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Binary Darboux Transformations for Manakov Triad

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Abstract. The binary Darboux transformations for Manakov triads of second order are constructed, which generalize the work for KP equation and Novikov-Veselov equation. An equation given by [1] is discussed.

1. Introduction.

A Manakov triad is a generalization of a Lax pair. It can represent more integrable systems than the Lax pair. In the present Letter, we discuss the Manakov triad (L, N, W) with

$$L_t = WL - LN, \tag{1.1}$$

$$\begin{aligned} &L\Psi = 0 \\ &\Psi_t = N\Psi, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} &L = \sum_{i=0}^{2} L_i \partial_x^i, \qquad L_i = \sum_{j=0}^{2-i} L_{ij} \partial_y^j, \\ &N = N_1 \partial_x + N_0, \qquad N_i = \sum_{j=0}^{n} N_{ij} \partial_y^j, \\ &W = W_1 \partial_x + W_0, \qquad W_i = \sum_{j=0}^{n} W_{ij} \partial_y^j, \end{aligned} \tag{1.3}$$

and the coefficients L_{ij}, N_{ij}, W_{ij} are $r \times r$ matrix functions of (x, y, t). Moreover, suppose $L_{20} = I$.

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(1.1) is the integrability condition of (1.2). We can determine W explicitly as

$$W_1 = N_1, \qquad W_0 = N_0 + 2N_{1,x} + [L_1, N_1], \tag{1.4}$$

and N satisfies

$$\begin{aligned} &L_{1,t} + [L_0, N_1] - [L_1, N_1]L_1 + [L_1, N_0] - 2N_{1,x}L_1 - N_1L_{1,x} + L_1N_{1,x} \\ &\qquad + 2N_{0,x} + N_{1,xx} = 0 \\ &L_{0,t} + [L_0, N_0] - [L_1, N_1]L_0 - 2N_{1,x}L_0 + L_1N_{0,x} - N_1L_{0,x} + N_{0,xx} = 0. \end{aligned} \tag{1.5}$$

This system contains several interesting equations.

Example 1.

$$\begin{aligned} L &= \partial_x^2 + \alpha\partial_y + u \qquad (\alpha = 1 \text{ or } i) \\ N &= 4\alpha\partial_x\partial_y - 2u\partial_x + u_x + 3\alpha w \\ W &= N - 4u_x. \end{aligned}$$

Equation (1.5) gives

$$\begin{aligned} &u_t + 6uu_x + u_{xxx} = -3\alpha^2 w_y, \\ &w_x = u_y, \end{aligned}$$

which is the well-known KP equation. This Manakov triad is equivalent to the usual Lax pair.

Example 2.

$$\begin{aligned} L &= \partial_x^2 + \partial_y^2 + u, \\ N &= -8\partial_x\partial_y^2 + (3v_R - 2u)\partial_x + 3v_I\partial_y - 2u_x \\ W &= N + 6v_{R,x} - 4u_x. \end{aligned}$$

We have the equations

$$\begin{aligned} &u_t = 2u_{xxx} - 6u_{xyy} + 3v_Ru_x + 3v_Iu_y + 2v_{R,x}u - 6uu_x \\ &v_{R,x} - v_{I,y} = 2u_x \\ &v_{R,y} + v_{I,x} = -2u_y, \end{aligned}$$

which is called the Novikov-Veselov equation, and is discussed in [4].

In [3,4], the binary Darboux transformation for DS equation, whose L is of first order, and KP equation, Novikov-Veselov equation, which are special cases of (1.2), are discussed. In the present Letter, we use binary Darboux transformation to discuss the more general system (1.2).

Remark. Any system in (1.2) can be transformed to a linear system of first order whose coefficients are $3r \times 3r$ matrices. However, since there are many constraints among these coefficients, (e.g., some coefficients should be given constants), it is more difficult to solve these systems than to solve (1.2). Therefore we would rather study (1.2) than study a general system of first order with many reductions.

