Inverse Problems **12** (1996) 89–109. Printed in the UK

Soliton solutions for some equations in the $(1+2)$-dimensional hyperbolic $su(N)$ AKNS system*

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Received 10 July 1995

Abstract. For the $(1+2)$-dimensional AKNS system of lower order, a linear system which separates variables is constructed. It is found out that any nonlinear equation generated by such a linear hyperbolic $su(N)$ system admits solutions which tend to zero in all directions in the space. The solutions given by nth Darboux transformation split up into more than n solitons as $t \to \infty$. The application to the Davey–Stewartson I equation is considered.

1. Introduction

We consider the following linear system:

$$\Psi_y = J\Psi_x + U(x,y,t)\Psi \qquad \Psi_t = \sum_{j=0}^{n} V_j(x,y,t)\partial^{n-j}\Psi \tag{1.1}$$

where $\partial = \partial/\partial x$, $J = \mathrm{diag}(J_1,\ldots,J_N)$ is a real constant diagonal $N \times N$ matrix with mutually different diagonal entries, $U(x,y,t)$ is off-diagonal with $U^* = -U$. In this case, we call (1.1) a hyperbolic $su(N)$ AKNS system, since J is real and $U \in su(N)$.

The system (1.1) is overdetermined, whose integrability conditions are

$$[J, V_{j+1}^A] = V_{j,y}^A - JV_{j,x}^A - [U, V_j]^A + \sum_{k=0}^{j-1} C_{n-k}^{n-j}(V_k\partial^{j-k}U)^A \tag{1.2}$$

$$V_{j,y}^D - JV_{j,x}^D = [U, V_j^A]^D - \sum_{k=0}^{j-1} C_{n-k}^{n-j}(V_k\partial^{j-k}U)^D \tag{1.3}$$

$$U_t = V_{n,y}^A - JV_{n,x}^A - [U, V_n]^A + \sum_{k=0}^{n-1}(V_k\partial^{n-k}U)^A \tag{1.4}$$

where the superscripts D and A refer to the diagonal and off-diagonal parts of a matrix. We regard (1.3), (1.4) as the nonlinear equations of unknowns U, V_j^D $(j = 0, 1, \ldots, n)$'s in which V_j^A $(j = 0, 1, \ldots, n)$'s are determined by (1.2).

* Partially supported by National Research Project 'Nonlinear Science', FEYUT-SEDC-CHINA and Fok Ying-Tung Education Foundation of China.

0266-5611/96/010089+21$19.50

Equations (1.3), (1.4) include several important equations in $(1+2)$ dimensions, such as the N-wave equation, Davey–Stewartson (DS) equation, and so on.

Equation (1.1) can be solved by various methods. In [5, 16], it is related with another linear system which separates variables, and some examples are discussed in [6, 9, 12, 15]. In these papers, the solutions do not decay at infinity in all directions. In N-wave and DSI cases, they correspond to the one column P in (2.1). In the present paper, we shall use a similar method, but introduce an $N\times N$ matrix P, to construct concrete solutions of (1.3), (1.4). We find out that for any equations in the system (1.3), (1.4) with $n\leqslant 3$, there exist solutions which tend to zero in all directions in the (x, y) plane. The solutions are obtained by Darboux transformations from a trivial solution with a suitable choice of parameters.

For the DSI equation, the soliton solutions which tends to zero in all directions were first discovered by [1] and then have been discussed in various papers such as [2–4, 7, 8, 10, 11, 13, 14]. Here we use the above-mentioned purely algebraic method to obtain these solutions. Moreover, the explicit conditions on the parameters are given to determine the appearance of each peaks in our cases.

2. A linear system related with $(1+2)$-dimensional AKNS system

We propose the following overdetermined system:

$$\begin{aligned}
&\Phi_x = \begin{pmatrix} \mathrm{i}\lambda I & \mathrm{i}P \\ \mathrm{i}P^* & 0 \end{pmatrix}\Phi \qquad \Phi_y = \begin{pmatrix} \mathrm{i}\lambda J + U & \mathrm{i}JP \\ \mathrm{i}P^*J & 0 \end{pmatrix}\Phi \\
&\Phi_t = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}\Phi = \sum_{j=0}^{n}\begin{pmatrix} W_j & X_j \\ -X_j^* & Z_j \end{pmatrix}\lambda^{m-j}\Phi
\end{aligned} \tag{2.1}$$

where P, W_j, X_j, Z_j are $N\times N$ matrices to be determined by the integrability condition of (2.1), and satisfy $W_j^* = -W_j$, $Z_j^* = -Z_j$.

The integrability conditions of (2.1) are

$$P_y = JP_x + UP \qquad U_x = [J, PP^*] \tag{2.2}$$

$$W_{j,x} = -\mathrm{i}PX_j^* - \mathrm{i}X_jP^* \tag{2.3a}$$

$$X_{j,x} = \mathrm{i}X_{j+1} - \mathrm{i}W_jP + \mathrm{i}PZ_j \tag{2.3b}$$

$$Z_{j,x} = \mathrm{i}P^*X_j + \mathrm{i}X_j^*P \tag{2.3c}$$

$$\mathrm{i}P_t = X_{n,x} - \mathrm{i}PZ_n + \mathrm{i}W_nP \tag{2.3d}$$

$$0 = W_{n,x} + \mathrm{i}PX_n^* + \mathrm{i}X_nP^* \tag{2.3e}$$

$$W_{j,y} = \mathrm{i}[J, W_{j+1}] + [U, W_j] - \mathrm{i}JPX_j^* - \mathrm{i}X_jP^*J \tag{2.4a}$$

$$X_{j,y} = \mathrm{i}JX_{j+1} + UX_j + \mathrm{i}JPZ_j - \mathrm{i}W_jJP \tag{2.4b}$$

$$Z_{j,y} = \mathrm{i}P^*JX_j + \mathrm{i}X_j^*JP \qquad (j = 0, 1, \ldots, n-1) \tag{2.4c}$$

$$\mathrm{i}JP_t = X_{n,y} - UX_n - \mathrm{i}JPZ_n + \mathrm{i}W_nJP \tag{2.4d}$$

$$U_t = W_{n,y} - [U, W_n] + \mathrm{i}JPX_n^* + \mathrm{i}X_nP^*J\,. \tag{2.4e}$$

For $n\leqslant 3$, we can choose proper integral constants in (2.3c) and (2.3a) such that W_j, Z_j $(j\leqslant 3)$ are differential polynomials of P and U.

For small j, we have

$$
\begin{aligned}
&X_0 = 0 \qquad Z_0 = 0 \qquad W_0 = \mathrm{i}K_0(t) \\
&X_1 = \mathrm{i}K_0 P \qquad Z_1 = 0 \qquad W_1 = U^{[0]} + \mathrm{i}K_1(t) \\
&X_2 = K_0 P_x + U^{[0]}P + \mathrm{i}K_1 P \qquad Z_2 = \mathrm{i}P^* K_0 P \\
&W_2 = -\mathrm{i}(\mathrm{ad}\, J)^{-1}\big(U_y^{[0]} - [U, U^{[0]}] - K_0 P P^* J + J P P^* K_0\big) \\
&\qquad\qquad + U^{[1]} - \mathrm{i}\big(K_0(PP^*)^D + (U(\mathrm{ad}\, J)^{-1}U^{[0]})^D\big) + \mathrm{i}K_2(t) \\
&X_3 = -\mathrm{i}K_0 P_{xx} - \mathrm{i}(U^{[0]} + \mathrm{i}K_1)P_x \\
&\qquad\qquad - \mathrm{i}(\mathrm{ad}\, J)^{-1}\big(U_y^{[0]} + JK_0PP^* + JPP^*K_0 - 2K_0PP^*J - [U, U^{[0]}]\big)P \\
&\qquad\qquad + U^{[1]}P - \mathrm{i}\big(2K_0(PP^*)^D + (U(\mathrm{ad}\, J)^{-1}U^{[0]})^D\big)P + \mathrm{i}K_2 P
\end{aligned}
\tag{2.5}
$$

where $U^{[i]} = (\mathrm{ad}\, J)^{-1}[K_i, U]$, $(\mathrm{ad}\, J)^{-1}(M) = M'$ if M' is off-diagonal and $[J, M'] = M$.

