Nonlinear constraints and soliton solutions of 1+2-dimensional three-wave equation

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Using the nonlinear constraint method, the explicit expressions of localized solitons of the 1+2-dimensional three-wave equation are obtained. After the lth Darboux transformation, each component of the solution has, at most, l^2 peaks when the parameters for each single Darboax transformation are large enough. This gives a corresponding asymptotic property for the DSI equation, although all the peaks of each component move in the same velocity here. © *1998 American Institute of Physics.* [S0022-2488(98)03701-3]

I. INTRODUCTION

The 1+2-dimensional three-wave equation describes the nonlinear interaction of three waves and is widely investigated.[1] The inverse scattering transformation has been developed systematically[2–6] and there are many other considerations for this equation.[7,8]

Since the work of Ref. 9 on the 1+1-dimensional problem, the method of nonlinear constraint has been discussed in various papers. In 1+2 dimensions, this method is first applied to the KP equation by Refs. 10 and 11. Generally, the nonlinear constraint in 1+2 dimensions is a method to reduce the original problem to a new 1+2-dimensional problem that can be dealt with by a 1+1-dimensional method. For that new problem, the Darboux transformation has been well developed.[12,13] Using the nonlinear constraint and Darboux transformation methods, Ref. 14 proved that for any equation in the hyperbolic $su(n)$ AKNS system, which includes the three-wave equation, there are localized solitons (i.e., the solitons that decay at infinity in all directions).

If a certain condition [condition (16) of Remark 1] is satisfied so that different solitons have different velocities, the asymptotic solution as $t\to\pm\infty$ has, at most, l^2 peaks if the solution is given by the lth Darboux transformation from the zero solution. This is true for the DSI equation. However, for the three-wave equation, that condition never holds, and all the solitons in each component move in the same velocity. Therefore, the asymptotic solution is still very complicated (not split up into several single solitons). In this paper, we show that when certain parameters are large enough, these peaks can also be separated. This is true, not only for the asymptotic solution (under a weaker condition), but also for the solution at finite time t (under a stronger condition).

In Sec. II of this paper we describe the Lax pair of the three-wave equation, whose nonlinear constraint and the Darboux transformation. The main conclusion has been obtained in Ref. 14. In Sec. III the behavior of a single-soliton solution is discussed in detail. In Sec. IV, the asymptotic behavior of the multisoliton solutions is obtained when the parameters of each single Darboux transformation (not time t) are large enough. This gives a corresponding property for the asymptotic solutions of the DSI equation, although all the solitons move in the same velocity here. Some figures describing the behaviors and interactions of the ''two-soliton'' solutions are shown.

II. NONLINEAR CONSTRAINT AND DARBOUX TRANSFORMATION

The 1+2-dimensional three-wave equation is

$$\begin{aligned} f_{1,t}&=\alpha_1 f_{1,y}+\beta_1 f_{1,x}+(\alpha_3-\alpha_2)f_2\overline{f_3},\\ f_{2,t}&=\alpha_2 f_{2,y}+\beta_2 f_{2,x}+(\alpha_1-\alpha_3)f_1 f_3,\\ f_{3,t}&=\alpha_3 f_{3,y}+\beta_3 f_{3,x}+(\alpha_2-\alpha_1)\overline{f_1}f_2,\end{aligned} \tag{1}$$

where α_i, β_i are real constants.

It is known that this equation has a Lax pair

$$\begin{aligned}\Psi_y&=J\Psi_x+U(x,y,t)\Psi,\\ \Psi_t&=K\Psi_x+V(x,y,t)\Psi,\end{aligned}\tag{2}$$

where

$$J=\begin{pmatrix}J_1&0&0\\0&J_2&0\\0&0&J_3\end{pmatrix},\quad K=\begin{pmatrix}K_1&0&0\\0&K_2&0\\0&0&K_3\end{pmatrix},\quad U=(u_{ij})=\begin{pmatrix}0&f_1&f_2\\-\overline{f_1}&0&f_3\\-\overline{f_2}&-\overline{f_3}&0\end{pmatrix},$$

J_i, K_i are real constants with $J_i\neq J_j$ for $i\neq j$, and V is a 3×3 matrix whose diagonal entries are zero. Moreover, we suppose that

$$\det\begin{pmatrix}1&J_1&K_1\\1&J_2&K_2\\1&J_3&K_3\end{pmatrix}\neq0.\tag{3}$$

The integrability conditions of (2) are

$$[J,V]=[K,U],\quad U_t-V_y+JV_x-KU_x+[U,V]=0.\tag{4}$$

Eliminating V, we get an equation of U, which is just (1) with $\alpha_1=(K_1-K_2)/(J_1-J_2)$, $\alpha_2=(K_1-K_3)/(J_1-J_3)$, $\alpha_3=(K_2-K_3)/(J_2-J_3)$, $\beta_i=K_i-J_i\alpha_i$ $(i=1,2,3)$.

There is another linear system related with (2), which is

$$\Phi_x=\begin{pmatrix}i\lambda I&iP\\iP^*&0\end{pmatrix}\Phi,\quad \Phi_y=\begin{pmatrix}i\lambda J+U&iJP\\iP^*J&0\end{pmatrix}\Phi,\quad \Phi_t=\begin{pmatrix}i\lambda K+V&iKP\\iP^*K&0\end{pmatrix}\Phi,\tag{5}$$

where $P(x,y,t)$ is a 3×3 matrix. The integrability conditions of (5) give

$$P_y=JP_x+UP,\quad P_t=KP_x+VP,\tag{6}$$

$$U_x=[J,PP^*],\tag{7}$$

$$U_t-V_y+[U,V]-JPP^*K+KPP^*J=0.\tag{8}$$

Clearly the system (6) is the same as (2), except that Ψ is replaced by P. If (7) is considered, (8) is just the three-wave equation (4). Moreover, (7) gives a nonlinear constraint between U and P [P is a solution of the Lax pair (2)]. Therefore, we relate the 1+2-dimensional problem (2) with the problem (5) by the nonlinear constraint (7). For (5), the standard method in 1+1 dimensions can be applied.

