Psi)_y - \partial U(\partial)\Psi = S(\Psi_y - U(\partial)\Psi). \tag{2.47}$$

Hence

$$\begin{aligned} & (V_y(\partial) + V(\partial)U(\partial))\Psi \\ = \ & (V(\partial)\Psi)_y - V(\partial)(\Psi_y - U(\partial)\Psi) \\ = \ & (V(S)\Psi)_y - V(S)(\Psi_y - U(S)\Psi) \\ = \ & V(S)_y\Psi + V(S)U(S)\Psi. \end{aligned} \tag{2.48}$$

Similarly,

$$(U_t(\partial) + U(\partial)V(\partial))\Psi = U(S)_t\Psi + U(S)V(S)\Psi. \tag{2.49}$$

Since $\det\Psi \neq 0$, the integrability condition (2.28) gives

$$U(S)_t - V(S)_y + [U(S), V(S)] = 0. \tag{2.50}$$

Hence the integrability condition $H_{yt} = H_{ty}$ for (2.46) holds.

Theorem 2.4 gives the other two integrability conditions $H_{xy} = H_{yx}$ and $H_{xt} = H_{tx}$. Hence (2.46) is integrable. For given initial value $H = H_0$ at $(t, x, y) = (t_0, x_0, y_0)$, (2.46) has a solution H. If H_0 is non-degenerate, H is also non-degenerate in a neighborhood of (t_0, x_0, y_0). That is, (2.46) has a non-degenerate matrix solution H such that $S = H_xH^{-1}$. The theorem is proved.

If there is no reduction, this theorem shows that any Darboux operator in the form $\partial - S$ can be expressed explicitly by the solutions of the Lax pair. Darboux transformation exists as long as the Lax pair has a non-degenerate $N \times N$ matrix solution. Under the Darboux transformation, U_j is transformed to U'_j given by (2.40). V'_j's have similar expressions.

Thus, we have constructed the Darboux transformation

$$(U, V, \Phi) \longrightarrow (U', V', \Phi'). \tag{2.51}$$

This process can be continued by algebraic and differential operations to get infinite number of solutions provided that the set of solutions of the Lax pair for the seed solution is big enough.

For the AKNS system (2.17), the action of the Darboux operator $\partial - S$ gives

$$(\partial - S)(J\partial + P) - S_x = (J\partial + P')(\partial - S). \tag{2.52}$$

The coefficients of ∂^2 on both sides are equal. Comparing the coefficient of ∂, we have

$$P' = P + [J, S]. \tag{2.53}$$

For practical problems, the entries of U and V often have some constraint relations. In that case, H in the theorem should also satisfy certain conditions so that (U', V') and (U, V) satisfy the same constraints. If so, we can obtain a transformation from a solution of a nonlinear partial differential equation to a solution of the same equation.

Remark 19 For the KP equation, the construction for the Darboux operator is completely the same as in Section 2.1. However, for the Davey-Stewartson equation, it is more difficult because we should consider the relations among the entries of P. We shall discuss it in Section 2.4.

Similar with the 1+1 dimensional case, we can also compose several Darboux transformations of degree one to a Darboux transformation of higher degree. However, they can be constructed directly with explicit formulae.

2.3.3 Darboux transformation of higher degree and the theorem of permutability

Now we discuss a Darboux operator of higher degree. It is a differential operator in the form

$$D(\partial) = \sum_{j=0}^{r} D_{r-j}\partial^j, \qquad D_0 = I \tag{2.54}$$

such that

$$\begin{aligned} D_y(\partial) &= U'(\partial)D(\partial) - D(\partial)U(\partial), \\ D_t(\partial) &= V'(\partial)D(\partial) - D(\partial)V(\partial). \end{aligned} \tag{2.55}$$

Here $U'(\partial)$ and $V'(\partial)$ are differential operators with respect to x.

For simplicity, we only discuss the Darboux operator of degree two. When $r > 2$, the Darboux operator can also be written down explicitly, but is more complicated.

THEOREM 2.6 *Let H_1 and H_2 be two $N \times N$ non-degenerate matrix solutions of (2.26). Let F be the block matrix*

$$\begin{pmatrix} H_1 & H_2 \\ \partial H_1 & \partial H_2 \end{pmatrix}. \tag{2.56}$$

Suppose $\det F \neq 0$*, then the following conclusions hold:*

(1) There is a unique differential operator of degree two

$$D(H_1, H_2, \partial) = \partial^2 + D_1\partial + D_2 \tag{2.57}$$

satisfying

$$D(H_1, H_2, \partial)H_i = 0 \qquad (i = 1, 2). \tag{2.58}$$

It is a Darboux operator.

(2) The theorem of permutability holds:

$$D(H_1, H_2, \partial) = D(H_2, H_1, \partial). \tag{2.59}$$

(3) There is a decomposition

$$D(H_1, H_2, \partial) = D(D(H_1, \partial)H_2, \partial)D(H_1, \partial). \tag{2.60}$$

Proof. Since $\det F \neq 0$, the linear algebraic system

$$D_1\partial H_1 + D_2H_1 = -\partial^2 H_1, \quad D_1\partial H_2 + D_2H_2 = -\partial^2 H_2 \tag{2.61}$$

for D_1, D_2 has a unique solution, which determines $D(\partial)$ uniquely and $D(\partial)$ satisfies (2.58). Since (2.61) is symmetric with respect to H_1 and H_2, (2) holds.

By the definitions of $D(D(H_1, \partial)H_2, \partial)$ and $D(H_1, \partial)$,

$$\begin{aligned} &D(D(H_1, \partial)H_2, \partial)D(H_1, \partial)H_1 = 0, \\ &D(D(H_1, \partial)H_2, \partial)D(H_1, \partial)H_2 = 0. \end{aligned} \tag{2.62}$$

Hence (2.60) holds. From (2.60) it is seen that $D(H_1, H_2, \partial)$ is a Darboux operator because it is the composition of two Darboux operators of degree one.

Similar to (1.134), the theorem of permutability can be expressed by the following diagram:

$$
\begin{array}{ccccc}
 & & (U^{(1)}, V^{(1)}, \Phi^{(1)}) & & \\
 & \overset{H_1}{\nearrow} & & \overset{H_2}{\searrow} & \\
 & & & & (U^{(1,2)}, V^{(1,2)}, \Phi^{(1,2)}) \\
(U, V, \Phi) & & & & \| \\
 & & & & (U^{(2,1)}, V^{(2,1)}, \Phi^{(2,1)}) \\
 & \underset{H_2}{\searrow} & & \underset{H_1}{\nearrow} & \\
 & & (U^{(2)}, V^{(2)}, \Phi^{(2)}) & &
\end{array}
\tag{2.63}
$$

EXAMPLE 2.7 *For the KP equation, $N = 1$, we can get the expression of u after the Darboux transformation. Denote $H_i = h_i$. Suppose the Darboux operator is*

$$\sum_{j=0}^{r} D_{r-j}\partial^j, \tag{2.64}$$

then Theorem 2.6 implies

$$\sum_{j=0}^{r-1} D_{r-j}\partial^j h_i = -\partial^r h_i, \tag{2.65}$$

i.e.,

$$(D_r, \cdots, D_1)F_r = -(\partial^r h_1, \cdots, \partial^r h_r). \tag{2.66}$$

Solving this system, we have

$$
\begin{aligned}
D_1 &= -\det \begin{pmatrix} h_1 & \partial h_1 & \cdots & \partial^{r-2} h_1 & \partial^r h_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_r & \partial h_r & \cdots & \partial^{r-2} h_r & \partial^r h_r \end{pmatrix} \cdot \\
&\quad \cdot \left(\det \begin{pmatrix} h_1 & \partial h_1 & \cdots & \partial^{r-2} h_1 & \partial^{r-1} h_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_r & \partial h_r & \cdots & \partial^{r-2} h_r & \partial^{r-1} h_r \end{pmatrix} \right)^{-1} \\
&= -(\ln \det F_r)_x.
\end{aligned}
\tag{2.67}
$$

Therefore, for the KP equation, the transformation between two solutions is

$$u' = u - 2D_{1,x} = u + 2(\ln\det F_r)_{xx}. \tag{2.68}$$

Many solutions can be obtained in this way [77, 80, 89].

2.4 Darboux transformation and binary Darboux transformation for DS equation

2.4.1 Darboux transformation for DSII equation

In Section 2.2 we introduced the DSI and DSII equations (2.25) and their Lax pairs (2.17) and (2.21). Since the reductions in DSI equation and in DSII equation are different, the method of solving these two equations are also quite different.

First, consider the DSII equation, i.e., $\epsilon = 1$, $\alpha = -\mathrm{i}$ [120].