For $n = 1$, the equations are

$$
\begin{aligned}
&P_y = JP_x + UP \\
&P_t = K_0 P_x + U^{[0]}P + \mathrm{i}K_1 P \\
&U_t = U_y^{[0]} + J(\mathrm{ad}\, J)^{-1}U_x K_0 - K_0(\mathrm{ad}\, J)^{-1}U_x J - [U, U^{[0]}] + \mathrm{i}[K_1, U]
\end{aligned}
\tag{2.6}
$$

with the constraint

$$
U_x = [J, PP^*]\,. \tag{2.7}
$$

The equations (2.6) are the N-wave equations with its Lax pairs [15].

For $n = 2$,

$$
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}
$$

and $K_0 = -2J$, $K_1 = K_2 = 0$, we have

$$
\begin{aligned}
&P_y = JP_x + UP \\
&P_t = 2\mathrm{i}JP_{xx} + 2\mathrm{i}UP_x + \mathrm{i}\begin{pmatrix} |u|^2 + 4Q_1 & u_x + u_y \\ -\bar{u}_x + \bar{u}_y & -|u|^2 - 4Q_2 \end{pmatrix} P \\
&-\mathrm{i}u_t = u_{xx} + u_{yy} + 2|u|^2 u + 4\mathrm{i}(Q_1 + Q_2)u \\
&Q_{1x} - Q_{1y} = -\tfrac{1}{2}(|u|^2)_x \qquad Q_{2x} + Q_{2y} = -\tfrac{1}{2}(|u|^2)_x
\end{aligned}
\tag{2.8}
$$

with the constraints

$$
(PP^*)^D = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \qquad [J, (PP^*)^A] = U_x\,. \tag{2.9}
$$

This is the Davey–Stewartson I (DSI) equation [16].

3. Darboux transformation

As in [9, 16], the Darboux transformation is still valid for (2.1) and its integrability conditions. We have

Proposition 1. Suppose (P, U, W_j, X_j, Z_j) satisfy (2.2)–(2.4). Let $\lambda_0 \in \mathbf{C}$ such that $\operatorname{Im}\lambda_0 \neq 0$. Let $\lambda_1 = \cdots = \lambda_N = \lambda_0$, $\lambda_{N+1} = \cdots = \lambda_{2N} = \bar{\lambda}_0$, h_i be a solution of (2.1) with $\lambda = \lambda_i$ which satisfies $h_i^* h_j = 0$ if $\lambda_i \neq \lambda_j$ and $\det H \neq 0$. Denote $S = H\Lambda H^{-1}$ by

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

where S_{ij}'s are $N \times N$ matrices, then $\widetilde{\Phi} = (\lambda - S)\Phi$ satisfies

$$\widetilde{\Phi}_x = \begin{pmatrix} \mathrm{i}\lambda I & \mathrm{i}\widetilde{P} \\ \mathrm{i}\widetilde{P}^* & 0 \end{pmatrix} \widetilde{\Phi} \qquad \widetilde{\Phi}_y = \begin{pmatrix} \mathrm{i}\lambda J + \widetilde{U} & \mathrm{i}J\widetilde{P} \\ \mathrm{i}\widetilde{P}^* J & 0 \end{pmatrix} \widetilde{\Phi}$$
$$\widetilde{\Phi}_t = \sum_{j=0}^{n} \begin{pmatrix} \widetilde{W}_j & \widetilde{X}_j \\ -\widetilde{X}_j^* & \widetilde{Z}_j \end{pmatrix} \lambda^{m-j} \widetilde{\Phi} \tag{3.1}$$

where

$$\widetilde{U} = U + \mathrm{i}[J, S_{11}] \qquad \widetilde{P} = P + S_{12} \tag{3.2}$$

and $\widetilde{W}_j$, $\widetilde{X}_j$, $\widetilde{Z}_j$ are uniquely determined by

$$\sum_j \begin{pmatrix} \widetilde{W}_j & \widetilde{X}_j \\ -\widetilde{X}_j^* & \widetilde{Z}_j \end{pmatrix} \lambda^{m-j}(\lambda - S) = \sum_j \begin{pmatrix} W_j & X_j \\ -X_j^* & Z_j \end{pmatrix} \lambda^{m-j}(\lambda - S) - S_t\,.$$

Remark. The H satisfying the above conditions always exists, because $\det H \neq 0$ and $h_i^* h_j = 0$ for $\lambda_i \neq \lambda_j$ hold identically if they hold at one point (x_0, y_0, t_0) [9].

4. Single-soliton solutions

Take the seed solution $P = 0$, $U = 0$. From (2.3), (2.4), we have $W_j = \mathrm{i}\omega_j(t)$, $X_j = 0$, $Z_j = \mathrm{i}\zeta_j(t)$ where $\omega_j(t)$'s are real diagonal and $\zeta_j(t)$'s are real. For simplicity, take ω_j to be independent of t and $\zeta_j = 0$ for all j.

Let $\omega(\lambda) = \sum_{j=0}^{m} w_j \lambda^{m-j}$. By the choice of $\Lambda = (\lambda_0, \ldots, \lambda_0, \bar{\lambda}_0, \ldots, \bar{\lambda}_0)$ as in section 3, we can take, without loss of generality,

$$H = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\lambda_0(x+Jy)+\mathrm{i}\omega(\lambda_0)t} & -\mathrm{e}^{\mathrm{i}\bar{\lambda}_0(x+Jy)+\mathrm{i}\omega(\bar{\lambda}_0)t} C^* \\ C & I \end{pmatrix} \tag{4.1}$$

where C is a constant $N \times N$ matrix. Moreover, choose C to be non-degenerate. Let $\phi = \lambda_0(x + Jy) + \omega(\lambda_0)t$, $\Gamma = C^*C$, then

$$S_{11} = (\lambda_0 \mathrm{e}^{\mathrm{i}\phi} + \bar{\lambda}_0 \mathrm{e}^{\mathrm{i}\bar{\phi}}\Gamma)\Delta^{-1}$$
$$S_{21} = (\lambda_0 - \bar{\lambda}_0)C\Delta^{-1} \qquad S_{12} = -S_{21}^* \tag{4.2}$$
$$S_{22} = \bar{\lambda}_0 + (\lambda_0 - \bar{\lambda}_0)C\Delta^{-1}\mathrm{e}^{\mathrm{i}\bar{\phi}}C^*$$

where

$$\Delta = e^{i\phi} + e^{i\bar{\phi}}\Gamma . \tag{4.3}$$

Since Γ is positive definite, Δ is non-degenerate everywhere.

The transformations of U and P are given by

$$\begin{aligned} \widetilde{U} &= U + i[J, S_{11}] = i[J, (\lambda_0 e^{i\phi} + \bar{\lambda}_0 e^{i\bar{\phi}}\Gamma)\Delta^{-1})] \\ \widetilde{P} &= P + (\lambda_0 - \bar{\lambda}_0)(C\Delta^{-1})^* . \end{aligned} \tag{4.4}$$

Remark. We assume $\lambda_1 = \cdots = \lambda_N = \lambda_0$ and C is non-degenerate so that the solution decays exponentially in all directions, as discussed from now on.

For definiteness, here we suppose $\mathrm{Im}\,\lambda_0 > 0$. All the following discussions hold for $\mathrm{Im}\,\lambda_0 < 0$ provided that $x, y \to \pm\infty$ is replaced by $x, y \to \mp\infty$.