In Ref. 15, this method was applied for 3×1 matrix P. In that case, nonlocalized solutions of the 1+2-dimensional three-wave equation are obtained. However, localized solutions can only be obtained by choosing P as a $3\times s$ $(s\geqslant3)$ matrix with rank 3.

If we can solve (6)–(8), the explicit solutions (f_i) of (1) are obtained, although they are not all solutions of (1).

Reference 14 showed that the equations (1) have localized soliton solutions that tend to zero exponentially at infinity. These solutions can be obtained by a Darboux transformation for (5), provided that the parameters in the Darboux transformation are chosen properly. More precisely, the Darboux transformation can be constructed as follows from the zero seed solution.

Take l complex numbers $\lambda_1,\ldots,\lambda_l$ such that $\lambda_j\neq\lambda_k$ $(j\neq k)$ and $\lambda_j\neq\overline{\lambda_k}$ for all $1\leqslant j,k\leqslant l$. Let

$$H_j=\begin{pmatrix}e^{\sqrt{-1}\lambda_j(x+Jy+Kt)}\\C^{(j)}\end{pmatrix},\tag{9}$$

where $C^{(j)}$'s are 3×3 nondegenerate constant matrices. Let $\Sigma=(\Sigma_{jk})_{1\le j,k\le l}$ be a block matrix given by

$$\Sigma_{jk}=\frac{H_j^*H_k}{\lambda_k-\lambda_j}. \tag{10}$$

Then

$$G(\lambda)=\prod_{j=1}^{l}(\lambda-\overline{\lambda_j})\left(1-\sum_{j,k=1}^{l}\frac{H_j}{\lambda-\lambda_k}(\Sigma^{-1})_{jk}H_k^*\right) \tag{11}$$

is a Darboux matrix of degree l for (5) with $U=P=0$. That is, for any solution Φ of (5) with $U=P=0$, $\widetilde{\Phi}=G\Phi$ satisfies (5) for certain U and P. It is known that the action of the Darboux matrix of degree l is equivalent to the successive actions of l Darboux matrices of degree one.

From the standard calculation of a Darboux transformation, we have

$$\begin{aligned} u_{\alpha\beta}&=2\sqrt{-1}\sum_{j,k=1}^{l}(h_j(\Sigma^{-1})_{jk}h_k^*)_{\alpha\beta}\\ &=2\sqrt{-1}\sum_{j,k=1}^{l}(\Sigma^{-1})_{\alpha\beta}e^{\sqrt{-1}(\lambda_j-\overline{\lambda_k})x+\sqrt{-1}(\lambda_jJ_\alpha-\overline{\lambda_k}J_\beta)y+\sqrt{-1}(\lambda_jK_\alpha-\overline{\lambda_k}K_\beta)t}. \end{aligned} \tag{12}$$

Reference 14 proved that when $C^{(j)}$'s are nondegenerate, the solution (12) is localized, and it decays at infinity exponentially. Moreover, the peaks in $u_{\alpha\beta}$ move in the velocity (v_x,v_y) given by

$$v_x=-\frac{J_\alpha K_\beta-J_\beta K_\alpha}{J_\alpha-J_\beta},\quad v_y=-\frac{K_\alpha-K_\beta}{J_\alpha-J_\beta}. \tag{13}$$

For a three-wave equation, these velocities are independent of λ. Therefore, all the solitons move in the same velocity for given J_α, K_α.

Remark 1: For a more general linear system [it comes from the hyperbolic $su(N)$ AKNS system),

$$\begin{gathered} \Phi_x=\begin{pmatrix} i\lambda I & iP\\ iP^* & 0\end{pmatrix}\Phi,\quad \Phi_y=\begin{pmatrix} i\lambda J+U & iJP\\ iP^*J & 0\end{pmatrix}\Phi,\\ \Phi_t=\begin{pmatrix} W(x,y,t,\lambda) & X(x,y,t,\lambda)\\ -X^*(x,y,t,\lambda) & Z(x,y,t,\lambda)\end{pmatrix}\Phi, \end{gathered} \tag{14}$$

the multisoliton solutions are taken from $U=P=0$, $X=0$, $Z=0$,

$$W(x,y,t,\lambda)=\begin{pmatrix} i\beta_1(\lambda) & & \\ & \ddots & \\ & & i\beta_N(\lambda)\end{pmatrix}. \tag{15}$$

In that case, Proposition 4 of Ref. 14 shows that if

$$\det\begin{pmatrix} 1 & J_i & \beta_{iI}(\lambda_\alpha)/\lambda_{\alpha I}\\ 1 & J_j & \beta_{jI}(\lambda_\beta)/\lambda_{\beta I}\\ 1 & J_k & \beta_{kI}(\lambda_\gamma)/\lambda_{\gamma I}\end{pmatrix}\neq 0 \tag{16}$$

for any mutually different pairs (i,α), (j,β), and (k,γ), then as $t\to\infty$, each entry $U_{ij}^{[l]}$ of the solution $U^{[l]}$ given by the lth Darboux transformations splits up into at most l^2 peaks. Here z_I is the imaginary part of a complex number z. Actually, (16) implies that different solitons have different velocities.

For a three-wave equation, $\beta_i(\lambda)=K_i$, which is independent of λ. (16) is always violated by taking pairs (i,α), (i,β), (i,γ) for different α,β,γ. Hence Proposition 4 of Ref. 14 is no longer valid. From (13), the formula for the velocities for the solitons, the solitons will not split as $t\to\infty$.

Remark 2: Since $u_{\beta\alpha}^{*}=-u_{\alpha\beta}$, we will always consider $u_{\alpha\beta}$ $(\alpha<\beta)$ from now on.