In this case, we should have $w_2 = \bar{w}_1$. Hence $v = -|u|^2 + \frac{\mathrm{i}}{2}(w_1 - \bar{w}_1)$, and J, P, V_j $(j = 0, 1, 2)$ are

$$\begin{aligned} &J = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}, \\ &V_0 = 2J, \quad V_1 = 2P, \quad V_2 = \begin{pmatrix} w_1 & u_x - \mathrm{i}u_y \\ -\bar{u}_x - \mathrm{i}\bar{u}_y & \bar{w}_1 \end{pmatrix}. \end{aligned} \tag{2.69}$$

J, P and V_j have the properties

$$\bar{J} = \sigma J \sigma^{-1}, \quad \bar{P} = \sigma P \sigma^{-1}, \quad \bar{V}_j = \sigma V_j \sigma^{-1} \quad (j = 0, 1, 2), \tag{2.70}$$

where

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.71}$$

$\bar{P}$ is the matrix each of whose entry is the complex conjugate of the corresponding entry of P. Now (2.25) becomes

$$\begin{aligned} &\mathrm{i}u_t = -u_{xx} + u_{yy} + 2|u|^2 u + 2uv, \\ &v_{xx} + v_{yy} + 2(|u|^2)_{xx} = 0. \end{aligned} \tag{2.72}$$

Its Lax pair is

$$\begin{aligned}\Phi_y =&\ \mathrm{i}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Phi_x + \begin{pmatrix}0 & u\\ -\bar{u} & 0\end{pmatrix}\Phi,\\ \Phi_t =&\ 2\mathrm{i}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Phi_{xx} + 2\begin{pmatrix}0 & u\\ -\bar{u} & 0\end{pmatrix}\Phi_x\\ &+\begin{pmatrix}w_1 & u_x - \mathrm{i}u_y\\ -\bar{u}_x - \mathrm{i}\bar{u}_y & \bar{w}_1\end{pmatrix}\Phi,\end{aligned} \tag{2.73}$$

with

$$v = -|u|^2 + \frac{i}{2}(w_1 - \bar{w}_1). \tag{2.74}$$

The Darboux operator for (2.73) is constructed as follows.

Suppose $(\xi,\,\eta)^T$ is a solution of (2.73), then $(-\bar{\eta},\,\bar{\xi})^T$ is also its solution. Hence we can choose

$$H = \begin{pmatrix}\xi & -\bar{\eta}\\ \eta & \bar{\xi}\end{pmatrix}, \tag{2.75}$$

$$S = H_x H^{-1} = \frac{1}{|\xi|^2 + |\eta|^2}\begin{pmatrix}\bar{\xi}\xi_x + \eta\bar{\eta}_x & \bar{\eta}\xi_x - \xi\bar{\eta}_x\\ \bar{\xi}\eta_x - \eta\bar{\xi}_x & \xi\bar{\xi}_x + \bar{\eta}\eta_x\end{pmatrix}. \tag{2.76}$$

Since $\bar{H} = \sigma H \sigma^{-1}$, we have $\bar{S} = \sigma S \sigma^{-1}$. The equations

$$\begin{aligned}U'(\partial)(\partial - S) &= (\partial - S)U(\partial) - S_y,\\ V'(\partial)(\partial - S) &= (\partial - S)V(\partial) - S_t\end{aligned} \tag{2.77}$$

imply

$$\begin{aligned}\bar{U}' &= \sigma U' \sigma^{-1},\\ \bar{V}' &= \sigma V' \sigma^{-1}.\end{aligned} \tag{2.78}$$

This means that the Darboux transformation keeps the reduction relations (2.70) invariant.

After the action of the Darboux operator $\partial - S$,

$$\begin{aligned}P' &= P + [J,S],\\ V_2' &= V_2 + V_{1,x} + 2V_0 S_x + [V_0,S]S + [V_1,S].\end{aligned} \tag{2.79}$$

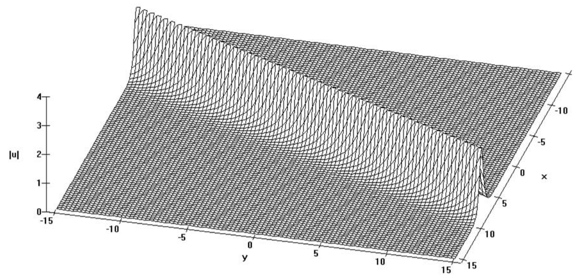


Figure 2.1. Single line-soliton, $t = 0$

Hence the new solution of the DSII equation is

$$\begin{aligned} u' &= u + 2\mathrm{i}S_{12} = u + 2\mathrm{i}\frac{\bar{\eta}\xi_x - \xi\bar{\eta}_x}{|\xi|^2 + |\eta|^2}, \\ v' &= v - 2(\operatorname{Re} S_{11})_x = v - (\ln(|\xi|^2 + |\eta|^2))_{xx}. \end{aligned} \tag{2.80}$$

EXAMPLE 2.8 *Take the seed solution* $u = 0$*, then we can choose* $v = 0$ *(*$w_1 = 0$*),* $\xi = \xi(x + \mathrm{i}y, t)$*,* $\eta = \eta(x - \mathrm{i}y, t)$ *(i.e.,* ξ *is analytic with respect to* $x + \mathrm{i}y$ *and* η *is analytic with respect to* $x - \mathrm{i}y$*) satisfying* $\xi_t = 2\mathrm{i}\xi_{xx}$*,* $\eta_t = -2\mathrm{i}\eta_{xx}$*. For these* (ξ, η)*,* (u, v) *given by (2.80) are all solutions of DSII equation. Especially, let* $\xi = e^{\alpha x + \mathrm{i}\alpha y + 2\mathrm{i}\alpha^2 t}$*,* $\eta = e^{\bar{\beta}x - \mathrm{i}\bar{\beta}y - 2\mathrm{i}\bar{\beta}^2 t}$*, then*

$$\begin{aligned} u &= \frac{2\mathrm{i}(\alpha - \beta)e^{(\alpha+\beta)x + \mathrm{i}(\alpha+\beta)y + 2\mathrm{i}(\alpha^2+\beta^2)t}}{e^{2\mathrm{Re}\alpha x - 2\mathrm{Im}\alpha y - 2\mathrm{Im}(\alpha^2)t} + e^{2\mathrm{Re}\beta x - 2\mathrm{Im}\beta y - 2\mathrm{Im}(\beta^2)t}}, \\ v &= -\frac{4(\operatorname{Re}\alpha - \operatorname{Re}\beta)^2 e^{2\mathrm{Re}(\alpha+\beta)x - 2\mathrm{Im}(\alpha+\beta)y - 2\mathrm{Im}(\alpha^2+\beta^2)t}}{(e^{2\mathrm{Re}\alpha x - 2\mathrm{Im}\alpha y - 2\mathrm{Im}(\alpha^2)t} + e^{2\mathrm{Re}\beta x - 2\mathrm{Im}\beta y - 2\mathrm{Im}(\beta^2)t})^2}. \end{aligned} \tag{2.81}$$

When t *is fixed, the solution* u *is a constant along the line with slope* $x : y = \mathrm{Im}(\beta - \alpha) : \mathrm{Re}(\beta - \alpha)$*, and tends to zero in any other directions. This kind of solution also belongs to "line-soliton".*

Multi-line-solitons can be obtained by successive Darboux transformations. They tend to zero at infinity except for finitely many directions.

Figures 2.1 – 2.4 show the single line-soliton and multi-line-solitons, where the parameters are $\alpha_1 = 3 + 2\mathrm{i}$, $\beta_1 = 1 + \mathrm{i}$, $\alpha_2 = \mathrm{i}$, $\beta_2 = (2 + \mathrm{i})/4$. (For the single line-soliton, only (α_1, β_1) is used.)

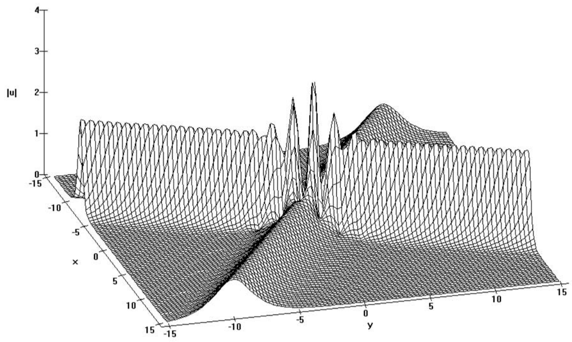


Figure 2.2. Double line-soliton, $t = 0$

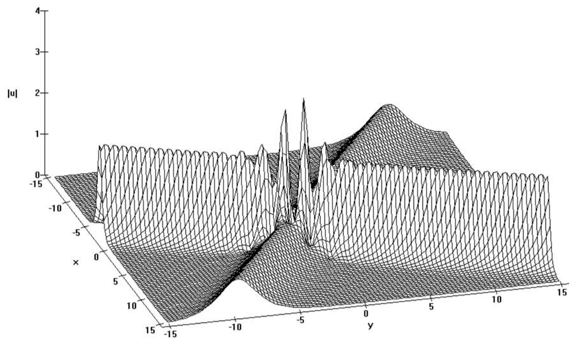


Figure 2.3. Double line-soliton, $t = 0.5$

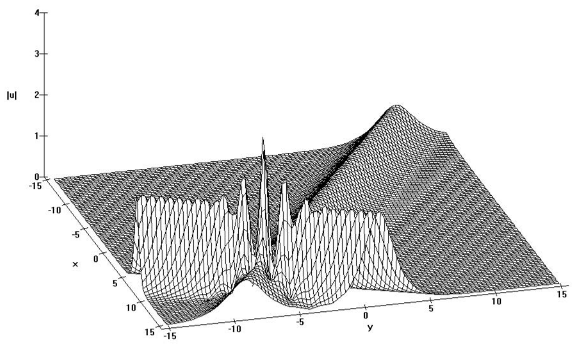


Figure 2.4. Double line-soliton, $t = 1$

Remark 20 Comparing to the general method discussed in the last section, the key point in the construction of Darboux transformation for the DSII equation is the choice of H in (2.75). Although it is successful to the DSII equation, it can not be applied to the DSI equation.