Now we consider the asymptotic behaviour of the solution $\widetilde{U}$ along a straight line L. If L is parallel with x-axis, it is easy to show that $S_{11} \to \bar{\lambda}_0 I$, $S_{12} \to 0$ exponentially as $x \to +\infty$ and $S_{11} \to \lambda_0 I$, $S_{12} \to 0$ exponentially as $x \to -\infty$. Hence $\widetilde{U} \to 0$ exponentially as $x \to \pm\infty$. In what follows, we suppose that L is not parallel with the x-axis. For given k, let $z = x - ky$ $(k \neq 0)$. By reordering the indices of the matrices, we can assume that

$$J_1 + k \leqslant \cdots \leqslant J_p + k < 0 < J_{p+1} + k \leqslant \cdots \leqslant J_{N-1} + k \qquad J_N + k = 0 \tag{4.5}$$

or

$$J_1 + k \leqslant \cdots \leqslant J_p + k < 0 < J_{p+1} + k \leqslant \cdots \leqslant J_N + k . \tag{4.6}$$

Under this choice of indices, write any $N \times N$ matrix M as the block matrix

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{matrix} p \\ q-p \\ N-q \end{matrix} \tag{4.7}$$

where $q = N - 1$ or $q = N$. (If $q = N$, the terms with subscript 3 disappear.) Using such a division, we have

$$\phi = \begin{pmatrix} \phi_1 & & \\ & \phi_2 & \\ & & \phi_3 \end{pmatrix}. \tag{4.8}$$

Then $\mathrm{Re}(i\phi_1) \to +\infty$, $\mathrm{Re}(i\phi_2) \to -\infty$ and $\mathrm{Re}(i\phi_3)$ stays finite as $y \to +\infty$ when z is finite. Here $\mathrm{Re}(i\phi_1) \to +\infty$ and $\mathrm{Re}(i\phi_2) \to -\infty$ mean that each entry of $\mathrm{Re}(i\phi_1)$ tends to $+\infty$ and each entry of $\mathrm{Re}(i\phi_2)$ tends to $-\infty$.

The matrix S is computed as follows. The first block of S is

$$\begin{aligned} S_{11} &= \frac{\lambda_0 + \bar{\lambda}_0}{2} + \frac{\lambda_0 - \bar{\lambda}_0}{2}(e^{i\phi} - e^{i\bar{\phi}}\Gamma)(e^{i\phi} + e^{i\bar{\phi}}\Gamma)^{-1} \\ &\equiv \frac{\lambda_0 + \bar{\lambda}_0}{2} + \frac{\lambda_0 - \bar{\lambda}_0}{2}\Sigma . \end{aligned} \tag{4.9}$$

To calculate the inverse of a block matrix, we need

Lemma 1. Suppose $T = (T_{ij})_{1\leqslant i,j\leqslant 3}$ is a positively definite block matrix, where T_{ij} are sub-matrices, then

$$T^{-1} = \begin{pmatrix} \widetilde{T}_{11} & \widetilde{T}_{12} & \widetilde{T}_{13} \\ \widetilde{T}_{21} & \widetilde{T}_{22} & \widetilde{T}_{23} \\ \widetilde{T}_{31} & \widetilde{T}_{32} & \widetilde{T}_{33} \end{pmatrix} \tag{4.10}$$

where

$$\begin{aligned} &\widetilde{T}_{11} = T_{11}^{-1} + T_{11}^{-1}T_{12}R_{22}^{-1}T_{21}T_{11}^{-1} + R_{13}R_{33}^{-1}R_{31} \\ &\widetilde{T}_{12} = -T_{11}^{-1}T_{12}R_{22}^{-1} + R_{13}R_{33}^{-1}R_{32}R_{22}^{-1} \\ &\widetilde{T}_{13} = -R_{13}R_{33}^{-1} \\ &\widetilde{T}_{21} = R_{22}^{-1}R_{23}R_{33}^{-1}R_{31} - R_{22}^{-1}T_{21}T_{11}^{-1} \qquad \widetilde{T}_{22} = R_{22}^{-1} + R_{22}^{-1}R_{23}R_{33}^{-1}R_{32}R_{22}^{-1} \\ &\widetilde{T}_{23} = -R_{22}^{-1}R_{23}R_{33}^{-1} \\ &\widetilde{T}_{31} = -R_{33}^{-1}R_{31} \qquad \widetilde{T}_{32} = -R_{33}^{-1}R_{32}R_{22}^{-1} \qquad \widetilde{T}_{33} = R_{33}^{-1} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &R_{33} = T_{33} - T_{31}T_{11}^{-1}T_{13} - R_{32}R_{22}^{-1}R_{23} \qquad R_{13} = T_{11}^{-1}T_{13} - T_{11}^{-1}T_{12}R_{22}^{-1}R_{23} \\ &R_{31} = T_{31}T_{11}^{-1} - R_{32}R_{22}^{-1}T_{21}T_{11}^{-1} \\ &R_{23} = T_{23} - T_{21}T_{11}^{-1}T_{13} \qquad R_{32} = T_{32} - T_{31}T_{11}^{-1}T_{12} \qquad R_{22} = T_{22} - T_{21}T_{11}^{-1}T_{12}\,. \end{aligned} \tag{4.12}$$

Moreover,

$$\det T = \det T_{11} \cdot \det R_{22} \cdot \det R_{33}\,. \tag{4.13}$$

Proof. Since T is positively definite, T_{11}, R_{22} and R_{33} are non-degenerate. The conclusion follows from direct calculation.

Denote $\beta_j = e^{i\bar{\phi}_j}$, $\hat{\beta}_j = e^{i\phi_j} = 1/\bar{\beta}_j$, then $\beta_1 \to 0$, $\beta_2 \to \infty$ exponentially as $y \to +\infty$ along L,

$$\Delta = \begin{pmatrix} \hat{\beta}_1 + \beta_1\Gamma_{11} & \beta_1\Gamma_{12} & \beta_1\Gamma_{13} \\ \beta_2\Gamma_{21} & \hat{\beta}_2 + \beta_2\Gamma_{22} & \beta_2\Gamma_{23} \\ \beta_3\Gamma_{31} & \beta_3\Gamma_{32} & \hat{\beta}_3 + \beta_3\Gamma_{33} \end{pmatrix}. \tag{4.14}$$

After tedious calculation, we derive the asymptotic behaviour of the blocks of Δ^{-1} as

$$\begin{aligned} &(\Delta^{-1})_{11} = \hat{\beta}_1^{-1} + \cdots \\ &(\Delta^{-1})_{12} = -\hat{\beta}_1^{-1}\beta_1(\Gamma_{12} - \theta_{13}h^{-1}\beta_3\Gamma_{32})\Gamma_{22}^{-1}\beta_2^{-1} + \cdots \\ &(\Delta^{-1})_{13} = -\hat{\beta}_1^{-1}\beta_1\theta_{13}h^{-1} + \cdots \\ &(\Delta^{-1})_{21} = -\Gamma_{22}^{-1}(\Gamma_{21} - \Gamma_{23}h^{-1}\beta_3\theta_{31})\hat{\beta}_1^{-1} + \cdots \\ &(\Delta^{-1})_{22} = \Gamma_{22}^{-1}(I + \Gamma_{23}h^{-1}\beta_3\Gamma_{32}\Gamma_{22}^{-1})\beta_2^{-1} + \cdots \\ &(\Delta^{-1})_{23} = -\Gamma_{22}^{-1}\Gamma_{23}h^{-1} + \cdots \\ &(\Delta^{-1})_{31} = -h^{-1}\beta_3\theta_{31}\hat{\beta}_1^{-1} + \cdots \\ &(\Delta^{-1})_{32} = -h^{-1}\beta_3\Gamma_{32}\Gamma_{22}^{-1}\beta_2^{-1} + \cdots \\ &(\Delta^{-1})_{33} = h^{-1} + \cdots \end{aligned} \tag{4.15}$$

when $y \to +\infty$ along L, where

$$\theta_{ij} = \Gamma_{ij} - \Gamma_{i2}\Gamma_{22}^{-1}\Gamma_{2j} \qquad h = \hat{\beta}_3 + \beta_3\theta_{33}\,. \tag{4.16}$$