III. SINGLE-SOLITON SOLUTIONS

Take $l=1$,

$$H=\begin{pmatrix} e^{\sqrt{-1}\lambda_0(x+Jy+Kt)} \\ C \end{pmatrix}, \tag{17}$$

where $C=(C_{jk})$ is a 3×3 nondegenerate constant matrix, then

$$\Sigma=\frac{1}{2b\sqrt{-1}}\,(e^{-2b(\zeta+J_\mu\eta)-2b\pi_\mu t}\delta_{\mu\nu}+g_{\mu\nu})_{3\times 3}, \tag{18}$$

with

$$\begin{gathered} x=v_xt+\xi,\quad y=v_yt+\eta, \\ \pi_\mu=v_x+J_\mu v_y+K_\mu,\quad a=\mathrm{Re}(\lambda_0),\quad b=\mathrm{Im}(\lambda_0),\quad g_{\mu\nu}=C_{\sigma\mu}^{*}C_{\sigma\nu}. \end{gathered} \tag{19}$$

Since the solitons moves in a fixed velocity for each component $u_{\alpha\beta}$, we consider the solution in a moving frame with velocity (v_x,v_y). The asymptotic behavior of the soliton is clearly seen in this frame.

When (v_x,v_y) takes the value in (13), $\pi_\alpha=\pi_\beta=0$. Without loss of generality, suppose $\pi_2=\pi_3=0$, then $\pi_1\neq 0$ by considering (3).

Let $(h_{\mu\nu})=(g_{\mu\nu})^{-1}$, $g=(g_{\mu\nu})$, $h=(h_{\mu\nu})$, then by direct calculation from (12), we get the new solution of (1) as follows:

$$\begin{aligned} u_{12}&=-\frac{4b}{\Delta}\,(h_{12}\sqrt{\det g}\,e^{b\xi+\theta_3}-g_{12}\sqrt{\det h}\,e^{-b\xi-\theta_3})e^{\sqrt{-1}a(J_1-J_2)\eta+\sqrt{-1}a\pi_1t}, \\ u_{13}&=-\frac{4b}{\Delta}\,(h_{13}\sqrt{\det g}\,e^{b\xi+\theta_2}-g_{13}\sqrt{\det h}\,e^{-b\xi-\theta_2})e^{\sqrt{-1}a(J_1-J_3)\eta+\sqrt{-1}a\pi_1t}, \\ u_{23}&=-\frac{4b}{\Delta}\,(h_{23}e^{s(t)}-g_{23}e^{-s(t)})e^{\sqrt{-1}a(J_2-J_3)\eta}, \end{aligned} \tag{20}$$

where

$$\begin{gathered} \begin{aligned} \Delta=&\,e^{s(t)}(e^{\rho}+g_{11}\det h e^{-\rho}+h_{22}e^{\theta_3-\theta_2}+h_{33}e^{\theta_2-\theta_3}) \\ &+e^{-s(t)}(e^{-\rho}+h_{11}\det g e^{\rho}+g_{22}e^{\theta_2-\theta_3}+g_{33}e^{\theta_3-\theta_2}), \end{aligned} \\ s(t)=b(\xi+J_1\eta+\pi_1t)+\tfrac{1}{2}\ln(\det g), \\ \rho=b(2\xi+J_2\eta+J_3\eta),\quad \theta_j=bJ_j\eta\quad (j=1,2,3). \end{gathered} \tag{21}$$

Whenever $b\neq 0$, u_{23} tends to a single localized peak as $t\to\pm\infty$, and both u_{12}, u_{13} tend to zero exponentially. More precisely, when $b\pi_1t\to+\infty$ (ξ, η bounded),

$$u_{12}\sim 0,\quad u_{13}\sim 0,$$
$$u_{23}\sim \frac{-4bh_{23}e^{\sqrt{-1}a(J_2-J_3)\eta}}{e^{\rho}+g_{11}\det he^{-\rho}+h_{22}e^{\theta_3-\theta_2}+h_{33}e^{\theta_2-\theta_3}},\tag{22}$$

and when $b\pi_1 t\to-\infty$ (ξ, η bounded),

$$u_{12}\sim 0,\quad u_{13}\sim 0,$$
$$u_{23}\sim \frac{4bg_{23}e^{\sqrt{-1}a(J_2-J_3)\eta}}{e^{-\rho}+h_{11}\det ge^{\rho}+g_{22}e^{\theta_2-\theta_3}+g_{33}e^{\theta_3-\theta_2}}.\tag{23}$$

The asymptotic solutions of u_{23} for $t\to-\infty$ and $t\to+\infty$ differ in the amplitudes as well as the phases. That is because there is an interaction of u_{12}, u_{13} and u_{23} around $t=0$. The total energy $\int(|u_{12}|^2+|u_{13}|^2+|u_{23}|^2)dx\,dy$ still conserves.

At an extreme case, let $h_{23}=0$ and $g_{23}\neq 0$ [this is always possible since the only constraints on $(g_{\mu\nu})$ are Hermitian and positivity], then a u_{23} wave disappears as $t\to+\infty$ although it exists as $t\to-\infty$.

On the other hand, if $(g_{\mu\nu})$ is unitary, then $h_{\mu\nu}=\overline{g_{\nu\mu}}$ and the asymptotic behavior of $|u_{23}|$ as $t\to-\infty$ and $t\to+\infty$ are the same.

The asymptotic behavior of u_{12} is similar when $\pi_1=\pi_2=0$, so is that of u_{13} when $\pi_1=\pi_3=0$.

IV. MULTISOLITON SOLUTIONS

Since the velocities of solitons do not depend on the choice of λ, Proposition 4 of Ref. 14 that is explained in Remark 1 fails. However, we can still see several peaks after twice Darboux transformations, although all these peaks move in the same velocity.

For $l>1$, we have obtained the solution $u_{\alpha\beta}^{[l]}$ given by (12). Now that the limit $t\to\pm\infty$ cannot separate these peaks, we consider the asymptotic behavior of the solutions when $C\to\infty$ in some way. This describes the separation of the peaks when the phase of each single soliton is large enough.