2. Binary Darboux transformation.

In order to discuss the binary Darboux transformation, we need the following lemma about a closed 1-form. Let

$$\mathcal{LP}[\Psi] = \operatorname{span}\{\, \partial_x^i \partial_y^j L\Psi,\ \partial_x^i \partial_y^j (\Psi_t - N\Psi) \,|\, i, j \geq 0 \,\} \tag{2.1}$$

and

$$\mathcal{LP}_2[\Psi] = \mathcal{LP}[\Psi] \otimes \operatorname{span}\{\, dx \wedge dy,\ dx \wedge dt,\ dy \wedge dt \,\}. \tag{2.2}$$

Lemma. *For given L, N in (1.3), there exists a 1-form*

$$\omega[\Psi] = \alpha\Psi\, dx + \beta\Psi\, dy + \gamma\Psi\, dt \tag{2.3}$$

where the differential operators

$$\begin{aligned} &\alpha = \alpha_{01}\partial_y + \alpha_{00}, \qquad \beta = \beta_1\partial_x + \beta_0, \\ &\beta_1 = \beta_{10}, \qquad \beta_0 = \beta_{01}\partial_y + \beta_{00}, \\ &\gamma = \gamma_1\partial_x + \gamma_0, \qquad \gamma_i = \sum_{j=0}^{n} \gamma_{ij}\partial_y^j \end{aligned} \tag{2.4}$$

(α_{0i}, β_{ij}, γ_{ij} are $r \times r$ matrix functions of (x, y, t)) such that $d\omega[\Psi] \in \mathcal{LP}_2[\Psi]$, provided that the system

$$\begin{aligned} &L^*\lambda^* = 0 \\ &\lambda_t^* = -W^*\lambda^* \end{aligned} \tag{2.5}$$

has a nondegenerate solution λ.

Here L^*, W^* are the formal conjugations of L and W.

Proof. Denote $dw[\Psi] = D_{xy} dx \wedge dy + D_{xt} dx \wedge dt + D_{yt} dy \wedge dt$. The highest order of ∂_x in D_{xy} does not exceed 2, and the coefficients of $\partial_x^2 \partial_y^k$ ($k \geq 1$) are all zero. Hence let $D_{xy} = \lambda L$, where λ is an $r \times r$ matrix function to be determined later. Comparing the coefficients in both sides, we have

$$\begin{aligned} &\alpha_{01} = -\lambda L_{02}, \qquad \alpha_{00} = -\lambda L_{01} + (\lambda L_{11})_x + (\lambda L_{02})_y, \\ &\beta_{10} = \lambda, \qquad \beta_{00} = \lambda L_{10} - \lambda_x, \qquad \beta_{01} = \lambda L_{11}, \end{aligned} \tag{2.6}$$

and λ satisfies

$$\sum_{j+l \leq 2} (-1)^{j+l} \partial_x^j \partial_y^l (\lambda L_{jl}) = 0, \tag{2.7}$$

which is the first equation of (2.5).

Next we consider D_{yt}. Since

$$\beta N\Psi = (\beta_1\partial_x + \beta_0)(N_1\partial_x + N_0)\Psi \equiv (-\beta_1 N_1 L_1 + \beta_1 N_{1,x} + \beta_0 N_1 + \beta_1 N_0)\partial_x \\ + (-\beta_1 N_1 L_0 + \beta_1 N_{0,x} + \beta_0 N_0)\Psi \pmod{\mathcal{LP}_2[\Psi]}$$

and

$$D_{yt} = \gamma\Psi_y + \gamma_y\Psi - \beta N\Psi - \beta_t\Psi,$$

the request $D_{yt} \equiv 0 \pmod{\mathcal{LP}_2[\Psi]}$ implies

$$-\beta_{1,t} + \gamma_1\partial_y + \gamma_{1,y} = -\beta_1 N_1 L_1 + \beta_1 N_{1,x} + \beta_0 N_1 + \beta_1 N_0 \tag{2.8.1}$$

$$-\beta_{0,t} + \gamma_0\partial_y + \gamma_{0,y} = -\beta_1 N_1 L_0 + \beta_1 N_{0,x} + \beta_0 N_0. \tag{2.8.2}$$