Here '$\cdots$' represents the terms of lower order comparing with the leading terms, whose ratio with the leading term tends to 0 exponentially. Moreover, the blocks of

$$\Sigma = -I + 2\mathrm{e}^{\mathrm{i}\phi}\Delta^{-1} = I - 2\mathrm{e}^{\mathrm{i}\bar{\phi}}\Gamma\Delta^{-1} \tag{4.17}$$

are

$$\begin{aligned}
\Sigma_{11} &= I - 2\beta_1(\theta_{11} - \theta_{13}h^{-1}\beta_3\theta_{31})\hat{\beta}_1^{-1} + \cdots \\
\Sigma_{22} &= -I + 2\hat{\beta}_2\Gamma_{22}^{-1}(I + \Gamma_{23}h^{-1}\beta_3\Gamma_{32}\Gamma_{22}^{-1})\beta_2^{-1} + \cdots \\
\Sigma_{33} &= (\hat{\beta}_3 - \beta_3\theta_{33})h^{-1} + \cdots \\
\Sigma_{12} &= -2\beta_1(\Gamma_{12} - \theta_{13}h^{-1}\beta_3\Gamma_{32})\Gamma_{22}^{-1}\beta_2^{-1} + \cdots \\
\Sigma_{13} &= -2\beta_1\theta_{13}h^{-1} + \cdots \\
\Sigma_{23} &= 2\hat{\beta}_2\Gamma_{22}^{-1}\Gamma_{23}h^{-1} + \cdots
\end{aligned} \tag{4.18}$$

with $\Sigma_{ij}^* = \Sigma_{ji}$. In these formulae, all the terms with subscript 3 disappear if $q = N$. Actually, equations (4.15) and (4.18) hold even when the order of Γ_{33} is greater than 1. This fact will be used later for the asymptotic behaviour of the solutions.

Hence, $\Sigma^A \to 0$ exponentially as $y \to +\infty$ along L. Moreover,

$$S_{11} \to \begin{pmatrix} \lambda_0 I_p & & \\ & \bar{\lambda}_0 I_{q-p} & \\ & & \mu I_{N-q} \end{pmatrix} \tag{4.19}$$

where μ is a constant.

On the other hand, all the entries of $C\Delta^{-1}$ in the first and second columns of its block division tends to 0 exponentially as $y \to +\infty$ along L. Hence, the limit of the entry (i, j) of S as $y \to +\infty$ along L is zero if $i \leqslant q$ or $j \leqslant q$ with $i \neq j$. The same conclusion holds as $y \to -\infty$ along L since we only need to exchange β_1 and β_2. Therefore, we have

Proposition 2. The one-soliton solution $\widetilde{U}$ constructed above tends to 0 exponentially when $(x, y) \to \infty$ along any straight line.

5. Multi-soliton solutions

To construct multi-soliton solutions, take $\lambda_0 = \lambda_0^{(i)}$, $C = C^{(i)}$ $(i = 1, \ldots, r)$, respectively, with $C^{(i)}$ non-degenerate. Here we assume $\lambda_0^{(i)} \neq \lambda_0^{(j)}$ and $\lambda_0^{(i)} \neq \bar{\lambda}_0^{(j)}$ for $i \neq j$. Let $\Gamma^{(i)} = C^{(i)*}C^{(i)}$ and $H^{(i)}$ be given by (4.1) with $\lambda_0 = \lambda_0^{(i)}$ and $C = C^{(i)}$. Then we can construct a Darboux matrix of lth order with respect to the parameters $\lambda_0^{(i)}$, $C^{(i)}$ $(i = 1, \ldots, l)$. Let $S^{(i)} = H^{(i)}\Lambda^{(i)}H^{(i)-1}$, and suppose the lth Darboux matrix is

$$G_l(\lambda) = \lambda^l - G_1^{(l)}\lambda^{l-1} + \cdots + (-1)^l G_l^{(l)} \tag{5.1}$$

then the new solution $U^{(l)}$ given by G_l is

$$U^{(l)} = U + \mathrm{i}[J, G_1^{(l)}]\,. \tag{5.2}$$

Denote

$$G_l(M) = M^l - G_1^{(l)}M^{l-1} + \cdots + (-1)^l G_l^{(l)} \tag{5.3}$$

for a $2N \times 2N$ matrix M. We have

Lemma 2.

$$G_l(\bar\lambda)^* G_l(\lambda) = \prod_{i=1}^{l} (\lambda - \lambda_0^{(i)})(\lambda - \bar\lambda_0^{(i)}) \tag{5.4}$$

and

$$G_{l+1}(\lambda) = \big(\lambda - G_l(S^{(l+1)})S^{(l+1)}(G_l(S^{(l+1)})^{-1}\big)G_l(\lambda) \tag{5.5}$$

is well defined.

First, we show that (5.4) implies that $G_l(S^{(l+1)})$ is non-degenerate, which leads to (5.5). Let $h_a^{(l+1)}$ be the ath column of $H^{(l+1)}$, then $G_l(\lambda_0^{(l+1)})h_i^{(l+1)}$ are solutions of the Lax pair (1.1) with $U = U^{(l)}$, $\lambda = \lambda_0^{(l+1)}$ for $a \leqslant N$, and $G_l(\bar\lambda_0^{(l+1)})h_i^{(l+1)}$ are solutions of the Lax pair (1.1) with $U = U^{(l)}$, $\lambda = \bar\lambda_0^{(l+1)}$ for $a \geqslant N+1$. Define

$$\begin{aligned}\widetilde H^{(l+1)} &= (G_l(\lambda_0^{(l+1)})h_1^{(l+1)}, \ldots, G_l(\lambda_0^{(l+1)})h_N^{(l+1)}, G_l(\bar\lambda_0^{(l+1)})h_{N+1}^{(l+1)}, \ldots, G_l(\bar\lambda_0^{(l+1)})h_{2N}^{(l+1)}) \\ &= G_l(S^{(l+1)})H^{(l+1)} .\end{aligned} \tag{5.6}$$

Since

$$\begin{aligned}&(G_l(\bar\lambda_0^{(l+1)})h_{N+a}^{(l+1)})^*(G_l(\lambda_0^{(l+1)})h_b^{(l+1)}) \\ &\qquad = \prod_{i=1}^{l}(\lambda_0^{l+1} - \lambda_0^{(i)})(\lambda_0^{l+1} - \bar\lambda_0^{(i)})h_{N+a}^{(l+1)*}h_b^{(l+1)} = 0\end{aligned} \tag{5.7}$$

for $1 \leqslant a, b \leqslant N$, and $G_l(\lambda_0^{(l+1)})$, $G_l(\bar\lambda_0^{(l+1)})$ are non-degenerate, we know that

$$\widetilde H^{(l+1)*}\widetilde H^{(l+1)} = \begin{pmatrix} \big((G_l(\lambda_0^{(l+1)})h_a^{(l+1)})^* G_l(\lambda_0^{(l+1)})h_b^{(l+1)}\big)_{1\leqslant a,b\leqslant N} & 0 \\ 0 & \big((G_l(\bar\lambda_0^{(l+1)})h_{N+a}^{(l+1)})^* G_l(\bar\lambda_0^{(l+1)})h_{N+b}^{(l+1)}\big)_{1\leqslant a,b\leqslant N}\end{pmatrix} \tag{5.8}$$

is non-degenerate. Therefore, equation (5.6) implies that $G_l(S^{(l+1)})$ is non-degenerate and the new Darboux matrix is given by

$$\widetilde S^{(l+1)} = \widetilde H^{(l+1)}\Lambda^{(l+1)}\widetilde H^{(l+1)-1} = G_l(S^{(l+1)})S^{(l+1)}(G_l(S^{(l+1)}))^{-1} . \tag{5.9}$$

The compound Darboux matrix is $G_{l+1}(\lambda) = (\lambda - \widetilde S^{(l+1)})G_l(\lambda)$. Next, we turn to prove (5.4). Clearly, it holds for $l = 0$. Suppose (5.4) is true for $l = j-1$, then the above discussion implies that $G_{j-1}(S^{(j)})$ and $\widetilde S^{(j)}$ are well defined, and $G_j(\lambda) = (\lambda - \widetilde S^{(j)})G_{j-1}(\lambda)$. Hence

$$G_j(\bar\lambda)^* G_j(\lambda) = G_{j-1}(\bar\lambda)^*(\lambda - \widetilde S^{(j)*})(\lambda - \widetilde S^{(j)})G_{j-1}(\lambda) = \prod_{a=1}^{j}(\lambda - \lambda_0^{(a)})(\lambda - \bar\lambda_0^{(a)}) \tag{5.10}$$

by using (5.8). This proves the lemma.