It is well known that H_j and $H_j\cdot\mathrm{diag}(\gamma_1^{(j)},\gamma_2^{(j)},\gamma_3^{(j)})$ ($\gamma_1^{(j)},\gamma_2^{(j)},\gamma_3^{(j)}$ are constants) give the same solution $u_{\alpha\beta}^{[l]}$ [see also (10) and (11)]. Therefore, we can rewrite H_j in (9) as

$$H_j=\begin{pmatrix} e^{\sqrt{-1}\lambda_j(x+Jy+Kt)-QL^{(j)}} \\ D^{(j)}\end{pmatrix}e^{QL^{(j)}},\tag{24}$$

where $L^{(j)}=\mathrm{diag}(L_1^{(j)},L_2^{(j)},L_3^{(j)})$ are arbitrary real constant diagonal matrices, Q is a real number, and $D^{(j)}=C^{(j)}e^{-QL^{(j)}}$.

The discussion in Secs. IV and VI of Ref. 14 can be used directly here. Let $x=\xi_0+Qw_\xi$, $y=\eta_0+Q\omega_\eta$ where w_ξ, w_η are real constants (‘‘velocity’’ with respect to Q). Let

$$\begin{aligned}\sqrt{-1}\phi_{i\alpha}&\equiv\sqrt{-1}\lambda_i(x+J_\alpha y+K_\alpha t)-QL_\alpha^{(i)}\\&=\sqrt{-1}\lambda_i(\xi_0+J_\alpha\eta_0+K_\alpha t)+(\sqrt{-1}\lambda_i(w_\xi+J_\alpha w_\eta)-L_\alpha^{(i)})Q.\end{aligned}$$

Suppose

$$\det\begin{pmatrix}1 & J_\alpha & L_\alpha^{(i)}/\lambda_{iI}\\ 1 & J_\beta & L_\beta^{(j)}/\lambda_{jI}\\ 1 & J_\gamma & L_\gamma^{(k)}/\lambda_{kI}\end{pmatrix}\neq 0\tag{25}$$

for mutually different pairs (α,i), (β,j), (γ,k). Here λ_{iI} is the imaginary part of λ_α.

Now fix $D^{(j)}$, $L^{(j)}$, and let $Q\to+\infty$ or $-\infty$. Note that

$$\mathrm{Re}(\sqrt{-1}\lambda_i(w_\xi+J_\alpha w_\eta)-L_\alpha^{(i)})=-\lambda_{iI}(w_\xi+J_\alpha w_\eta)-L_\alpha^{(i)}.$$

Therefore, Im $\phi_{i\alpha}$ keeps finite if and only if $\lambda_{iI}(w_\xi+J_\alpha w_\eta)+L_\alpha^{(i)}=0$.

The discussion in Secs. IV and VI of Ref. 14 implies the following.

Proposition 1: Suppose the matrix H_j in (9), (10), (11) can be written as

$$H_j=\begin{pmatrix} e^{QA^{(j)}+B^{(j)}} \\ D^{(j)} \end{pmatrix}$$

where Q is a parameter, $A^{(j)}=diag(A_1^{(j)},\ldots,A_N^{(j)}),B^{(j)}=diag(B_1^{(j)},\ldots,B_N^{(j)})$ are $N\times N$ diagonal matrices independent of $Q,D^{(j)}$ are $N\times N$ matrices independent of Q. Let $u_{\alpha\beta}^{[l]}$ be the solution given by l Darboux transformations. Then the following occurs.

(1) If there is at most one pair (α,i) such that $A_\alpha^{(i)}=0$, then $u_{\mu\nu}^{[l]}\to 0$ for all $\mu<\nu$ when $Q\to+\infty$ or $-\infty$.

(2) If there are exactly two pairs (α,i)≠(β,j) (α<β) such that $A_\alpha^{(i)}=A_\beta^{(j)}=0$, then, when $Q\to+\infty$ or $-\infty$, $u_{\mu\nu}^{[l]}\to 0$ for all $\mu<\nu$ except $\mu=\alpha$, $\nu=\beta$.

For the H_j's in (24), Proposition 1 implies the following.

(1) If (w_ξ,w_η) is chosen so that there is at most one pair (α,i), such that $\lambda_{iI}(w_\xi+J_\alpha w_\eta)+L_\alpha^{(i)}=0$, then $u_{\mu\nu}^{[l]}\to 0$ for all $\mu<\nu$ when $Q\to+\infty$ or $-\infty$.

(2) If (w_ξ,w_η) is chosen so that there are exactly two pairs $(\alpha,i)\neq(\beta,j)(\alpha<\beta)$ such that

$$\begin{aligned}&\lambda_{iI}(w_\xi+J_\alpha w_\eta)+L_\alpha^{(i)}=0,\\ &\lambda_{jI}(w_\xi+J_\beta w_\eta)+L_\beta^{(j)}=0,\end{aligned}\tag{26}$$

i.e.,

$$w_\xi=-\frac{L_\alpha^{(i)}/\lambda_{iI}-L_\beta^{(j)}/\lambda_{jI}}{J_\alpha-J_\beta},\quad w_\eta=-\frac{J_\alpha L_\beta^{(j)}/\lambda_{jI}-J_\beta L_\alpha^{(i)}/\lambda_{iI}}{J_\alpha-J_\beta},\tag{27}$$

then, when $Q\to+\infty$ or $-\infty$, $u_{\mu\nu}^{[l]}\to 0$ for all $\mu<\nu$ except $\mu=\alpha$, $\nu=\beta$.

(3) If (25) holds for mutually different pairs (α,i), (β,j), (γ,k), then for any (w_ξ,w_η), there are at most two pairs (α,i), (β,j) such that (26) holds.