γ_1 can be solved recursively by comparing the coefficients of ∂_y^{n+1}, $\cdots$, ∂_y in (2.8.1). With (1.4), it can be checked that the right-hand side of (2.8.1) equals to $\lambda W_0 - (\lambda W_1)_x$. Hence the terms without ∂_y in (2.8.1) are equal identically by the second equation of (2.5). Similarly, γ_0 can be solved from (2.8.2) and (2.8.2) holds identically because of (1.5) and (2.5). Therefore, we have proved $D_{yt} \equiv 0 \pmod{\mathcal{LP}_2[\Psi]}$. The definition of D_{xy}, D_{xt} and D_{yt} leads to $\partial_y D_{xt} \equiv 0 \pmod{\mathcal{LP}_2[\Psi]}$. Suppose

$$D_{xt} \equiv \sum_{i=0}^{1}\sum_{j=0}^{J} d_{ij}\partial_x^i\partial_y^j\Psi \quad \pmod{\mathcal{LP}_2[\Psi]},$$

then $\sum_{i=0}^{1}\sum_{j=0}^{J}(d_{ij} + \partial_y d_{i,j-1})\partial_y^{j+1}\Psi + d_{oo}\Psi \equiv 0 \pmod{\mathcal{LP}_2[\Psi]}$, which means $d_{ij} \equiv 0$ for all i, j. The lemma is proved.

Now we proceed to the discussion of binary Darboux transformations for (1.2).

Theorem. *Let π and λ be nondegenerate $r \times r$ matrix solutions of (1.2) and (2.5) respectively, $S(x,y,t)$ be an arbitrary nondegenerate matrix function and C be a constant $r \times r$ matrix. Suppose Ψ is a solution of (1.2). Let $\Omega[\Psi] = \int \omega[\Psi] + C$, $\Omega_0 = \Omega[\pi]$, $R = S\Omega_0\pi^{-1}$, and*

$$\widetilde{\Psi} = R\Psi - S\Omega[\Psi]. \tag{2.9}$$

Then, at the points where $\det\Omega_0 \neq 0$ and $\det(1 - R^{-1}S\lambda L_{11} + (R^{-1}S\lambda)^2 L_{02}) \neq 0$, $\widetilde{\Psi}$ satisfies

$$\begin{aligned} \widetilde{L}\widetilde{\Psi} &= 0 \\ \widetilde{\Psi}_t &= \widetilde{N}\widetilde{\Psi}. \end{aligned} \tag{2.10}$$

Here

$$\widetilde{L} = \sum_{i=0}^{2} \widetilde{L}_i \partial_x^i, \qquad \widetilde{L}_i = \sum_{j=0}^{2-i} \widetilde{L}_{ij} \partial_y^j,$$
$$\widetilde{N} = \widetilde{N}_1 \partial_x + \widetilde{N}_0 \qquad \widetilde{N}_i = \sum_{j=0}^{n} \widetilde{N}_{ij} \partial_y^j, \tag{2.11}$$