2.4.2 Darboux transformation and binary Darboux transformation for DSI equation

When $\epsilon = 1$ and $\alpha = 1$, (2.25) becomes

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + u_{yy} + 2|u|^2 u + 2uv = 0, \\ &v_{xx} - v_{yy} + 2(|u|^2)_{xx} = 0, \end{aligned} \tag{2.82}$$

and its Lax pair is

$$\begin{aligned} \Phi_y =& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_x + \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \Phi, \\ \Phi_t =& 2\mathrm{i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_{xx} + 2\mathrm{i} \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \Phi_x \\ &+ \mathrm{i} \begin{pmatrix} w_1 & u_x + u_y \\ -\bar{u}_x + \bar{u}_y & w_2 \end{pmatrix} \Phi, \end{aligned} \tag{2.83}$$

where

$$v = -|u|^2 + \frac{1}{2}(w_1 - w_2), \tag{2.84}$$

w_1 and w_2 are real functions.

Since we can not find the solution H of (2.83) like that of (2.75) to construct the Darboux transformation for (2.82), we should use the binary Darboux transformation. The binary Darboux transformation was first introduced by V. B. Matveev et al and has many applications [4, 81, 80, 72, 125, 126]. Here we show its application to DSI equation for constructing new solutions. For the general case, please refer to [80].

For simplicity, rewrite (2.83) as

$$\begin{aligned} &\Phi_y = J\Phi_x + P\Phi, \\ &\Phi_t = 2\mathrm{i}J\Phi_{xx} + 2\mathrm{i}P\Phi_x + \mathrm{i}V_2\Phi. \end{aligned} \tag{2.85}$$

Apart from this Lax pair, consider its adjoint equations

$$\begin{aligned} &\Psi_y = \Psi_x J - \Psi P, \\ &\Psi_t = -2\mathrm{i}\Psi_{xx}J + 2\mathrm{i}(\Psi P)_x - \mathrm{i}\Psi V_2. \end{aligned} \tag{2.86}$$

With $P^* = -P$ and $V_2^* = V_2 - 2P_x$, we know that if Φ is a solution of (2.85), then $\Psi = \Phi^*$ is a solution of (2.86), and vice versa. Therefore, as soon as a solution of (2.85) or (2.86) is known, a solution of its adjoint equations is also known.

Similar to Section 2.3, we first take Darboux transformation for the adjoint equation (2.86):

$$\begin{aligned} &\Psi' = \Psi_x - \Psi S, \\ &P' = P + [J, S], \\ &V_2' = V_2 - 2P_x + 2[P, S] - 2S[J, S] + 2(JS_x + S_x J) \end{aligned} \tag{2.87}$$

where

$$S = \Psi_0^{-1}\Psi_{0,x}, \tag{2.88}$$

Ψ_0 is a non-degenerate 2×2 matrix solution of (2.86). Notice that P' does not satisfy $P'^* = -P'$, and Ψ'^* is not a solution of (2.85) with P replaced by P'. In order to preserve the reduction, the binary Darboux transformation is a useful tool. To get a new solution of the DSI equation, it needs the following steps:

Step 1: For a solution Φ of (2.85) and a solution Ψ of the adjoint equations (2.86), define 1-form

$$\omega(\Psi, \Phi) = \Psi\Phi\,\mathrm{d}x + \Psi J\Phi\,\mathrm{d}y + 2\mathrm{i}(\Psi P\Phi + \Psi J\Phi_x - \Psi_x J\Phi)\,\mathrm{d}t. \tag{2.89}$$

It can be verified that $\omega(\Psi, \Phi)$ is a closed 1-form, that is, its exterior differential $\mathrm{d}\omega(\Psi, \Phi) = 0$. Hence, in a simply connected region, the integral of ω along any closed curve is zero. In $\mathbf{R}^{2,1}$, define

$$\Omega(\Psi, \Phi)(x, y, t) = \int_{(x_0, y_0, t_0)}^{(x,y,t)} \omega(\Psi, \Phi), \tag{2.90}$$

which is independent of the path of integration, and $\omega(\Psi, \Phi) = \mathrm{d}\Omega(\Psi, \Phi)$.

Step 2: Let $\Phi' = \Psi_0^{-1}\Omega(\Psi_0, \Phi)$, then we can verify that Φ' is a solution of (2.85) with (P, V_2) replaced by (P', V_2').

Step 3: Let $\Phi_0' = \Psi_0^{-1}\Omega(\Psi_0, \Psi_0^*)$. Acting the Darboux operator $\partial - \Phi_0'\Phi_0'^{-1}$ on Φ', we get the Darboux transformation

$$\begin{aligned} &\Phi'' = \Phi_x' - \Phi_{0,x}'\Phi_0'^{-1}\Phi' = \Phi - \Psi_0^*\Omega(\Psi_0, \Psi_0^*)^{-1}\Omega(\Psi_0, \Phi), \\ &P'' = P' + [J, \Psi_{0,x}'\Psi_0'^{-1}] = P + [J, \Psi_0^*\Omega(\Psi_0, \Psi_0^*)^{-1}\Psi_0]. \end{aligned} \tag{2.91}$$

$P^* = -P$ leads to $P''^* = -P''$. Therefore, we get a new solution of the DSI equation.

The process in Step 1 and Step 2 is called a binary Darboux transformation. For the DSI equation, a new solution is obtained by the composition of a Darboux transformation and a binary Darboux transformation. It needs differentiation and integration in this procedure.

2.5 Application to 1+1 dimensional Gelfand-Dickey system

In this section, we use Theorem 2.5 to discuss the Darboux transformation for the (1+1 dimensional) Gelfand-Dickey system

$$\begin{aligned}\lambda\Phi &= U(x,t,\partial)\Phi,\\ \Phi_t &= V(x,t,\partial)\Phi\end{aligned} \tag{2.92}$$

where

$$\begin{aligned}U(\partial) &= \sum_{j=0}^{m} U_{m-j}(x,t)\partial^j,\\ V(\partial) &= \sum_{j=0}^{n} V_{n-j}(x,t)\partial^j.\end{aligned} \tag{2.93}$$

From the first equation of (2.92), we can compute Φ_t and it should be the same as that given by the second equation of of (2.92). This gives the integrability condition

$$U_t(\partial) + [U(\partial), V(\partial)] = 0 \tag{2.94}$$

of (2.92).

Let $D(x,t,\partial)$ be a differential operator. If for any solution Φ of (2.92), $\Phi' = D(\partial)\Phi$ satisfies

$$\begin{aligned}\lambda\Phi' &= U'(x,t,\partial)\Phi',\\ \Phi'_t &= V'(x,t,\partial)\Phi',\end{aligned} \tag{2.95}$$

where U' and V' are differential operators of the form

$$\begin{aligned}U'(\partial) &= \sum_{j=0}^{m} U'_{m-j}(x,t)\partial^j,\\ V'(\partial) &= \sum_{j=0}^{n} V'_{n-j}(x,t)\partial^j,\end{aligned} \tag{2.96}$$

then $D(x,t,\partial)$ is called a Darboux operator for (2.92).

For a differential operator $D(x,t,\partial) = \partial - S(x,t)$, we have the following theorem.

THEOREM 2.9 *$\partial - S(x,t)$ is a Darboux operator for (2.92) if and only if $S = H_x H^{-1}$, where H is an $N \times N$ non-degenerate matrix solution of*

$$\begin{aligned} H\Lambda &= U(\partial)H, \\ H_t &= V(\partial)H, \end{aligned} \tag{2.97}$$

and Λ is a constant upper-triangular matrix.

Proof. Introduce a new variable y and consider the system

$$\begin{aligned} \Psi_y &= U(x,t,\partial)\Psi, \\ \Psi_t &= V(x,t,\partial)\Psi. \end{aligned} \tag{2.98}$$

If $\partial - S(x,t)$ is a Darboux operator for (2.92), then there exist $U'(\partial)$ and $V'(\partial)$ such that

$$\begin{aligned} 0 &= (\partial - S)U(\partial) - U'(\partial)(\partial - S), \\ S_t &= (\partial - S)V(\partial) - V'(\partial)(\partial - S). \end{aligned} \tag{2.99}$$

Since S is independent of y and (2.37) holds, $\partial - S$ is a Darboux operator for (2.98) which is independent of y. According to Theorem 2.5, there exists an $N \times N$ non-degenerate matrix solution H_0 of (2.98) such that $S = H_{0,x}H_0^{-1}$. Here H_0 may depend on y.

Let $L_0 = H_0^{-1}H_{0,y}$, (2.36) leads to

$$\begin{aligned} L_0 = {} & H_0^{-1}U(S)H_0, \\ L_{0,x} = {} & -H_0^{-1}H_{0,x}H_0^{-1}U(S)H_0 + H_0^{-1}(U(S))_x H_0 \\ & +H_0^{-1}U(S)H_{0,x} \\ = {} & H_0^{-1}\{(U(S))_x - [S, U(S)]\}H_0 = 0, \end{aligned}$$

and (2.50) leads to

$$\begin{aligned} L_{0,y} = {} & -H_0^{-1}H_{0,y}H_0^{-1}U(S)H_0 + H_0^{-1}U(S)H_{0,y} = 0, \\ L_{0,t} = {} & -H_0^{-1}H_{0,t}H_0^{-1}U(S)H_0 + H_0^{-1}(U(S))_t H_0 \\ & +H_0^{-1}U(S)H_{0,t} \\ = {} & H_0^{-1}\{(U(S))_t + [U(S), V(S)]\}H_0 = 0. \end{aligned}$$

Hence L_0 is a constant matrix. Therefore, there exists a constant upper-triangular matrix Λ and a constant matrix T such that $L_0 = T\Lambda T^{-1}$. According to the definition of L_0,

$$H_{0,y} = H_0 T\Lambda T^{-1}.$$

Hence

$$H_0(x,y,t) = H(x,t)\exp(\Lambda y)T^{-1}$$

where H satisfies (2.97) and $S = H_x H^{-1}$.