Therefore, for any α, β, unless (w_ξ,w_η) takes the value in (27), $u_{\alpha\beta}^{[l]}\to 0$. Since i,j in (27) can be $1,\ldots,l$, respectively, $u_{\alpha\beta}^{[l]}$ has at most l^2 peaks. Thus, we have the following.

Theorem 1: *Let λ_j (j=1,...,l) be complex numbers such that $\lambda_j\neq\lambda_k(j\neq k),\lambda_j\neq\overline{\lambda_k}$ for all $1\leqslant j,k\leqslant l$. Let $L^{(j)}$'s be 3×3 constant diagonal matrices such that (25) holds, $D^{(j)}$'s be 3×3 nondegenerate constant matrices. Let $C^{(j)}=D^{(j)}e^{QL^{(j)}}$ and construct H_j as in (9). Then as $Q\to+\infty$ or $-\infty$, each component $u_{\alpha\beta}^{[l]}$ of the solution derived by the Darboux transformation of degree l has at most l^2 peaks for fixed time t.*

Remark 3: Generally, when (26) holds, $u_{\alpha\beta}^{[l]}$ tends to a nonzero function unless its coefficient tends to zero. Hence, usually there are exact l^2 peaks for each $u_{\alpha\beta}^{[l]}$ when Q is large enough. Of course, if a certain coefficient vanishes (e.g., $h_{23}=0$ or $g_{23}=0$ for u_{23} in a single-soliton case), the corresponding peak disappears.

When we only consider the asymptotic behavior of $u_{\alpha\beta}^{[l]}$, a condition weaker than (25) is enough to separate the peaks as $Q\to\infty$. This is because as $t\to\infty$, the interaction between $u_{\alpha\beta}^{[l]}$ and any other $u_{\mu\nu}^{[l]}$ vanishes.

As in Sec. III consider u_{23} and choose the velocity such that $\pi_2=\pi_3=0$, then $\pi_1\neq 0$. Take the coordinates ξ, η given by (19). Since $b_j\equiv\lambda_{jI}\neq 0$, $b_j\pi_1 t\to\infty$ as $t\to\infty$. Therefore, the limit Darboux matrix can be constructed by

$$H_j(-\infty)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{\sqrt{-1}\lambda_j(\xi+J_2\eta)-QL_2^{(j)}} & 0 \\ 0 & 0 & e^{\sqrt{-1}\lambda_j(\xi+J_3\eta)-QL_3^{(j)}} \\ D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}$$

if $b\pi_1 t\to-\infty$, or

$$H_j(+\infty)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\sqrt{-1}\lambda_j(\xi+J_2\eta)-QL_2^{(j)}} & 0 \\ 0 & 0 & e^{\sqrt{-1}\lambda_j(\xi+J_3\eta)-QL_3^{(j)}} \\ 0 & D_{12} & D_{13} \\ 0 & D_{22} & D_{23} \\ 0 & D_{32} & D_{33} \end{pmatrix}$$

if $b\pi_1 t\to+\infty$. Now let $\xi=\xi_0+Qw_\xi$, $\eta=\eta_0+Qw_\eta$. Clearly, if $(H_j(-\infty))_{(1,1)}$ is replaced by $e^{-|Q|}$ and $(H_j(+\infty))_{(1,1)}$ is replaced by $e^{|Q|}$, the limit Darboux transformation is the same when $Q\to\infty$. Then, the coefficients of Q in the power of $(H_j(\pm\infty))_{(1,1)}$ are always nonzero. Therefore, according to Proposition 1, we only need to guarantee that (25) holds for $(\alpha,i)\neq(\beta,j)$ with $\alpha,\beta\in\{2,3\}$ so that $u_{\alpha\beta}^{[l]}(-\infty)$, the asymptotic solution for $t\to-\infty$, does not tend to zero, and $u_{\mu\nu}^{[l]}(-\infty)\to 0$ for any $(\mu,\nu)\neq(\alpha,\beta)$. This implies that we only need that (25) holds for all mutually different pairs (α,i), (β,j), (γ,k) with $\alpha,\beta,\gamma\in\{2,3\}$.

Since α, β, γ can only take two values and J_α's are mutually different, (25) is equivalent to

$$L_\alpha^{(i)}/\lambda_{iI}\neq L_\alpha^{(j)}/\lambda_{jI}\quad(\alpha=2,3,\quad i\neq j).$$

The same argument can be applied to $u_{23}^{[l]}(+\infty)$, the asymptotic solution for $t\to+\infty$.

Clearly, there are corresponding conclusions for $u_{12}^{[l]}$, $u_{13}^{[l]}$. Therefore, we have the following.

Theorem 2: *Given two integers α, β with $1\leq\alpha<\beta\leq 3$. Let λ_j $(j=1,...,l)$ be as in Theorem 1, $D^{(j)}$'s be 3×3 nondegenerate constant matrices. Let $L^{(j)}$'s be 3×3 constant diagonal matrices such that $L_\alpha^{(i)}/\lambda_{iI}\neq L_\alpha^{(j)}/\lambda_{jI}$, $L_\beta^{(i)}/\lambda_{iI}\neq L_\beta^{(j)}/\lambda_{jI}$ for $i\neq j$. Let $C^{(j)}=D^{(j)}e^{QL^{(j)}}$ and construct H_j as in (9). Then as $Q\to+\infty$ or $-\infty$, the asymptotic solutions $u_{\alpha\beta}^{[l]}(-\infty)$ and $u_{\alpha\beta}^{[l]}(+\infty)$ have at most l^2 peaks respectively.*

Remark 4: Both Theorem 1 and Theorem 2 are valid for the N-wave equation, except that 3 is replaced by N. Here we discuss three-wave equation for its simplicity and importance.