Denote

$$\mathcal{M} = \{M = (M_{ij})_{1\leqslant i,j\leqslant 2N} \mid M_{ij} = 0 \text{ if } i \leqslant q \text{ or } j \leqslant q \text{ with } i \neq j\} . \tag{5.11}$$

Evidently, $\mathcal{M}$ is a ring. Therefore, by the same procedure as in section 4, we can show that $\lim_{(x,y)\in L,\,(x,y)\to\infty} S_{22}^{(i)}$ exists and $S_{22}^{(i)}$ tends to its limit exponentially. Hereafter, we call a limit $\lim_{(x,y)\in L,\,(x,y)\to\infty} f(x,y)$ exists exponentially if the limit exists and $f(x,y)$ tends to its limit exponentially. Hence $S^{(i)} \to S_0^{(i)} \in \mathcal{M}$ exponentially as $(x,y) \to \infty$ along L. Clearly, $G_0(\lambda) = I \in \mathcal{M}$. Suppose that $G_{j-1}(\lambda) \in \mathcal{M}$ and $\lim_{(x,y)\in L,\,(x,y)\to\infty} G_{j-1}(\lambda)$ exists exponentially. From equations (5.8), (5.10) and the equality $G_{j-1}(S^{(j)}) = \widetilde{H}^{(j)} H^{(j)-1}$, we have

$$G_{j-1}(S^{(j)})^* G_{j-1}(S^{(j)}) = \prod_{a=1}^{j-1} (\lambda_0^{(j)} - \lambda_0^{(a)})(\lambda_0^{(j)} - \bar{\lambda}_0^{(a)})\,. \tag{5.12}$$

Since both the limits of $G_{j-1}(\lambda)$ and $S^{(j)}$ as $(x,y) \to \infty$ along L exist exponentially, so is the limit of $G_{j-1}(S^{(j)})$. Moreover, from equation (5.12), we know that $\lim_{(x,y)\in L,\,(x,y)\to\infty} G_{j-1}(S^{(j)}) \in \mathcal{M}$ is z. From (5.4), $G_j(\lambda) \in \mathcal{M}$ and $\lim_{(x,y)\in L,\,(x,y)\to\infty} G_j(\lambda)$ exists exponentially. This proves the fact that $\lim_{(x,y)\in L,\,(x,y)\to\infty} G_l(\lambda) \in \mathcal{M}$ for all l. Especially, $\lim_{(x,y)\in L,\,(x,y)\to\infty} G_1(\lambda) \in \mathcal{M}$ exists exponentially, which means $U^{(l)} \to 0$ exponentially as $(x,y) \to \infty$ along L.

Therefore, we have proved

Proposition 3. For any equation in the hyperbolic $su(N)$ AKNS system, the multi-soliton solutions $U^{(l)}$ tends to zero exponentially as $(x,y) \to \infty$ in all directions.

6. Asymptotic behaviour of the multi-soliton solutions as $t \to \infty$

First, consider the single-soliton solution. Take a reference frame moving with velocity (v_1, v_2) such that $x = \xi + v_1 t,\ y = \eta + v_2 t$, then

$$\begin{aligned}\phi &= \lambda_0(x + Jy) + \omega(\lambda_0)t \\ &= (\lambda_0 v_1 + \lambda_0 J v_2 + \omega(\lambda_0))t + \lambda_0(\xi + J\eta)\,.\end{aligned} \tag{6.1}$$

Denote

$$\omega(\lambda_0) = \begin{pmatrix} \beta_1(\lambda_0) & & \\ & \ddots & \\ & & \beta_n(\lambda_0) \end{pmatrix} \tag{6.2}$$

and suppose

$$\det\begin{pmatrix} 1 & J_i & \beta_{iI}(\lambda_0) \\ 1 & J_j & \beta_{jI}(\lambda_0) \\ 1 & J_k & \beta_{kI}(\lambda_0) \end{pmatrix} \neq 0 \tag{6.3}$$

for all mutually different (i, j, k). (If $N = 2$, this is supposed to be always true.) If we keep ξ, η finite and let $t \to \infty$, the analysis in section 4 shows that

$$S_{11} \to \begin{pmatrix} \lambda_0 I_p & & \\ & \bar{\lambda}_0 I_{q-p} & \\ & & \mu I_{N-q} \end{pmatrix}$$

as in (4.19), and $S_{12}, S_{21} \to 0$, unless

$$v_1 = \frac{-J_j\beta_{iI}(\lambda_0) + J_i\beta_{j\mathrm{I}}(\lambda_0)}{\lambda_{0\mathrm{I}}(J_j - J_i)} \qquad v_2 = \frac{-\beta_{j\mathrm{I}}(\lambda_0) + \beta_{i\mathrm{I}}(\lambda_0)}{\lambda_{0\mathrm{I}}(J_j - J_i)}. \tag{6.4}$$

Here the subscript 'I' refers to the imaginary part. Hence, if (6.4) fails, the solution $\widetilde{U} \to 0$ exponentially as $t \to \infty$ with (ξ, η) finite.

If equation (6.4) holds, we suppose, without loss of generality, that $i = N-1$, $j = N$, then

$$\begin{aligned}
\widetilde{U}_{N-1,N} &\to \frac{2b\theta_{12}\mathrm{e}^{\mathrm{i}a(J_{N-1}-J_N)\eta + \mathrm{i}(a(\beta_{j\mathrm{I}}-\beta_{iI})+b(\beta_{iR}-\beta_{jR}))t/b}}{\sqrt{\det\theta}\,\mathrm{ch}(2b\xi + b(J_{N-1}-J_N)\eta + \phi_1) + \sqrt{\theta_{11}\theta_{22}}\,\mathrm{ch}(b(J_N - J_{N-1})\eta + \phi_2)} \\
\widetilde{U}_{N,N-1} &= -\widetilde{U}^*_{N-1,N} \\
\widetilde{U}_{i,j} &\to 0 \qquad (i, j) \neq (N-1, N) \quad \text{or} \quad (N, N-1)
\end{aligned} \tag{6.5}$$

where a, b are real and imaginary parts of λ_0, $\theta = \theta_{33} = \Gamma_{33} - \Gamma_{32}\Gamma_{22}^{-1}\Gamma_{23}$ is a constant positive-definite 2×2 matrix as in (4.16), and $\phi_1 = \frac{1}{2}\ln\det\theta$, $\phi_2 = \frac{1}{2}\ln(\theta_{22}/\theta_{11})$.

For multi-soliton solutions in section 5, we suppose

$$\det\begin{pmatrix} 1 & J_i & \beta_{iI}(\lambda_\alpha)/\lambda_{\alpha I} \\ 1 & J_j & \beta_{j\mathrm{I}}(\lambda_\beta)/\lambda_{\beta I} \\ 1 & J_k & \beta_{kI}(\lambda_\gamma)/\lambda_{\gamma I} \end{pmatrix} \neq 0 \tag{6.6}$$

for any mutually different pairs (i, α), (j, β) and (k, γ). Here $\lambda_\alpha = \lambda_0^{(\alpha)}$. As in section 5, if

$$v_1 = \frac{-J_j\beta_{iI}(\lambda_\alpha)/\lambda_{\alpha I} + J_i\beta_{j\mathrm{I}}(\lambda_\beta)/\lambda_{\beta I}}{J_j - J_i} \qquad v_2 = \frac{-\beta_{j\mathrm{I}}(\lambda_\beta)/\lambda_{\beta I} + \beta_{iI}(\lambda_\alpha)/\lambda_{\alpha I}}{J_j - J_i} \tag{6.7}$$

fails, the solution $U^{(l)} \to 0$ exponentially as $t \to \infty$ with (ξ, η) finite. Moreover, if (6.7) holds for some (i, j, α, β), equation (6.6) guarantees that (6.7) cannot hold for other (i, j, α, β). Therefore, we have

Proposition 4. As $t \to \infty$, each entry $U_{ij}^{(l)}$ of the solution $U^{(l)}$ given by lth Darboux transformations splits up into at most l^2 peaks, whose velocities (v_1, v_2) are given by (6.7) with $1 \leqslant \alpha, \beta \leqslant l$.