Conversely, if H is a solution (2.97) and $S = H_x H^{-1}$, then S satisfies (2.99), i.e., $\partial - S$ is a Darboux operator for (2.92). The theorem is proved.

Remark 21 (1) When $N = 1$, H satisfies the Lax pair (2.92).

(2) If L_0 in the above theorem is diagonalizable, then each column in H satisfies the Lax pair (2.92) for specific λ.

EXAMPLE 2.10 *The original Darboux transformation for the KdV equation can also be deduced from the above theorem.*

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2.100}$$

has the Lax pair

$$\begin{aligned} \lambda\phi &= -\phi_{xx} - u\phi, \\ \phi_t &= 2(2\lambda - u)\phi_x + u_x\phi. \end{aligned} \tag{2.101}$$

From Theorem 2.9, the Darboux transformation is

$$\phi' = \phi_x - \frac{f_x}{f}\phi \tag{2.102}$$

where f is a solution of the Lax pair for $\lambda = \lambda_0$. The new solution given by this Darboux transformation is

$$u' = u + 2(\ln f)_{xx}. \tag{2.103}$$

EXAMPLE 2.11 *The Boussinesq equation*

$$(u_{xxx} + 6uu_x)_x + 3\epsilon u_{tt} = 0 \quad (\epsilon = \pm 1)$$

has the Lax pair

$$\begin{aligned} \lambda\phi &= \phi_{xxx} + \frac{3u}{2}\phi_x + w\phi, \\ \phi_t &= \sigma\phi_{xx} + \sigma u\phi \end{aligned} \tag{2.104}$$

($\sigma^2 = \epsilon$, $w_x = 3(\sigma u_{xx} + u_t)/4\sigma$). Theorem 2.9 also gives the Darboux transformation [73]

$$\phi' = \phi_x - \frac{f_x}{f}\phi \tag{2.105}$$

where f is a solution of the Lax pair for $\lambda = \lambda_0$. The new solution of the Boussinesq equation is

$$\begin{aligned} u' &= u + 2(\ln f)_{xx}, \\ w' &= w + \frac{3}{2}u_x + 3((\ln f)_{xxx} + (\ln f)_x(\ln f)_{xx}). \end{aligned} \tag{2.106}$$

2.6 Nonlinear constraints and Darboux transformation in 2+1 dimensions

Now we come back to the 2+1 dimensional AKNS system. In this section we will use the nonlinear constraint method and the Darboux transformation method to solve this system.

The basic idea of the nonlinear constraint method is:

(1) Find a suitable nonlinear relation between U and Ψ and express U as a nonlinear matrix function of Ψ: $U = f(\Psi)$.

(2) Substitute $U = f(\Psi)$ into the Lax pair so that the original Lax pair becomes a system of nonlinear partial differential equations of Ψ. In each equation, the derivative with respect to only one of x, y, t is concerned.

(3) The constraint $U = f(\Psi)$ is suitable so that the new system of nonlinear equations has a Lax set (generalized Lax pair).

Then by solving the new system of nonlinear equations and its Lax set, we can get solutions of the original problem.

This idea was first applied in 1+1 dimensional integrable systems [11] and was generalized to the (2+1 dimensional) KP equation [14, 71]. Here we pay our attention to the 2+1 dimensional AKNS system so that we can get localized soliton solutions. With this method, we can also get a lot of non-localized solutions [123, 124]. However, since localized solutions are more interesting, here we only consider localized solutions [127, 128].

In order to use the nonlinear constraint method, here we add some conditions on the 2+1 dimensional AKNS system. As in Section 2.2, the 2+1 dimensional AKNS system is

$$\begin{aligned} \Psi_y &= J\Psi_x + U(x,y,t)\Psi, \\ \Psi_t &= \sum_{j=0}^{n} V_j(x,y,t)\partial^{n-j}\Psi \end{aligned} \tag{2.107}$$

where $\partial = \partial/\partial x$, $J = \mathrm{diag}(J_1, \cdots, J_N)$ is a constant diagonal $N \times N$ matrix with distinct entries. $U(x,y,t)$ is off-diagonal. Moreover, here we want that all J_j's are real and $U^* = -U$. In this case, we call (2.107) a hyperbolic $u(N)$ AKNS system. The condition $U^* = -U$ will imply that the solutions are globally defined, and the condition J_j's are real will guarantee that there exist localized solutions.

As in Section 2.2, the integrability conditions of (2.107) are given by (2.20).

Now we introduce a new linear system

$$\begin{aligned}\Phi_x &= \begin{pmatrix} i\lambda I & iF \\ iF^* & 0 \end{pmatrix}\Phi, \qquad \Phi_y = \begin{pmatrix} i\lambda J + U & iJF \\ iF^*J & 0 \end{pmatrix}\Phi, \\ \Phi_t &= \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\Phi = \sum_{j=0}^{n} \begin{pmatrix} W_j & X_j \\ -X_j^* & Z_j \end{pmatrix}\lambda^{n-j}\Phi\end{aligned} \tag{2.108}$$

where F, W_j, X_j, Z_j are $N\times K$, $N\times N$, $N\times K$, $K\times K$ matrices respectively ($K\geq 1$) and satisfy $W_j^* = -W_j$, $Z_j^* = -Z_j$.

The integrability conditions of (2.108) consists of

$$\begin{aligned}F_y &= JF_x + UF, \\ iF_t &= X_{n,x} + iW_nF - iFZ_n,\end{aligned} \tag{2.109}$$

$$\begin{aligned}&iX_{j+1} = X_{j,x} + iW_jF - iFZ_j \quad (j = 0, 1, \cdots, n-1) \\ &W_{j,x} = -iFX_j^* - iX_jF^* \quad (j = 0, 1, \cdots, n) \\ &Z_{j,x} = iF^*X_j + iX_j^*F \quad (j = 0, 1, \cdots, n) \\ &i[J, W_{j+1}] = W_{j,y} - [U, W_j] + iJFX_j^* + iX_jF^*J \\ &\quad (j = 0, 1, \cdots, n-1) \\ &Z_{j,y} = iF^*JX_j + iX_j^*JF \quad (j = 0, 1, \cdots, n),\end{aligned} \tag{2.110}$$

$$U_x = [J, FF^*], \tag{2.111}$$

$$U_t = W_{n,y} - [U, W_n] + iJFX_n^* + iX_nF^*J. \tag{2.112}$$

For $U = 0$, $F = 0$, (2.110) implies that $W_j(\lambda) = i\Omega_j(t)$, $X_j = 0$, $Z_j = iZ_j^0(t)$ where $\Omega_j(t)$'s are real diagonal matrices and $Z_j^0(t)$'s are real matrices.

When $Z_j^0(t) = \zeta_j(t)I_K$ (I_K is the $K\times K$ identity matrix) where $\zeta_j(t)$ is a real function of t, (2.109) is just the Lax pair (2.107) for $n = 1, 2, 3$. (2.110) and (2.112) give the recursion relations to determine W_j, X_j, Z_j, together with the evolution equations corresponding to (2.18)–(2.20), which are the integrability conditions of (2.107). (2.111) gives a nonlinear constraint between U and F.

This system includes the DSI equation and the 2+1 dimensional N-wave equation as special cases.

In order to consider the asymptotic behavior of the solution U, here we suppose Ω_j is independent of t and $\zeta_j = 0$. Moreover, denote $\Omega = \sum_{j=0}^{n} \Omega_j \lambda^{n-j}$ and write $\Omega = \mathrm{diag}(\omega_1, \cdots, \omega_N)$.

Remark 22 (2.108) is a special case of the high-dimensional generalized AKNS system (3.1). Here we only consider this special system to find localized solutions. The general theory will be discussed in the next chapter.

The soliton solutions are obtained by Darboux transformations from $U = 0$, $F = 0$. In the present case, the Darboux transformation can be constructed as in Subsection 1.4.4 with $u(n)$ reduction. However, in order to get localized solutions, there should be more restrictions on the parameters of Darboux transformations.

Let λ_α $(\alpha = 1, 2, \cdots, r)$ be r non-real complex numbers such that $\lambda_\alpha \neq \lambda_\beta$ for $\alpha \neq \beta$ and $\lambda_\alpha \neq \bar{\lambda}_\beta$ for all α, β. Let

$$\Lambda_\alpha = \mathrm{diag}(\underbrace{\lambda_\alpha, \cdots, \lambda_\alpha}_{N}, \underbrace{\bar{\lambda}_\alpha, \cdots, \bar{\lambda}_\alpha}_{K}). \tag{2.113}$$

Considering the orthogonal relation (1.241), we can always take

$$H_\alpha = \begin{pmatrix} \exp(Q_\alpha) & -\exp(-Q_\alpha^*)C_\alpha^* \\ C_\alpha & I_K \end{pmatrix}, \tag{2.114}$$

where C_α's are $K \times N$ constant matrices,

$$Q_\alpha = \mathrm{diag}(q_1, \cdots, q_N), \quad q_j = \mathrm{i}\lambda_\alpha x + \mathrm{i}\lambda_\alpha J_j y + \mathrm{i}\omega_j(\lambda_\alpha, t). \tag{2.115}$$

According to Section 1.4, the derived solutions are always global. However, in order to get localized solutions, we choose special

$$C_\alpha = (0, \cdots, 0, \underset{l_\alpha}{\kappa_\alpha}, 0, \cdots, 0) \tag{2.116}$$

where κ_α is a constant $K \times 1$ non-zero vector being the l_α's column of C_α.