Remark 5: Return to the case of single-soliton solutions. In (20), (21), the phase of the soliton is characterized by

$$\phi_1=\ln(e^{s^{(j)}(t)}+e^{-s^{(j)}(t)}h_{11}^{(j)}\det g^{(j)})-\ln(e^{-s^{(j)}(t)}+e^{s^{(j)}(t)}g_{11}^{(j)}\det h^{(j)}),$$

$$\phi_2=\ln(e^{s^{(j)}(t)}h_{22}^{(j)}+e^{-s^{(j)}(t)}g_{33}^{(j)})-\ln(e^{-s^{(j)}(t)}g_{22}^{(j)}+e^{s^{(j)}(t)}h_{33}^{(j)}).$$

If we let $L_{\alpha\alpha}^{(j)}=g_{\alpha\alpha}^{(j)}/Q$, then, when $L^{(j)}$ is fixed, $Q\to+\infty$ implies $\phi_1\to+\infty$ and ϕ_2 is bounded.

Here we show some figures that describe the behavior of the solitons given by a Darboux transformation of degree 2. Figures 1, 3, 5, and 7 correspond to time $t=-16$, 0, 12, and 16, respectively. As discussed above, these solitons have at most four peaks. Here the vertical axis is $|u|^{1/2}\equiv(|u_{12}|^2+|u_{13}|^2+|u_{23}|)^{1/4}$. Therefore, all three components are shown in one figure. The corresponding parameters are