7. Example: solutions of DSI equation given by once and twice Darboux transformations

From equation (6.5), we know that the solution given by once Darboux transformation is

$$u = \frac{2bc_{12}\mathrm{e}^{2\mathrm{i}a\eta + 4\mathrm{i}(a^2+b^2)t}}{|\det C|\,\mathrm{ch}(2b\xi + \phi_1) + c_1c_2\,\mathrm{ch}(-2b\eta + \phi_2)}$$

where

$$c_1 = \sqrt{|C_{11}|^2 + |C_{21}|^2} \qquad c_2 = \sqrt{|C_{12}|^2 + |C_{22}|^2} \qquad c_{12} = \bar{C}_{11}C_{12} + \bar{C}_{21}C_{22}$$
$$\phi_1 = \ln|\det C| \qquad \phi_2 = \ln\frac{c_2}{c_1}$$

with $x = \xi$, $y = \eta + 4at$. Therefore, the velocity is $(0, 4a)$ and the amplitude is

$$\frac{2|b|\,|c_{12}|}{|\det C| + c_1c_2}\,.$$

The maximum of $|u|$ appears at $\xi = -\frac{1}{2b}\ln|\det C|$, $\eta = \frac{1}{2b}\ln\frac{c_2}{c_1}$.

The lth Darboux transformation can also be given by [15] as

$$G(\lambda) = \prod_{j=1}^{l}(\lambda - \bar{\lambda}_j)\left(1 - \sum_{j,k=1}^{l}\frac{h_j}{\lambda - \bar{\lambda}_k}(\Sigma^{-1})_{jk}h_k^*\right)$$

where

$$h_j = \begin{pmatrix} e^{i\lambda_j(x+Jy)+i\omega(\lambda_j)t} \\ C^{(j)} \end{pmatrix}$$

where $(\lambda_j, C^{(j)})$ are corresponding parameters (λ_0, C) in (4.1), and Σ is a block-matrix given by

$$\Sigma_{jk} = \frac{h_j^*h_k}{\lambda_k - \bar{\lambda}_j}\,.$$

Correspondingly,

$$\begin{aligned} u &= 2i\sum_{j,k=1}^{l}\left(h_j(\Sigma^{-1})_{jk}h_k^*\right)_{(1,2)} \\ &= 2i\sum_{j,k=1}^{l}\left((\Sigma^{-1})_{jk}\right)_{(1,2)} e^{i(\lambda_j-\bar{\lambda}_k)x+i(\lambda_j+\bar{\lambda}_k)y-2i(\lambda_j^2+\bar{\lambda}_k^2)t} \end{aligned}$$

which is the same as the solutions given by [13] with $\mu_n = \lambda_n$.

For the solutions given by twice Darboux transformations, there are at most four peaks as $t \to \pm\infty$. For simplicity, we suppose $a_1 \neq a_2$, $b_1 \neq \pm b_2$ where $a_i = \mathrm{Re}\,\lambda_i$, $b_i = \mathrm{Im}\,\lambda_i$. After tedious calculation, we can get the asymptotic properties of these four peaks, which are more direct than those in the previous papers.

Denote

$$\begin{aligned} g_{ij}^{\alpha\beta} &= \overline{C}_{1i}^{(\alpha)}C_{1j}^{(\beta)} + \overline{C}_{2i}^{(\alpha)}C_{2j}^{(\beta)} \\ f_{ij}^{\alpha\beta} &= C_{1i}^{(\alpha)}C_{2j}^{(\beta)} - C_{2i}^{(\alpha)}C_{1j}^{(\beta)}\,. \end{aligned}$$

(Note that $f_{12}^{\alpha\alpha} = \det C^{(\alpha)}$.) Let $x = \xi + v_1t$, $y = \eta + v_2t$ with certain velocity (v_1, v_2), then as ξ, η fixed and $t \to \pm\infty$, u behaves asymptotically as follows.

(i) $v_1 = 0$, $v_2 = 4a_1$:

$$u \to \frac{4b_1\gamma e^{i\theta}}{A_1e^{2b_1\xi} + A_2e^{-2b_1\xi} + B_1e^{2b_1\eta} + B_2e^{-2b_1\eta}}$$

where

$$
\begin{aligned}
&\theta = 2a_1\eta + 4(a_1^2 + b_1^2)t\\
&\gamma = \begin{cases} (\lambda_2-\lambda_1)(\lambda_2-\bar\lambda_1)(|\lambda_2-\bar\lambda_1|^2 g_{12}^{11}g_{22}^{22} - 4b_1b_2 g_{12}^{12}\bar g_{22}^{12}) & (a_2-a_1)b_2t \to +\infty \\ (\bar\lambda_2-\bar\lambda_1)(\bar\lambda_2-\lambda_1)(|\lambda_2-\lambda_1|^2 g_{12}^{11}g_{11}^{22} - 4b_1b_2 g_{11}^{12}\bar g_{21}^{12}) & (a_2-a_1)b_2t \to -\infty \end{cases}\\
&A_1 = \begin{cases} |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2|f_{12}^{11}|^2 g_{22}^{22} & (a_2-a_1)b_2t \to +\infty \\ |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2|f_{12}^{11}|^2 g_{11}^{22} & (a_2-a_1)b_2t \to -\infty \end{cases}\\
&A_2 = \begin{cases} |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2 g_{22}^{22} & (a_2-a_1)b_2t \to +\infty \\ |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2 g_{11}^{22} & (a_2-a_1)b_2t \to -\infty \end{cases}\\
&B_1 = \begin{cases} |\lambda_2-\bar\lambda_1|^2(|\lambda_2-\bar\lambda_1|^2|g_{11}^{11}|^2 g_{22}^{22} - 4b_1b_2|g_{12}^{12}|^2) & (a_2-a_1)b_2t \to +\infty \\ |\lambda_2-\bar\lambda_1|^2(|\lambda_2-\bar\lambda_1|^2|g_{22}^{11}|^2 g_{11}^{22} - 4b_1b_2|g_{21}^{12}|^2) & (a_2-a_1)b_2t \to -\infty \end{cases}\\
&B_2 = \begin{cases} |\lambda_2-\lambda_1|^2(|\lambda_2-\bar\lambda_1|^2|g_{22}^{11}|^2 g_{22}^{11} - 4b_1b_2|g_{22}^{12}|^2) & (a_2-a_1)b_2t \to +\infty \\ |\lambda_2-\lambda_1|^2(|\lambda_2-\bar\lambda_1|^2|g_{11}^{11}|^2 g_{11}^{22} - 4b_1b_2|g_{11}^{11}|^2) & (a_2-a_1)b_2t \to -\infty\,. \end{cases}
\end{aligned}
$$

(ii) $v_1 = 0$, $v_2 = 4a_2$. The solution is similar to that in case (i), except for the exchange of λ_1 and λ_2, and the exchange of ‘1’ and ‘2’ in the superscripts of $g_{ij}^{\alpha\beta}$, $f_{ij}^{\alpha\beta}$. For example, a_1 is changed to a_2, g_{11}^{12} is changed to $g_{11}^{21} = \bar g_{11}^{12}$, f_{11}^{12} is changed to $f_{11}^{21} = -f_{11}^{12}$ etc.