The Darboux matrices for such $\{\Lambda_\alpha, H_\alpha\}$ can be constructed as follows. Let

$$\begin{aligned}
&D^{(1)}(\lambda) = \lambda - H_1\Lambda_1 H_1^{-1}, \quad H_\alpha^{(1)} = D^{(1)}(\lambda_\alpha)H_\alpha \\
&\qquad (\alpha = 2, 3, \cdots, r), \\
&D^{(2)}(\lambda) = \lambda - H_2^{(1)}\Lambda_2 H_2^{(1)-1}, \quad H_\alpha^{(2)} = D^{(2)}(\lambda_\alpha)H_\alpha^{(1)} \\
&\qquad (\alpha = 3, 4, \cdots, r), \\
&\cdots \\
&D^{(r)}(\lambda) = \lambda - H_r^{(r-1)}\Lambda_r H_r^{(r-1)-1},
\end{aligned} \tag{2.117}$$

$$D(\lambda) = D^{(r)}(\lambda)D^{(r-1)}(\lambda)\cdots D^{(1)}(\lambda), \tag{2.118}$$

then $D(\lambda)$ is a polynomial of λ of degree r. The permutability (Theorem 1.12) implies that if $(\Lambda_\alpha, H_\alpha)$ and (Λ_β, H_β) are interchanged, $D(\lambda)$ is invariant.

Let

$$m_j = \#\{\,\alpha \,|\, 1 \le \alpha \le r,\, l_\alpha = j\,\} \qquad m = (m_1, \cdots, m_N) \tag{2.119}$$

then $m_1 + \cdots + m_N = r$.

Suppose

$$D(\lambda) = \lambda^r - D_1\lambda^{r-1} + \cdots + (-1)^r D_r. \tag{2.120}$$

The solution given by this Darboux matrix is

$$U^{[m]} = \mathrm{i}[J, (D_1)_{B_N}]. \tag{2.121}$$

Here $(D_1)_{B_N}$ denotes the first $N \times N$ principal submatrix of D_1.

In order to consider the localization, the asymptotic behavior as $t \to \infty$ and the asymptotic behavior as the phase difference tends to infinity uniformly, we write

$$q_j = a_{\alpha j}s + b_{\alpha j} \tag{2.122}$$

where $a_{\alpha j}$ and $b_{\alpha j}$ are independent of s. Here s can be a linear parameter of a straight line in (x, y) plane, or time t, or any other parameters.

Moreover, denote

$$\begin{aligned}
\rho_\alpha &= \mathrm{Re}(a_{\alpha,l_\alpha}), \qquad & \phi_\alpha &= \mathrm{Im}(a_{\alpha,l_\alpha}), \\
\pi_\alpha &= \mathrm{Re}(b_{\alpha,l_\alpha}), \qquad & \psi_\alpha &= \mathrm{Im}(b_{\alpha,l_\alpha}).
\end{aligned} \tag{2.123}$$

In order to prove the following theorem, we need some symbols and simple facts.

If M_1, M_2 are $j \times k$ matrices, we write $M_1 \doteq M_2$ if there is a non-degenerate diagonal $k \times k$ matrix A such that $M_2 = M_1 A$.

If L is a $k\times k$ diagonal matrix, M_1 and M_2 are $k\times k$ matrices with $M_1 \doteq M_2$ and $\det M_1\neq 0$, then $M_1LM_1^{-1}=M_2LM_2^{-1}$.

Let

$$M=\begin{pmatrix} a & -v^*/\bar{a} \\ v & I_K \end{pmatrix} \tag{2.124}$$

where $v\neq 0$ is an $K\times 1$ vector, $a\neq 0$ is a number. Let

$$\Lambda=\begin{pmatrix} \lambda_0 & \\ & \bar{\lambda}_0 I_K \end{pmatrix}. \tag{2.125}$$

Then we have

$$M^{-1}=\frac{1}{\Delta}\begin{pmatrix} \bar{a} & v^* \\ -\bar{a}v & \Delta I_K - vv^* \end{pmatrix}, \tag{2.126}$$

$$M\Lambda M^{-1}=\frac{1}{\Delta}\begin{pmatrix} \bar{\lambda}_0\Delta+(\lambda_0-\bar{\lambda}_0)|a|^2 & (\lambda_0-\bar{\lambda}_0)av^* \\ (\lambda_0-\bar{\lambda}_0)\bar{a}v & \bar{\lambda}_0\Delta I_K+(\lambda_0-\bar{\lambda}_0)vv^* \end{pmatrix} \tag{2.127}$$

where $\Delta=v^*v+|a|^2$. Moreover,

$$\lim_{a\to\infty} M\Lambda M^{-1}=\begin{pmatrix} \lambda_0 & \\ & \bar{\lambda}_0 I_K \end{pmatrix}, \tag{2.128}$$

$$\lim_{a\to 0} M\Lambda M^{-1}=\begin{pmatrix} \bar{\lambda}_0 & \\ & \bar{\lambda}_0 I_K+(\lambda_0-\bar{\lambda}_0)\dfrac{vv^*}{v^*v} \end{pmatrix}. \tag{2.129}$$

THEOREM 2.12 *(1) If there is at most one* α $(1\leq\alpha\leq r)$ *such that* $\rho_\alpha=0$, *then* $\lim_{s\to\infty}U^{[m]}=0$.

(2) If $\rho_{\alpha_j}=0$ $(j=1,2,\cdots,q)$ *with* $\alpha_j\neq\alpha_k$ *for* $j\neq k$, $\rho_\gamma\neq 0$ *for all* $\gamma\neq\alpha_j$ $(j=1,\cdots,q)$ *and* $l_{\alpha_1}=\cdots=l_{\alpha_q}$, *then* $\lim_{s\to\infty}U^{[m]}=0$.

(3) If $\rho_\alpha=0$, $\rho_\beta=0$ $(\alpha\neq\beta)$, $\rho_\gamma\neq 0$ *for all* $\gamma\neq\alpha,\beta$, *and* $l_\alpha\neq l_\beta$, *then*

$$\lim_{s\to\infty}U^{[m]}_{ab}=0 \quad \textit{for } (a,b)\neq(l_\alpha,l_\beta) \tag{2.130}$$

and as $s\to\infty$,

$$U^{[m]}_{l_\alpha,l_\beta}\sim\frac{B_{\alpha\beta}\exp(\mathrm{i}(\psi_\alpha-\psi_\beta)+\mathrm{i}(\phi_\beta-\phi_\alpha)s)}{A_{\alpha\beta}\cosh(\pi_\alpha+\pi_\beta-\delta^{(1)}_{\alpha\beta})+\cosh(\pi_\alpha-\pi_\beta-\delta^{(2)}_{\alpha\beta})} \tag{2.131}$$

where $A_{\alpha\beta}$, $\delta^{(1)}_{\alpha\beta}$, $\delta^{(2)}_{\alpha\beta}$ *are real constants,* $A_{\alpha\beta} > 0$, *and* $B_{\alpha\beta}$ *are complex constants. Moreover, if* $K = 1$, *then* $B_{\alpha\beta} \neq 0$ *if and only if* $\kappa_\alpha \neq 0$ *and* $\kappa_\beta \neq 0$.

Proof. First suppose $\rho_\alpha \neq 0$. By (2.128) and (2.129),

$$\lim_{\rho_\alpha s \to \pm\infty} H_\alpha \Lambda_\alpha H_\alpha^{-1} = S_\alpha^{\pm\infty} \tag{2.132}$$

where

$$\begin{aligned} S_\alpha^{+\infty} &= \begin{pmatrix} \lambda_\alpha I_N & \\ & \bar\lambda_\alpha I_K \end{pmatrix}, \\ S_\alpha^{-\infty} &= \begin{pmatrix} \lambda_\alpha I_N + (\bar\lambda_\alpha - \lambda_\alpha) E_{l_\alpha l_\alpha} & \\ & \bar\lambda_\alpha I_K + (\lambda_\alpha - \bar\lambda_\alpha) \dfrac{\kappa_\alpha \kappa_\alpha^*}{\kappa_\alpha^* \kappa_\alpha}, \end{pmatrix}, \end{aligned} \tag{2.133}$$

E_{jk} is an $N \times N$ matrix whose (j, k)th entry is 1 and the rest entries are zero.