and the coefficients are given by

$$\begin{aligned}
&\left(\widetilde{L}_{20} \quad \widetilde{L}_{11} \quad \widetilde{L}_{02} \right) \begin{pmatrix} R & -S\alpha_{01} & 0 \\ 0 & R & -S\alpha_{01} \\ 0 & -S\beta_{10} & R - S\beta_{01} \end{pmatrix} = R \left(L_{20} \quad L_{11} \quad L_{02} \right) \\
&\left(\widetilde{L}_{10} \quad \widetilde{L}_{01} \right) \begin{pmatrix} R & -S\alpha_{01} \\ -S\beta_{10} & R - S\beta_{01} \end{pmatrix} = R \left(L_{10} \quad L_{01} \right) \\
&\quad - \left(\widetilde{L}_{20} \quad \widetilde{L}_{11} \quad \widetilde{L}_{02} \right) \begin{pmatrix} 2R_x - S\alpha_{00} & -2S_x\alpha_{01} - S\alpha_{01,x} \\ R_y - S_x\beta_{10} & R_x - S_x\beta_{01} - S_y\alpha_{01} - S\alpha_{00} - S\alpha_{01,y} \\ -2S_y\beta_{10} - S\beta_{10,y} & 2R_y - 2S_y\beta_{01} - S\beta_{00} - S\beta_{01,y} \end{pmatrix} \\
&\widetilde{L}_{00} R = R L_{00} - \left(\widetilde{L}_{10} \quad \widetilde{L}_{01} \right) \begin{pmatrix} R_x - S\alpha_{00} \\ R_y - S\beta_{00} \end{pmatrix} \\
&\quad - \left(\widetilde{L}_{20} \quad \widetilde{L}_{11} \quad \widetilde{L}_{02} \right) \begin{pmatrix} R_{xx} - 2S_x\alpha_{00} - S\alpha_{00,x} \\ R_{xy} - S_x\beta_{00} - S_y\alpha_{00} - S\alpha_{00,y} \\ R_{yy} - 2S_y\beta_{00} - S\beta_{00,y} \end{pmatrix}.
\end{aligned} \tag{2.12}$$

$\widetilde{N}$ *can be determined by the recursion formulae*

$$\begin{aligned}
\widetilde{N}_{1\mu} R &= R N_{1\mu} - S\gamma_{1\mu} - \sum_{k \geq \mu+1} C_k^\mu \widetilde{N}_{1k} (\partial_y^{k-\mu} R) \\
&\quad + \sum_{j \geq k \geq \mu+1} C_j^k C_{k-1}^\mu \widetilde{N}_{1j} (\partial_x \partial_y^{j-k} S)(\partial_y^{k-1-\mu} \beta_1) \\
&\quad + \sum_{j \geq k \geq \mu+1} C_j^k C_{k-1}^\mu \widetilde{N}_{0j} (\partial_y^{j-k} S)(\partial_y^{k-1-\mu} \beta_1) \\
\widetilde{N}_{0\mu} R - \widetilde{N}_{1\mu} S\alpha &= R N_{0\mu} - S\gamma_{0\mu} + R_t \delta_{0\mu} \\
&\quad - \sum_{k \geq \mu+1} C_k^\mu \widetilde{N}_{0k} (\partial_y^{k-\mu} R) + \sum_{k \geq \mu+1} C_k^\mu \widetilde{N}_{1k} (\partial_y^{k-\mu} (S\alpha)) \\
&\quad + \sum_{j \geq k \geq \mu+1} C_j^k C_{k-1}^\mu \widetilde{N}_{1j} (\partial_x \partial_y^{j-k} S)(\partial_y^{k-1-\mu} \beta_0) \\
&\quad + \sum_{j \geq k \geq \mu+1} C_j^k C_{k-1}^\mu \widetilde{N}_{0j} (\partial_y^{j-k} S)(\partial_y^{k-1-\mu} \beta_0).
\end{aligned} \tag{2.13}$$

Remark. For given π and λ, we can always have $\det \Omega_0 \neq 0$ and $\det(1 - R^{-1}S\lambda L_{11} + (R^{-1}S\lambda)^2 L_{02}) \neq 0$ locally by the suitable choice of the arbitrary matrices S and C.

Proof. Since $\det(1 - R^{-1}S\lambda L_{11} + (R^{-1}S\lambda)^2 L_{02}) \neq 0$, the coefficient matrices on the left-hand side of all the equations in (2.12) are nondegenerate. Hence (2.12) determines $\widetilde{L}_{ij}$ uniquely. Direct calculation shows that

$$\widetilde{L}\widetilde{\Psi} = (\widetilde{L}S)\Omega[\Psi] \tag{2.14}$$

holds for any solution Ψ of (1.2). Taking $\Psi = \pi$, we have $\widetilde{L}S = 0$. Hence $\widetilde{L}\widetilde{\Psi} \equiv 0$ for any solution Ψ of (1.2).