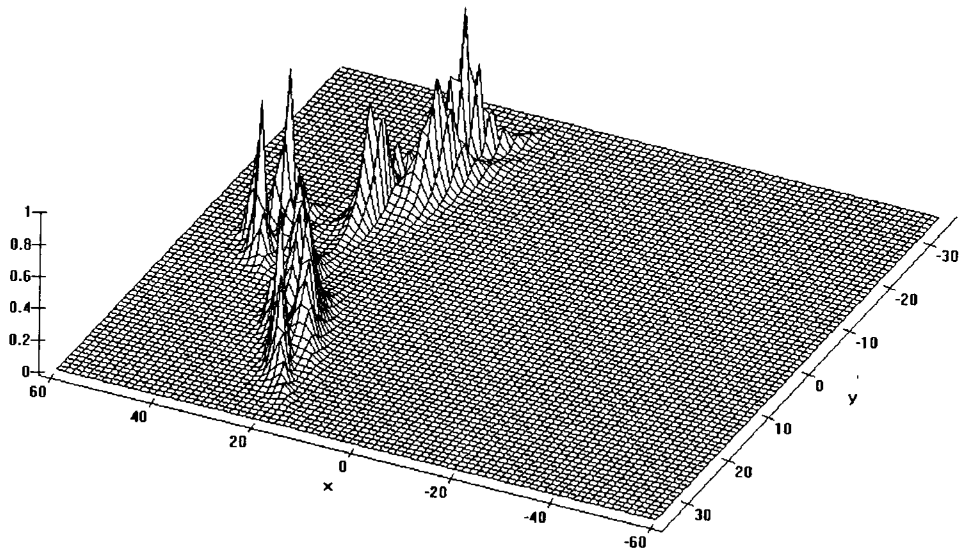

1
0.8
0.6
0.4
0.2
0
60
40
20
0
x
-20
-40
-60
-30
-20
-10
0
y
10
20
30

FIG. 1

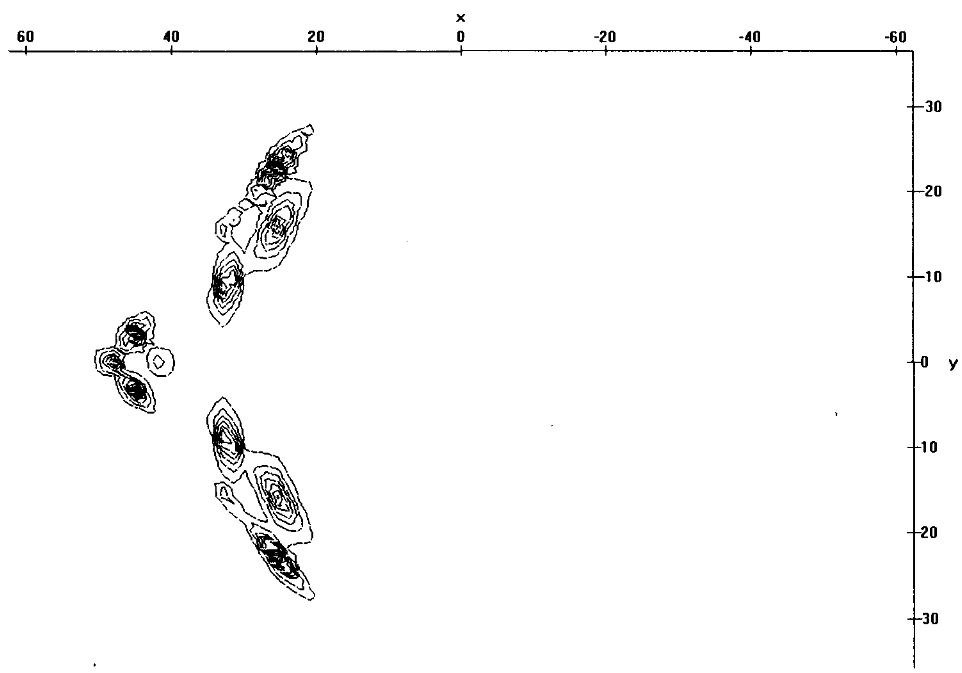

x
60
40
20
0
-20
-40
-60
30
20
10
0
y
-10
-20
-30

FIG. 2

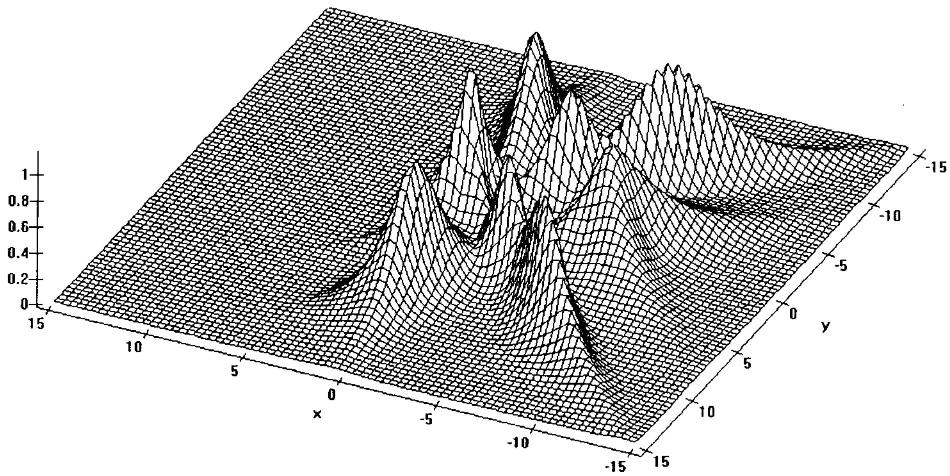

1
0.8
0.6
0.4
0.2
0
15
10
5
0
x
-5
-10
-15
15
10
5
0
y
-5
-10
-15

FIG. 3

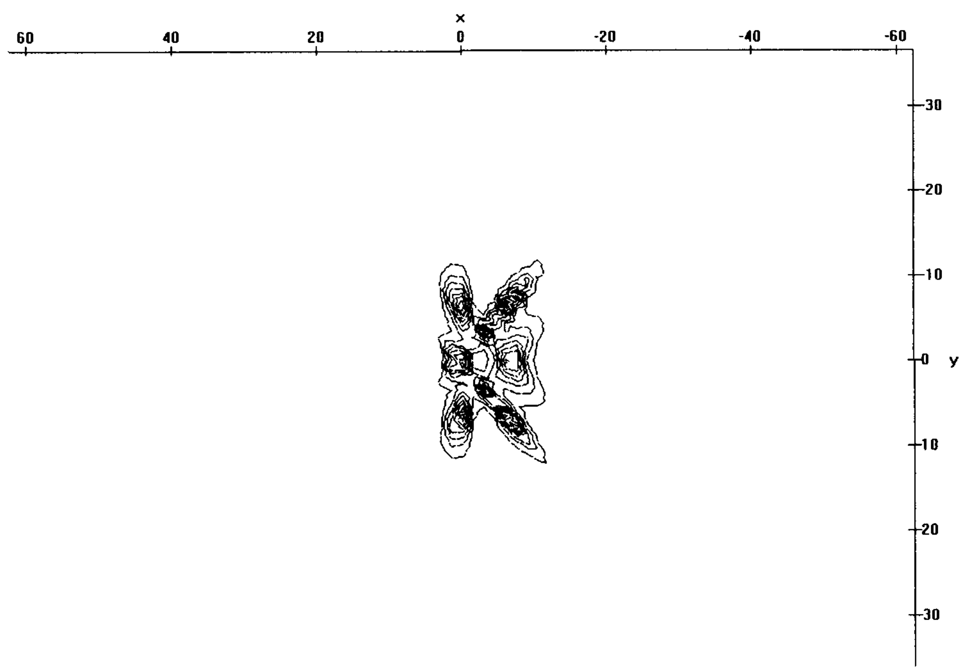
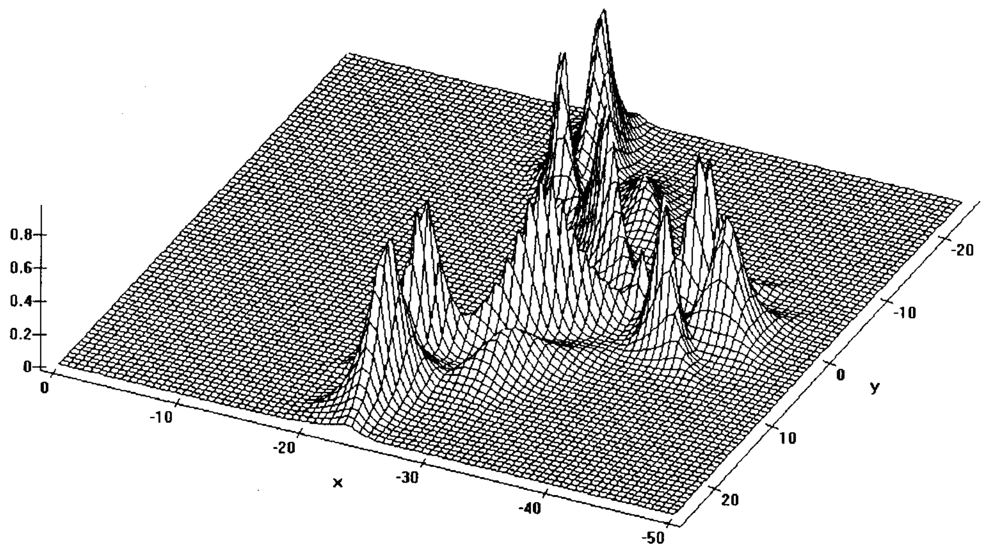
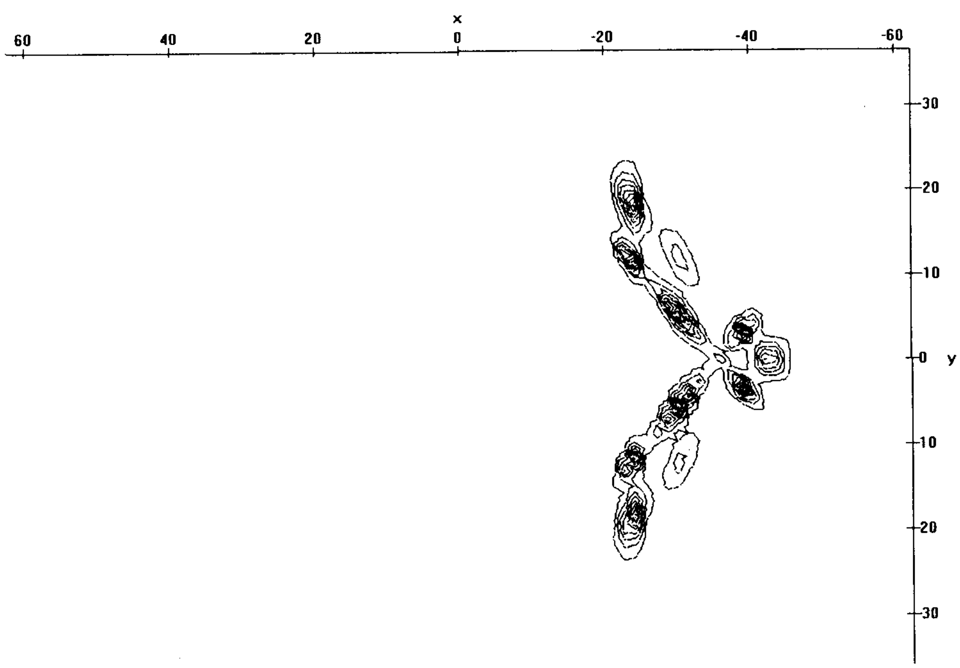