For $\beta \neq \alpha$,

$$(\lambda_\beta - S_\alpha^{\pm\infty}) H_\beta \doteq \begin{pmatrix} \exp(Q_\beta(s)) & -\exp(-Q_\beta(s)^*) \widetilde{C}_\beta^{\pm *} \\ \widetilde{C}_\beta^{\pm} & I_K \end{pmatrix} \tag{2.134}$$

where

$$\begin{aligned} \widetilde{C}_\beta^{\pm} &= (0, \cdots, 0, \underset{l_\beta}{\widetilde{\kappa}_\beta^{\pm}}, 0, \cdots, 0), \\ \widetilde{\kappa}_\beta^{+} &= \frac{\lambda_\beta - \bar\lambda_\alpha}{\lambda_\beta - \lambda_\alpha} \kappa_\beta, \\ \widetilde{\kappa}_\beta^{-} &= \begin{cases} \dfrac{\lambda_\beta - \bar\lambda_\alpha}{\lambda_\beta - \lambda_\alpha} \kappa_\beta - \dfrac{\lambda_\alpha - \bar\lambda_\alpha}{\lambda_\beta - \lambda_\alpha} \dfrac{\kappa_\alpha^* \kappa_\beta}{\kappa_\alpha^* \kappa_\alpha} \kappa_\alpha & \text{if } l_\beta \neq l_\alpha, \\ \kappa_\beta - \dfrac{\lambda_\alpha - \bar\lambda_\alpha}{\lambda_\beta - \bar\lambda_\alpha} \dfrac{\kappa_\alpha^* \kappa_\beta}{\kappa_\alpha^* \kappa_\alpha} \kappa_\alpha & \text{if } l_\beta = l_\alpha. \end{cases} \end{aligned} \tag{2.135}$$

Therefore, if $\rho_\alpha \neq 0$, the action of the limit Darboux matrix $\lambda - S_\alpha^{\pm\infty}$ on H_β ($\beta \neq \alpha$) does not change the form of H_β, but only changes the constant vector κ_β.

If $K = 1$, then $\kappa_\beta^{\pm *} \kappa_\gamma^{\pm} \neq 0$ implies $\widetilde{\kappa}_\beta^{\pm *} \widetilde{\kappa}_\gamma^{\pm} \neq 0$. When $K > 1$, this does not hold in general.

Now suppose $\rho_\alpha = 0$. Without loss of generality, suppose $l_\alpha = 1$. Then

$$H_\alpha \doteq \begin{pmatrix} \exp(\pi_\alpha) & & & & -\exp(-\bar{\pi}_\alpha)\kappa_\alpha^* \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ \kappa_\alpha & 0 & \cdots & 0 & I_K \end{pmatrix}. \tag{2.136}$$

By (2.127),

$$H_\alpha \Lambda_\alpha H_\alpha^{-1} = \frac{1}{\Delta} \cdot \begin{pmatrix} \bar{\lambda}_\alpha \Delta + (\lambda_\alpha - \bar{\lambda}_\alpha)\exp(\pi_\alpha + \bar{\pi}_\alpha) & & & & (\lambda_\alpha - \bar{\lambda}_\alpha)\exp(\pi_\alpha)\kappa_\alpha^* \\ & \lambda_\alpha & & & 0 \\ & & \ddots & & \vdots \\ & & & \lambda_\alpha & 0 \\ (\lambda_\alpha - \bar{\lambda}_\alpha)\exp(\bar{\pi}_\alpha)\kappa_\alpha & 0 & \cdots & 0 & \bar{\lambda}_\alpha \Delta I_K + (\lambda_\alpha - \bar{\lambda}_\alpha)\kappa_\alpha \kappa_\alpha^* \end{pmatrix} \tag{2.137}$$

where $\Delta = \exp(\pi_\alpha + \bar{\pi}_\alpha) + \kappa_\alpha^* \kappa_\alpha$.

Part (1) of the theorem is derived as follows. Owing to the permutability of Darboux transformations, we can suppose $\rho_1 \neq 0$, $\cdots$, $\rho_{r-1} \neq 0$, $\rho_r = 0$. Then, as $s \to \infty$, $D^{(\alpha)}$ tends to a diagonal matrix for $\alpha \leq r-1$. Considering (2.137), the limit of $(D^{(r)}(\lambda))_{B_N}$ is also diagonal, hence

$$U^{[m]} = \mathrm{i}[J, (D_1)_{B_N}] \to 0. \tag{2.138}$$

Now we turn to prove part (2). We use the construction of Darboux matrices in (1.258). However, the λ in (1.258) should be replaced by $\mathrm{i}\lambda$ because of its appearance in (2.108).

Let

$$\mathring{H}_\alpha = \begin{pmatrix} \exp(Q_\alpha(s)) \\ C_\alpha \end{pmatrix}, \quad \Gamma_{\alpha\beta} = \frac{\mathring{H}_\alpha^* \mathring{H}_\beta}{\lambda_\beta - \bar{\lambda}_\alpha}, \tag{2.139}$$

then the Darboux matrix is

$$D(\lambda) = \prod_{\alpha=1}^{r} (\lambda - \bar{\lambda}_\alpha) \left(1 - \sum_{\alpha,\beta=1}^{r} \frac{\mathring{H}_\alpha (\Gamma^{-1})_{\alpha\beta} \mathring{H}_\beta^*}{\lambda - \bar{\lambda}_\beta} \right) \tag{2.140}$$

and the new solution is

$$U^{[m]} = \mathrm{i} \left[J, \sum_{\alpha,\beta=1}^{r} \left(\mathring{H}_\alpha (\Gamma^{-1})_{\alpha\beta} \mathring{H}_\beta^* \right)_{B_N} \right]. \tag{2.141}$$

First, suppose $q = r$ and $\alpha_j = j$ $(j = 1, 2, \cdots, r)$.
Since

$$\overset{\circ}{H}_j \doteq \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \exp(\pi_j) & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ 0 & \cdots & 0 & \underset{l_j}{\kappa_j} & 0 & \cdots & 0 \end{pmatrix} \tag{2.142}$$

and $l_1 = \cdots = l_q$, we have

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \cdots & \Gamma_{1q} \\ \vdots & \ddots & \vdots \\ \Gamma_{q1} & \cdots & \Gamma_{qq} \end{pmatrix} \tag{2.143}$$

where Γ_{jk}'s are $N \times N$ diagonal matrices. Therefore,

$$\Gamma^{-1} = \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1q} \\ \vdots & \ddots & \vdots \\ \Sigma_{q1} & \cdots & \Sigma_{qq} \end{pmatrix} \tag{2.144}$$

where Σ_{jk}'s are also $N \times N$ diagonal matrices. This implies that $U^{[m]} = 0$.

When $r > q$, we use the permutability of Darboux transformations and suppose $\rho_1 \neq 0, \cdots, \rho_{r-q} \neq 0, \rho_{r-q+1} = \cdots = \rho_r = 0$. Then, after the action of $D^{(r-q)}(\lambda) \cdots D^{(1)}(\lambda)$, the derived $H^{(r-q)}_{r-q+1}$ and $H^{(r-q)}_r$ have the same asymptotic form as H_{r-q+1} and H_r respectively, provided that the constant vectors κ_{r-q+1} and κ_r are changed to $\kappa^{(r-q)}_{r-q+1}$ and $\kappa^{(r-q)}_r$. Therefore, as in the case $r = q$, the limits of the components of $U^{[m]}$ are all zero. This proved part (2).

Now we prove part (3). First consider the case $r = 2$. Suppose $\overset{\circ}{H}_j$ is given by (2.142) $(j = 1, 2)$ and $l_1 \neq l_2$. Denote

$$\theta_{12} = \frac{\kappa_1^* \kappa_2}{\sqrt{\kappa_1^* \kappa_1 \kappa_2^* \kappa_2}}, \tag{2.145}$$

$$g_{12} = 1 - \frac{4 \operatorname{Im} \lambda_1 \operatorname{Im} \lambda_2}{|\lambda_2 - \bar{\lambda}_1|^2} |\theta_{12}|^2 > 0, \tag{2.146}$$

then, by direct calculation, we have

$$\begin{aligned}
\lim_{s\to\infty} U^{[m]}_{l_1,l_2} \exp(\mathrm{i}(\phi_2 - \phi_1)s) &= \frac{2\mathrm{i}(J_{l_2} - J_{l_1}) \operatorname{Im} \lambda_1 \operatorname{Im} \lambda_2 \theta_{12}}{\bar{\lambda}_2 - \lambda_1} \cdot \\
&\cdot \frac{\exp(\mathrm{i}(\psi_1 - \psi_2))}{\sqrt{g_{12}} \cosh(\pi_1 + \pi_2 - \delta_1) + \cosh(\pi_1 - \pi_2 - \delta_2)}, \\
\delta_1 &= \frac{1}{2} \ln g_{12} + \frac{1}{2} \ln(\kappa_1^* \kappa_1 \kappa_2^* \kappa_2) + 2 \ln \left| \frac{\lambda_2 - \lambda_1}{\lambda_2 - \bar{\lambda}_1} \right|, \\
\delta_2 &= \frac{1}{2} \ln \frac{\kappa_1^* \kappa_1}{\kappa_2^* \kappa_2},
\end{aligned} \tag{2.147}$$

and $U^{[m]}_{\mu\nu} \to 0$ if $(\mu, \nu) \neq (l_1, l_2)$.

When $r > 2$, we still use the permutability of Darboux transformations and suppose $\rho_1 \neq 0, \cdots, \rho_{r-2} \neq 0, \rho_{r-1} = \rho_r = 0$. As in the proof of part (2), after the action of $D^{(r-2)}(\lambda) \cdots D^{(1)}(\lambda)$, the derived $H^{(r-2)}_{r-1}$ and $H^{(r-2)}_r$ have the same asymptotic form as H_{r-1} and H_r respectively, provided that the constant vectors κ_{r-1} and κ_r are changed to $\kappa^{(r-2)}_{r-1}$ and $\kappa^{(r-2)}_r$. Therefore, as in the case $r = 2$, the limit of U_{l_{r-1},l_r} has the desired form, and the limits of the other components of $U^{[m]}$ are all zero. The theorem is proved.

Now we can discuss the properties of the solution $U^{[m]}$.