$\widetilde{N}$ is uniquely determined by (2.13). Using (2.13), we can check that

$$(\partial_t - \widetilde{N})\widetilde{\Psi} = -(S_t - \widetilde{N}S)\Omega[\Psi]. \tag{2.15}$$

Let $\Psi = \pi$, then $S_t = \widetilde{N}S$. Hence $\widetilde{\Psi}_t = \widetilde{N}\widetilde{\Psi}$. The theorem is proved.

Remark. Since $L_t^* = (-N^*)L^* - L^*(-W^*)$, if (π, λ) creates a binary Darboux transformation as in the Theorem, then (λ^*, π^*) creates a binary Darboux transformation for (2.5). Therefore, not only the solution of (1.2) with respect to $(\widetilde{L}, \widetilde{N})$, but also the solution of (2.5) with respect to $(\widetilde{L}, \widetilde{W})$ are constructed. This means that the binary Darboux transformation can be constructed successively by using differentiation and integration, without solving any differential equations.

3. Examples.

For the KP equation and the Novikov-Veselov equation, the binary Darboux transformations given by the Theorem are just those given by [4].

Here we consider the following example.

Let

$$\begin{aligned} L &= \partial_x \partial_y + P\partial_y + Q \\ N &= \partial_x^2 + F, \end{aligned} \tag{3.1}$$

then (1.1) holds if and only if

$$\begin{aligned} &F_y = 2Q_x \\ &P_t - P_{xx} + 2P_x P + F_x + [P, F] = 0 \\ &Q_t + Q_{xx} + 2(PQ)_x + [Q, F] = 0. \end{aligned} \tag{3.2}$$

(3.1) is not of form (1.2). However, a simple change of variables can lead to $L_{20} = I$.

For $r = 1$, this is a special case of the equation discussed in [1,2], which will be discussed here. Under the reduction $Q = F = 0$, the system (3.2) becomes the matrix Burgers equation in 1+1 dimensions.

Let $\xi = x + y$, $\eta = x - y$, then (3.1) becomes

$$
\begin{aligned}
L &= \partial_\xi^2 - \partial_\eta^2 + P(\partial_\xi - \partial_\eta) + Q \\
N &= (\partial_\xi + \partial_\eta)^2 + F.
\end{aligned}
\tag{3.3}
$$

For $r = 1$,

$$
W = (\partial_\xi + \partial_\eta)^2 + F - 2P_\eta = \partial_x^2 + F - P_x + P_y,
\tag{3.4}
$$

$$
\begin{aligned}
&\alpha = \lambda\partial_\eta - \lambda_\eta + \lambda P, \qquad \beta = \lambda\partial_\xi - \lambda_\xi + \lambda P, \\
&\gamma = 2\lambda\partial_\xi\partial_\eta + 2\lambda\partial_\eta^2 - 2(\lambda_\xi + \lambda_\eta)(\partial_\xi + \partial_\eta) + \lambda P\partial_\xi + 3\lambda P\partial_\eta \\
&\quad + 2\lambda_{\xi\eta} + 2\lambda_{\eta\eta} + \lambda(P_\xi - P_\eta) - \lambda_\xi P - 3\lambda_\eta P - 2\lambda Q.
\end{aligned}
\tag{3.5}
$$

To construct the binary Darboux transformation for (3.3), let π be a solution of

$$
\begin{aligned}
&L\pi = 0 \\
&\pi_t = N\pi
\end{aligned}
\tag{3.6}
$$

and λ be a solution of

$$
\begin{aligned}
&L^*\lambda^* = 0 \\
&\lambda_t^* = -W^*\lambda^*.
\end{aligned}
\tag{3.7}
$$