(iii) $v_1 = 2(a_1 - a_2)$, $v_2 = 2(a_1 + a_2)$:

$$
u \to \frac{-8\mathrm{i}b_1b_2\gamma\mathrm{e}^{\mathrm{i}\theta}}{A_1\mathrm{e}^{(b_1+b_2)\xi+(b_1-b_2)\eta} + A_2\mathrm{e}^{-(b_1+b_2)\xi-(b_1-b_2)\eta} + B_1\mathrm{e}^{(b_1-b_2)\xi+(b_1+b_2)\eta} + B_2\mathrm{e}^{-(b_1-b_2)\xi-(b_1+b_2)\eta}}
$$

where

$$
\begin{aligned}
&\theta = (a_1-a_2)\xi + (a_1+a_2)\eta + (a_1^2 + b_1^2 + a_2^2 + b_2^2)t\\
&\gamma = \begin{cases} -|\lambda_2-\lambda_1|^2(\lambda_2-\bar\lambda_1)g_{12}^{12} & d(t) \to (+\infty,+\infty) \\ |\lambda_2-\bar\lambda_1|^2(\bar\lambda_2-\bar\lambda_1)\bar f_{11}^{12} f_{12}^{22} & d(t) \to (+\infty,-\infty) \\ |\lambda_2-\bar\lambda_1|^2(\lambda_2-\lambda_1) f_{22}^{12}\bar f_{12}^{11} & d(t) \to (-\infty,+\infty) \\ |\lambda_2-\lambda_1|^2(\bar\lambda_2-\lambda_1)\bar g_{21}^{12}\bar f_{12}^{11} f_{12}^{22} & d(t) \to (-\infty,-\infty) \end{cases}\\
&A_1 = \begin{cases} |\lambda_2-\bar\lambda_1|^2(|\lambda_2-\lambda_1|^2 g_{11}^{11}g_{22}^{22} + 4b_1b_2|f_{12}^{12}|^2) & d(t) \to (+\infty,+\infty) \\ |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2 g_{11}^{11}|f_{12}^{22}|^2 & d(t) \to (+\infty,-\infty) \\ |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2 g_{22}^{22}|f_{12}^{11}|^2 & d(t) \to (-\infty,+\infty) \\ |\lambda_2-\lambda_1|^4|f_{12}^{11}|^2|f_{12}^{22}|^2 & d(t) \to (-\infty,-\infty) \end{cases}\\
&A_2 = \begin{cases} |\lambda_2-\lambda_1|^4 & d(t) \to (+\infty,+\infty) \\ |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2 g_{11}^{22} & d(t) \to (+\infty,-\infty) \\ |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2 g_{22}^{11} & d(t) \to (-\infty,+\infty) \\ |\lambda_2-\bar\lambda_1|^2(|\lambda_2-\lambda_1|^2 g_{22}^{11}g_{11}^{22} + 4b_1b_2|f_{21}^{12}|^2) & d(t) \to (-\infty,-\infty) \end{cases}\\
&B_1 = \begin{cases} |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2 g_{11}^{11} & d(t) \to (+\infty,+\infty) \\ |\lambda_2-\lambda_1|^2(|\lambda_2-\bar\lambda_1|^2 g_{11}^{11}g_{11}^{22} - 4b_1b_2|g_{11}^{12}|^2) & d(t) \to (+\infty,-\infty) \\ |\lambda_2-\bar\lambda_1|^4|f_{12}^{11}|^2 & d(t) \to (-\infty,+\infty) \\ |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2 g_{11}^{22}|f_{12}^{11}|^2 & d(t) \to (-\infty,-\infty) \end{cases}
\end{aligned}
$$

$$B_2 = \begin{cases} |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2 g^{22}_{22} & d(t)\to(+\infty,+\infty) \\ |\lambda_2-\bar\lambda_1|^4|f^{22}_{12}|^2 & d(t)\to(+\infty,-\infty) \\ |\lambda_2-\lambda_1|^2(|\lambda_2-\bar\lambda_1|^2 g^{11}_{22}g^{22}_{22}-4b_1b_2|g^{12}_{22}|^2) & d(t)\to(-\infty,+\infty) \\ |\lambda_2-\bar\lambda_1|^2|\lambda_2-\lambda_1|^2 g^{11}_{22}|f^{22}_{12}|^2 & d(t)\to(-\infty,-\infty) \end{cases}$$

where $d(t)=((a_2-a_1)b_1t,(a_2-a_1)b_2t)$.

(iv) $v_1=2(a_2-a_1)$, $v_2=2(a_1+a_2)$.

The solution is similar to (iii), but the indices '1' and '2' are interchanged as in the case (ii).

The peak vanishes when $\gamma=0$. Write $\gamma^{(i)}$ as that in case (i), then,

$$\gamma^{(1)} \sim \begin{cases} |\lambda_2-\bar\lambda_1|^2 g^{11}_{12}g^{22}_{22}-4b_1b_2 g^{12}_{12}\bar g^{12}_{22} & d(t)\to(+\infty,+\infty) \text{ or } (-\infty,+\infty) \\ |\lambda_2-\lambda_1|^2 g^{11}_{12}g^{22}_{11}-4b_1b_2 g^{12}_{11}\bar g^{12}_{21} & d(t)\to(+\infty,-\infty) \text{ or } (-\infty,-\infty) \end{cases}$$

$$\gamma^{(2)} \sim \begin{cases} |\lambda_2-\lambda_1|^2 g^{22}_{12}g^{11}_{11}-4b_1b_2 g^{12}_{12}\bar g^{12}_{11} & d(t)\to(+\infty,+\infty) \text{ or } (+\infty,-\infty) \\ |\lambda_2-\bar\lambda_1|^2 g^{22}_{12}g^{11}_{22}-4b_1b_2 g^{12}_{22}\bar g^{12}_{21} & d(t)\to(-\infty,+\infty) \text{ or } (-\infty,-\infty) \end{cases}$$

$$\gamma^{(3)} \sim \begin{cases} g^{12}_{12} & d(t)\to(+\infty,+\infty) \\ \bar f^{12}_{11} & d(t)\to(+\infty,-\infty) \\ f^{12}_{22} & d(t)\to(-\infty,+\infty) \\ \bar g^{12}_{21} & d(t)\to(-\infty,-\infty) \end{cases}$$

$$\gamma^{(4)} \sim \begin{cases} g^{12}_{12} & d(t)\to(+\infty,+\infty) \\ \bar f^{12}_{11} & d(t)\to(+\infty,-\infty) \\ f^{12}_{22} & d(t)\to(-\infty,+\infty) \\ \bar g^{12}_{21} & d(t)\to(-\infty,-\infty)\,. \end{cases}$$

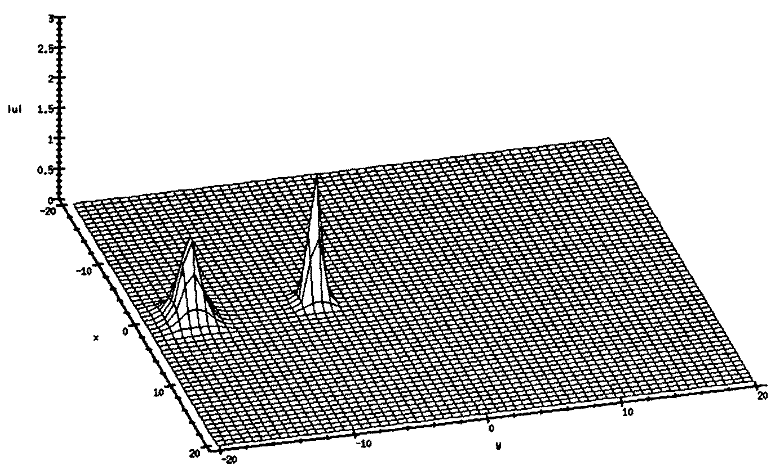

Figure 1. $t=-2$.