(1) Localization of the solutions

For the Lax pair (2.108),

$$Q_\alpha = \mathrm{i}\lambda_\alpha(x + Jy) + \mathrm{i}\omega(\lambda_\alpha)t. \tag{2.148}$$

Consider the limit of the solution as $(x, y) \to \infty$ along a straight line $x = \xi + v_x s$, $y = \eta + v_y s$ $(v_x^2 + v_y^2 > 0)$, then

$$Q_\alpha = \mathrm{i}\lambda_\alpha(\xi + J\eta) + \mathrm{i}\omega(\lambda_\alpha)t + \mathrm{i}\lambda_\alpha(v_x + Jv_y)s. \tag{2.149}$$

Now

$$\rho_\alpha = \operatorname{Re}(\mathrm{i}\lambda_\alpha(v_x + J_{l_\alpha} v_y)) = -\operatorname{Im} \lambda_\alpha(v_x + J_{l_\alpha} v_y). \tag{2.150}$$

If there is at most one $\rho_\alpha = 0$, then part (1) of Theorem 2.12 implies that $U^{[m]} \to 0$ as $s \to \infty$. If $\rho_\alpha = 0$, $\rho_\beta = 0$ $(\alpha \neq \beta)$, then $l_\alpha = l_\beta$ since $J_{l_\alpha} \neq J_{l_\beta}$ if $l_\alpha \neq l_\beta$. Hence, part (2) of Theorem 2.12 also implies $U^{[m]} \to 0$ as $s \to \infty$. Therefore, we have

THEOREM 2.13 *$U^{[m]} \to 0$ as $(x, y) \to \infty$ in any directions.*

(2) Asymptotic behavior of the solutions as $t \to \infty$

Now we use a frame (ξ, η) which moves in a fixed velocity (v_x, v_y), that is, let $x = \xi + v_x t$, $y = \eta + v_y t$, then

$$Q_\alpha = \mathrm{i}\lambda_\alpha(\xi + J\eta) + (\mathrm{i}\lambda_\alpha(v_x + Jv_y) + \mathrm{i}\omega(\lambda_\alpha))t, \tag{2.151}$$

$$\rho_\alpha = -\operatorname{Im}\lambda_\alpha(v_x + J_{l_\alpha}v_y) - \operatorname{Im}(\omega_{l_\alpha}(\lambda_\alpha)). \tag{2.152}$$

Suppose that for distinct α, β, γ,

$$\det\begin{pmatrix} 1 & J_{l_\alpha} & \sigma_\alpha \\ 1 & J_{l_\beta} & \sigma_\beta \\ 1 & J_{l_\gamma} & \sigma_\gamma \end{pmatrix} \neq 0 \tag{2.153}$$

where

$$\sigma_\alpha = \operatorname{Im}(\omega_{l_\alpha}(\lambda_\alpha))/\operatorname{Im}(\lambda_\alpha). \tag{2.154}$$

Then there are at most two $\rho_\alpha = 0$ $(\alpha = 1, \cdots, r)$. By Theorem 2.12, $U^{[m]}_{l_\alpha, l_\beta} \not\to 0$ only if $\rho_\alpha = 0$, $\rho_\beta = 0$. This leads to

$$v_x = \frac{J_{l_\alpha}\sigma_\beta - J_{l_\beta}\sigma_\alpha}{J_{l_\beta} - J_{l_\alpha}}, \qquad v_y = \frac{\sigma_\alpha - \sigma_\beta}{J_{l_\beta} - J_{l_\alpha}}. \tag{2.155}$$

For $U^{[m]}_{jk} \not\to 0$, α, β can take m_j and m_k values respectively, hence there are at most $m_j m_k$ velocities (v_x, v_y) such that $U^{[m]}_{jk} \not\to 0$. Therefore, we have

THEOREM 2.14 *Suppose (2.153) is satisfied. Then as $t \to \infty$, the asymptotic solution of $U^{[m]}_{jk}$ has at most $m_j m_k$ peaks whose velocities are given by (2.155) ($l_\alpha = j$, $l_\beta = k$). If a peak has velocity (v_x, v_y), then, in the coordinate $\xi = x - v_x t$, $\eta = y - v_y t$, $\lim_{t\to\infty} U_{ab} = 0$ for all $(a, b) \neq (j, k)$, and as $t \to \infty$*

$$\begin{aligned} U^{[m]}_{jk} &\sim \frac{B_{\alpha\beta}\exp(\mathrm{i}\operatorname{Re}(\lambda_\alpha - \lambda_\beta)\xi + \mathrm{i}(\lambda_\alpha J_j - \lambda_\beta J_k)\eta + \mathrm{i}(\phi_\alpha - \phi_\beta)t)}{\Delta}, \\ \Delta &= A_{\alpha\beta}\cosh(\operatorname{Im}(\lambda_\alpha + \lambda_\beta)\xi + \operatorname{Im}(\lambda_\alpha J_j + \lambda_\beta J_k)\eta + \delta^{(1)}_{\alpha\beta}) \\ &\quad + \cosh(\operatorname{Im}(\lambda_\alpha - \lambda_\beta)\xi + \operatorname{Im}(\lambda_\alpha J_j - \lambda_\beta J_k)\eta + \delta^{(2)}_{\alpha\beta}) \end{aligned} \tag{2.156}$$

where $A_{\alpha\beta}$, $\delta^{(1)}_{\alpha\beta}$, $\delta^{(2)}_{\alpha\beta}$ are real constants, $A_{\alpha\beta} > 0$, and $B_{\alpha\beta}$ are complex constants,

$$\phi_\gamma = \operatorname{Re}\lambda_\gamma(v_x + J_{l_\gamma}v_y) + \operatorname{Re}(\omega_{l_\gamma}(\lambda_\gamma)) \quad (\gamma = \alpha, \beta). \tag{2.157}$$

Remark 23 The condition (2.153) implies that the velocities of the solitons are all different. This is true for the DSI equation. However, for the 2+1 dimensional N-wave equation, all the solitons move in the same velocity. We shall discuss this problem later.

EXAMPLE 2.15 *DSI equation*

Let $n = 2$, $N = 2$,

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}, \qquad \omega = -2\mathrm{i}J\lambda^2, \quad (2.158)$$

then we have

$$\begin{aligned} &F_y = JF_x + UF, \\ &F_t = 2\mathrm{i}JF_{xx} + 2\mathrm{i}UF_x + \mathrm{i}\begin{pmatrix} |u|^2 + 2q_1 & u_x + u_y \\ -\bar{u}_x + \bar{u}_y & -|u|^2 - 2q_2 \end{pmatrix} F, \end{aligned} \quad (2.159)$$

$$\begin{aligned} &-\mathrm{i}u_t = u_{xx} + u_{yy} + 2|u|^2 u + 2(q_1 + q_2)u, \\ &q_{1,x} - q_{1,y} = q_{2,x} + q_{2,y} = -(|u|^2)_x, \end{aligned} \quad (2.160)$$

$$(FF^*)^D = \frac{1}{2}\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \qquad [J, FF^*] = U_x. \quad (2.161)$$

(2.160) is the DSI equation.

If we construct the solution $U^{[m]}$ as above, then Theorem 2.13 implies that $U^{[m]} \to 0$ as $(x, y) \to \infty$ in any directions. If $\mathrm{Re}\,\lambda_\alpha \neq \mathrm{Re}\,\lambda_\beta$ for $\alpha \neq \beta$ and $l_\alpha = l_\beta$, then, Theorem 2.14 implies that as $t \to \infty$, the derived solution u has at most $m_1 m_2$ peaks ($m_1 + m_2 = r$). From (2.154), $\sigma_\alpha = -4J_{l_\alpha}\,\mathrm{Re}\,\lambda_\alpha$, hence (2.155) implies that each peak has the velocity $v_x = 2\ \mathrm{Re}(\lambda_\alpha - \lambda_\beta)$, $v_y = 2\ \mathrm{Re}(\lambda_\alpha + \lambda_\beta)$ ($l_\alpha = 1$, $l_\beta = 2$). This is the (m_1, m_2) solitons [30]. When $K = 1$, these peaks do not vanish if and only if all κ_α's are non-zero.

Figures 2.5 – 2.7 show the solitons $u^{[1,3]}$, $u^{[2,3]}$ and $u^{[3,3]}$ respectively. The parameters are $K = 1$, $t = 2$, $\lambda_1 = 1 - 2\mathrm{i}$, $\lambda_2 = -3 - \mathrm{i}$, $\lambda_3 = 2 + \mathrm{i}$, $\lambda_4 = -1 + 3\mathrm{i}$, $\lambda_5 = 2 + 1.5\mathrm{i}$, $\lambda_6 = -0.5 - 1.5\mathrm{i}$, $C_1 = (1, 0)$, $C_2 = (0, 1)$, $C_3 = (0, 2)$, $C_4 = (0, -2)$, $C_5 = (2, 0)$, $C_6 = (-2, 0)$.

(3) Asymptotic solutions as the phases differences tend to infinity

For the equations whose solitons move in the same speed, like the 2+1 dimensional N-wave equation, the peaks do not separate as $t \to \infty$.