x
60
40
20
0
-20
-40
-60
30
20
10
0
y
10
20
30

FIG. 4



0.8
0.6
0.4
0.2
0
0
-10
-20
x
-30
-40
-50
-20
-10
0
y
10
20

FIG. 5



x
60
40
20
0
-20
-40
-60
30
20
10
0
y
10
20
30

FIG. 6

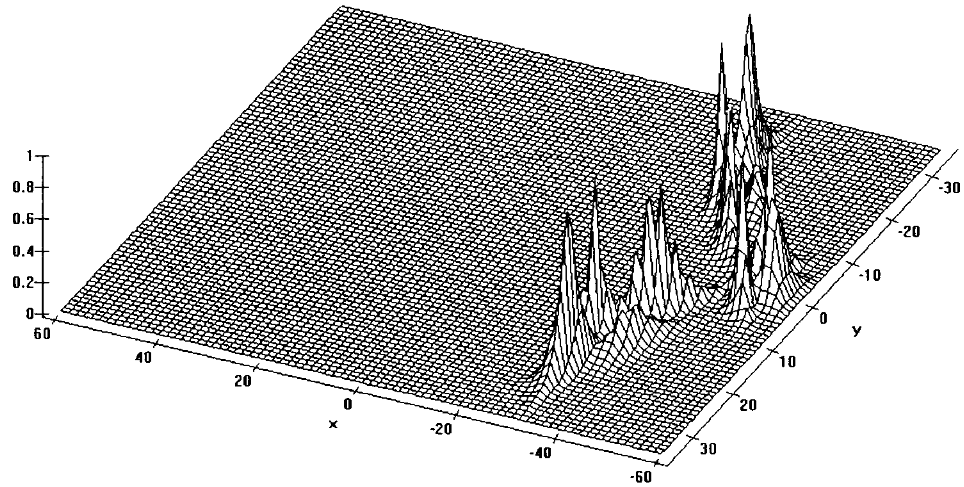

1
0.8
0.6
0.4
0.2
0
60
40
20
0
x
-20
-40
-60
30
20
10
0
y
-10
-20
-30

FIG. 7

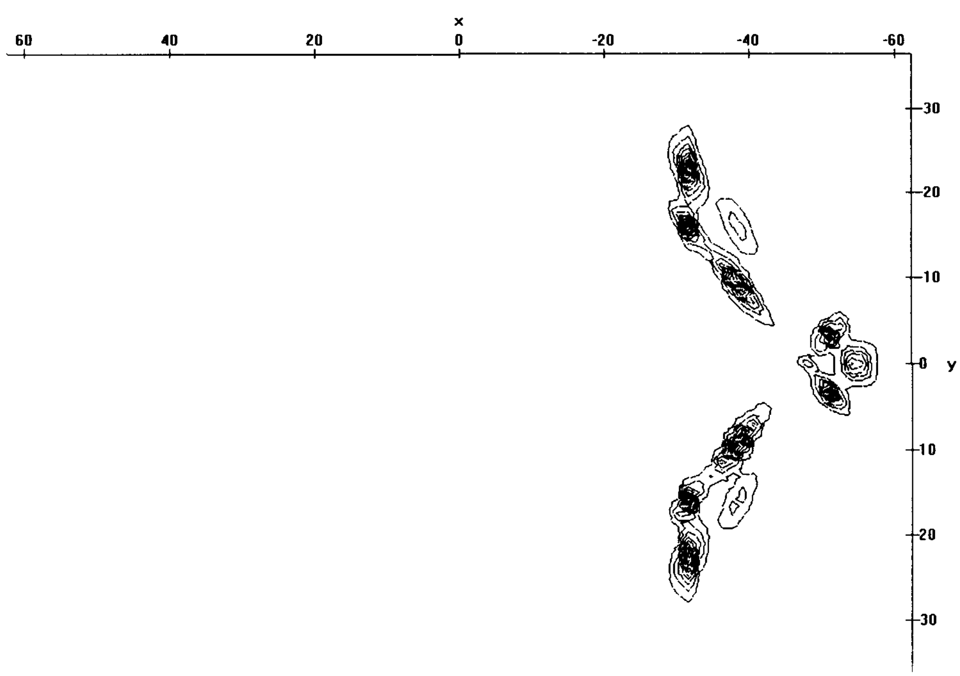

x
60
40
20
0
-20
-40
-60
30
20
10
0
y
10
20
30

FIG. 8

$$J=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$K=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$\lambda_1=1-2\sqrt{-1}, \quad \lambda_2=2+\sqrt{-1},$$

$$C_2=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad C_1=256C_2.$$

Figures 2, 4, 6, and 8 are the corresponding contour plot of Figs. 1, 3, 5, and 7. Note that in Fig. 3 and Fig. 5, the scales of the coordinates are different from the others so that the peaks can be seen more clearly.

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