Suppose $\Omega_0 \neq 0$ and $\Omega_0 \pm \lambda\pi \neq 0$, and let

$$
S = \frac{\pi}{\Omega_0 - \lambda\pi}.
\tag{3.8}
$$

As in the Theorem,

$$
R = \frac{\Omega_0}{\Omega_0 - \lambda\pi}.
\tag{3.9}
$$

(2.12) leads to

$$
\begin{aligned}
&\widetilde{L}_{20} = 1, \qquad \widetilde{L}_{11} = 0, \qquad \widetilde{L}_{02} = -1, \\
&\widetilde{L}_{10} = -\widetilde{L}_{01} \equiv \widetilde{P} = P - 2\frac{\Omega_0 - \lambda\pi}{\Omega_0 + \lambda\pi}(\partial_\xi + \partial_\eta)\frac{\lambda\pi}{\Omega_0 - \lambda\pi} \\
&\qquad = P - 2\frac{\Omega_0(\partial_\xi + \partial_\eta)(\lambda\pi) - \lambda\pi(\partial_\xi + \partial_\eta)\Omega_0}{\Omega_0^2 - \lambda^2\pi^2} \\
&\widetilde{L}_{00} \equiv \widetilde{Q} = Q - R^{-1}\widetilde{P}(S_\xi - S_\eta)\lambda \\
&\qquad - R^{-1}(R_{\xi\xi} - R_{\eta\eta} - 2(S_\xi - S_\eta)\lambda P - S((\lambda P)_\xi - (\lambda P)_\eta) + 2S_\xi\lambda_\eta - 2S_\eta\lambda_\xi),
\end{aligned}
\tag{3.10}
$$

which gives the new solution $(\widetilde{P}, \widetilde{Q})$ of (3.2).

We can obtain many explicit solutions using (3.10). For example, take the seed solution as $P = 2p > 0$ (p is a real constant), $Q = F = 0$, then (3.6) has a solution

$$\pi = e^{a(\xi+\eta)+4a^2t} + e^{b(\xi-\eta)-p(\xi+\eta)+4p^2t}, \tag{3.11}$$

and (3.7) has a solution

$$\lambda = -e^{c(\xi+\eta)-4c^2t} + e^{d(\xi-\eta)+p(\xi+\eta)-4p^2t}, \tag{3.12}$$

where the constants a, b, c and d satisfy

$$(p+a)(a+c) < 0, \qquad b(d+b) < 0.$$

Let

$$\mu = -\frac{p+a}{a+c} > 0, \qquad \nu = -\frac{b}{b+d} > 0,$$

and

$$\begin{aligned} A &= a(\xi+\eta) + 4a^2t, \qquad & B &= b(\xi-\eta) - p(\xi+\eta) + 4p^2t, \\ C &= c(\xi+\eta) - 4c^2t, \qquad & D &= d(\xi-\eta) + p(\xi+\eta) - 4p^2t, \end{aligned}$$

then direct calculation gives

$$\widetilde{P} = -2p + (a+c)\frac{\sigma_1}{\sigma_2}$$

where

$$\begin{aligned} \sigma_1 =& (\mu+1)^2 e^{2A+C+D} + \mu^2 e^{A+B+2C} + \mu\nu e^{A+B+2D} + (\mu+1)(\nu+1)e^{2B+C+D} \\ &+ (3\mu+\nu+2)e^{A+B+C+D}, \\ \sigma_2 =& \mu(\mu+1)e^{2A+2C} + \nu(\nu+1)e^{2B+2D} + (\mu+1)e^{2A+C+D} + \mu e^{A+B+2C} + \nu e^{A+B+2D} \\ &+ (\nu+1)e^{2B+C+D} + (2\mu\nu+\mu+\nu+2)e^{A+B+C+D}. \end{aligned}$$

Clearly, it is bounded for all (ξ, η, t). However, it does not decay at infinity. It is still interesting to look for solutions which decay at infinity.

Remark. In this Letter, we have only discussed the binary Darboux transformations for the Manakov triads of second order. It is naturally interesting to discuss the binary Darboux transformation for the Manakov triads of higher order. This is still open due to some technical problems.

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