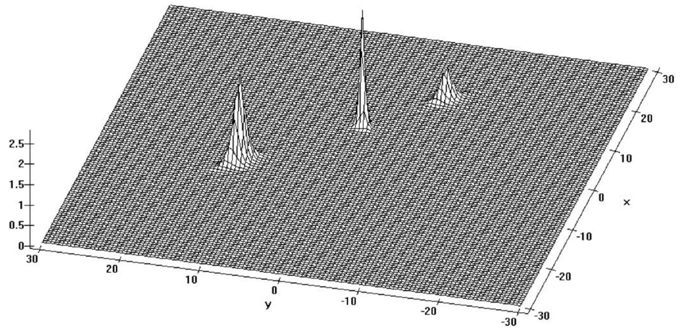


Figure 2.5. $u^{[1,3]}$ of the DSI equation

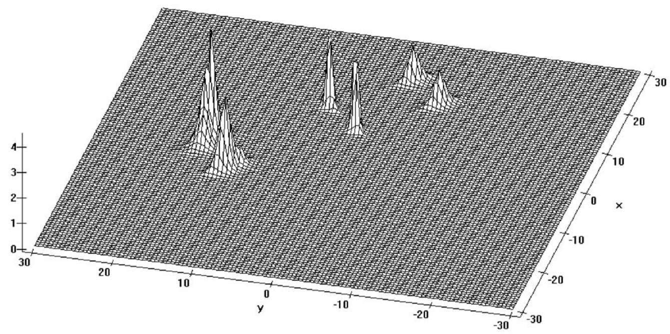


Figure 2.6. $u^{[2,3]}$ of the DSI equation

However, we can still see some peaks in the figures. Here we will get the corresponding asymptotic properties of the solitons.

THEOREM 2.16 *Let p_α $(\alpha = 1.\cdots, r)$ be constant real numbers satisfying*

$$\det \begin{pmatrix} 1 & J_{l_\alpha} & p_\alpha / \operatorname{Im} \lambda_\alpha \\ 1 & J_{l_\beta} & p_\beta / \operatorname{Im} \lambda_\beta \\ 1 & J_{l_\gamma} & p_\gamma / \operatorname{Im} \lambda_\gamma \end{pmatrix} \neq 0 \tag{2.162}$$

for distinct α, β, γ. Let d_α be complex constant $K \times 1$ vectors, $\kappa_\alpha = d_\alpha \exp(p_\alpha \tau)$ and construct the Darboux transformations as above. Let

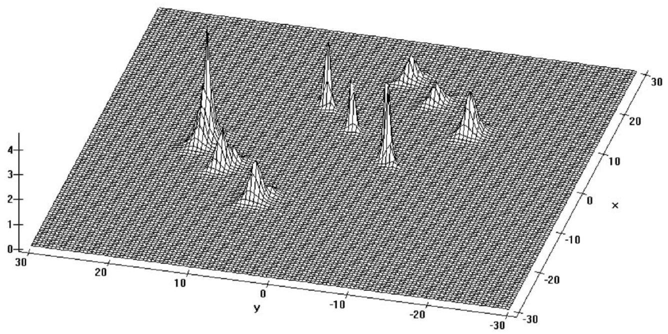


Figure 2.7. $u^{[3,3]}$ of the DSI equation

$x = \xi + v_x\tau$, $y = \eta + v_y\tau$, *then, for any* j, k *with* $1 \le j, k \le N$, $j \ne k$, $\lim_{\tau\to\infty} U^{[m]}_{jk} \ne 0$ *only if* (v_x, v_y) *takes specific* $m_j m_k$ *values.*

Proof. Here

$$Q_\alpha = \mathrm{i}\lambda_\alpha(\xi + J\eta) + \mathrm{i}\omega(\lambda_\alpha)t + \mathrm{i}\lambda_\alpha(v_x + Jv_y)\tau. \tag{2.163}$$

Hence

$$\mathring{H}_\alpha \doteq \begin{pmatrix} \exp(\widetilde{Q}_\alpha(\tau)) \\ D_\alpha \end{pmatrix} \tag{2.164}$$

where

$$D_\alpha = (0, \cdots, 0, \underset{l_\alpha}{d_\alpha}, 0, \cdots, 0), \tag{2.165}$$

$$\widetilde{Q}_\alpha(\tau) = \mathrm{i}\lambda_\alpha(\xi + J\eta) + \mathrm{i}\omega(\lambda_\alpha)t + (\mathrm{i}\lambda_\alpha(v_x + Jv_y) - p_\alpha)\tau. \tag{2.166}$$

The real part of the coefficient of τ in $\widetilde{Q}_\alpha(\tau)$ is

$$\widetilde{\rho}_\alpha = -\operatorname{Im}\lambda_\alpha(v_x + Jv_y) - p_\alpha. \tag{2.167}$$

Condition (2.162) implies that there are at most two $\widetilde{\rho}_\alpha$'s such that $\widetilde{\rho}_\alpha = 0$. According to Theorem 2.12, as $\tau \to \infty$, $U^{[m]}_{jk} \not\to 0$ only if there exist $\widetilde{\rho}_\alpha = 0$, $\widetilde{\rho}_\beta = 0$, $\alpha \ne \beta$, $l_\alpha = j$, $l_\beta = k$. Therefore, the theorem is verified.

When the condition (2.153) holds, this theorem is useless, because the evolution will always separate the peaks. However, when (2.153) does not hold, especially when it is never satisfied, this theorem reveals a fact of the separation of the peaks.

EXAMPLE 2.17 *2+1 dimensional N-wave equation*

Let $n = 1$, $\omega = L\lambda$ *where* $L = \mathrm{diag}(L_1, \cdots, L_N)$ *is a constant real diagonal matrix such that* $L_j \neq L_k$ *for* $j \neq k$. *Then, the integrability conditions (2.109) – (2.112) imply*

$$F_y = JF_x + UF, \quad F_t = LF_x + VF, \tag{2.168}$$
$$[J, V] = [L, U], \quad U_t - V_y + [U.V] + JV_x - LU_x = 0, \tag{2.169}$$
$$U_x = [J, FF^*]. \tag{2.170}$$

(2.169) is just the 2+1 dimensional N-wave equation.

Suppose $U^{[m]}$ *is constructed as above, then Theorem 2.13 implies that* $U^{[m]} \to 0$ *as* $(x, y) \to \infty$ *in any directions. Theorem 2.14 cannot be applied here. The reason is: the condition (2.153) holds only if* $l_\alpha \neq l_\beta$ *for* $\alpha \neq \beta$. *Hence for any* j, $m_j = 0$ *or* 1. *This implies that (2.153) does not hold generally unless* $m_j = 0$ *or* 1 *for all* $1 \leq j \leq N$. *Therefore, we apply Theorem 2.16 to the previous problem. Theorem 2.16 implies that if we choose* $\{p_\alpha\}$ *such that (2.162) is satisfied, then, for each* (j, k), $\lim_{\tau\to\infty} U_{jk}^{[m]}$ *has at most* $m_j m_k$ *peaks. When* $K = 1$, *these peaks do not vanish if and only if all* κ_α*'s are non-zero.*

Remark 24 *Here* $\tau \to \infty$ *means that the phase differences of different peaks tend to infinity. Therefore, the peaks are separated by enlarging the phase differences.*

Here are the figures describing the solutions $U^{[0,1,2]}$ *and* $U^{[1,1,2]}$ *of the 3-wave equation. The vertical axis is* $(|u_{12}|^2 + |u_{13}|^2 + |u_{23}|^2)^{1/4}$ *so that all the components are shown in one figure. The parameters are*

$$J = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \qquad L = \begin{pmatrix} 2 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

$K = 1$, $t = 10$, $\lambda_1 = 1 - 2\mathrm{i}$, $\lambda_2 = -3 - \mathrm{i}$, $\lambda_3 = 2 + i$, $\lambda_4 = -1 + 3\mathrm{i}$, $C_1 = (0, 1, 0)$, $C_2 = (0, 0, 1)$, $C_3 = (0, 0, 4096)$, $C_4 = (1, 0, 0)$. *Note that for* $U^{[0,1,2]}$, *only* U_{23} *has two peaks, and for* $U^{[1,1,2]}$, U_{12}, U_{13}, U_{23} *have one, two, two peaks respectively.*

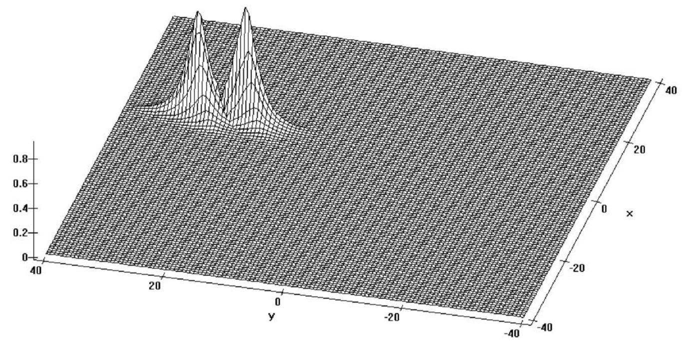


Figure 2.8. $U^{[0,1,2]}$ of the 3-wave equation

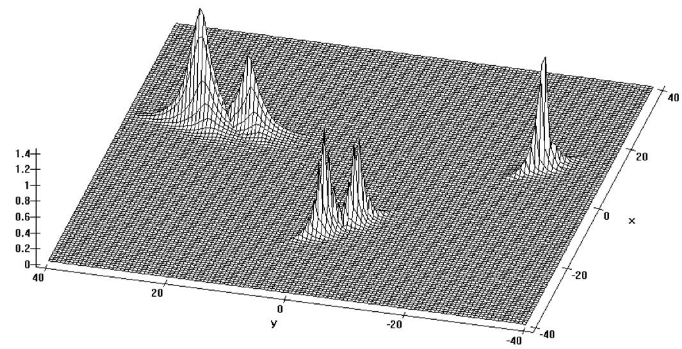


Figure 2.9. $U^{[1,1,2]}$ of the 3-wave equation

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