At the end of the paper, three sets of figures of these solutions are shown with $t=-2$, -1, -0.5, 0, 0.5, 1, 2, respectively. The corresponding parameters are figures 1–7:

$$\lambda_1 = 1+2\mathrm{i} \qquad \lambda_2 = 2-\mathrm{i} \qquad C_1=\begin{pmatrix}1 & 4\\ 0 & 3\end{pmatrix} \qquad C_2=\begin{pmatrix}1 & \frac{4}{3}\\ 0 & 1\end{pmatrix}\,.$$

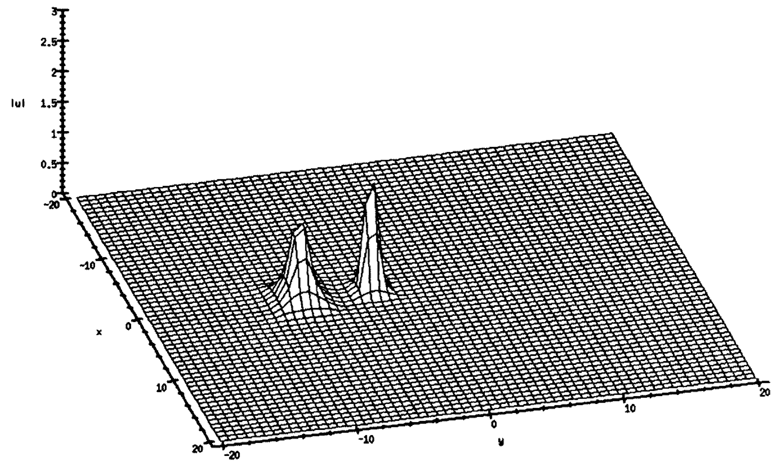

Figure 2. $t = -1$.

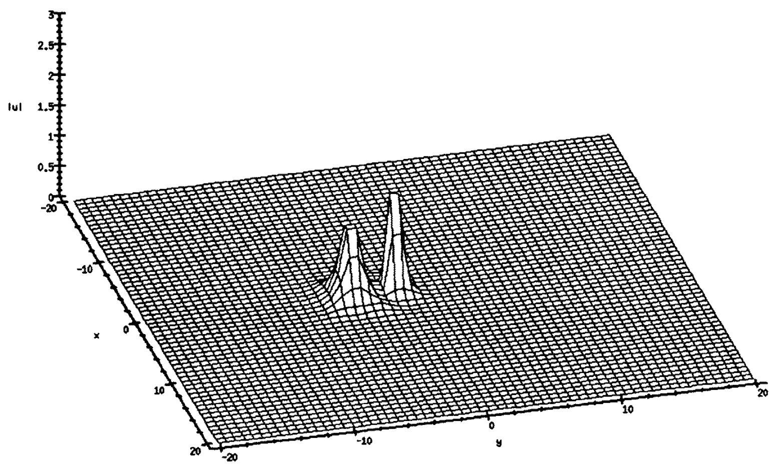

Figure 3. $t = -0.5$.

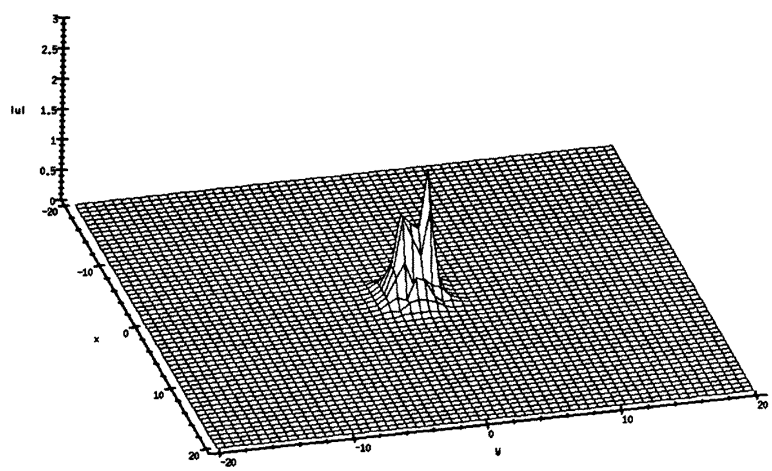

Figure 4. $t = 0$.

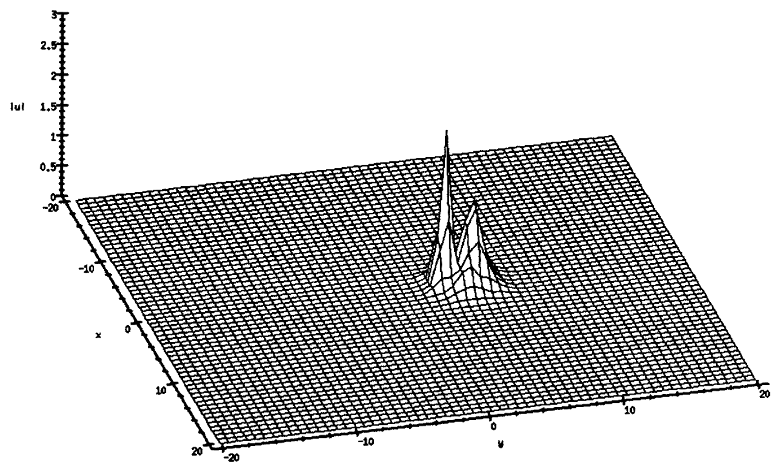

Figure 5. $t = 0.5$.

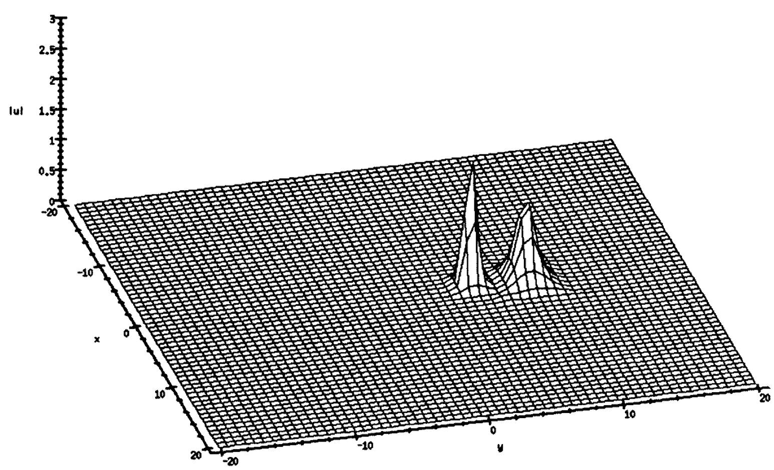

Figure 6. $t = 1$.

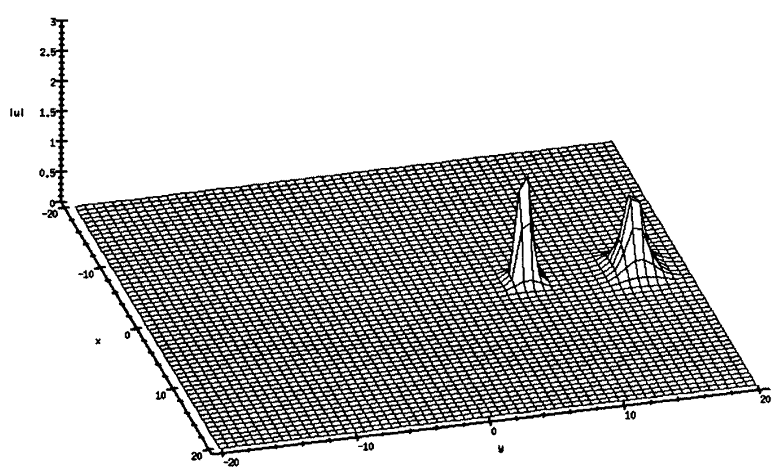

Figure 7. $t = 2$.

Figures 8–14:

$$\lambda_1 = 1 + 2\mathrm{i} \qquad \lambda_2 = 2 - \mathrm{i} \qquad C_1 = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{pmatrix}.$$

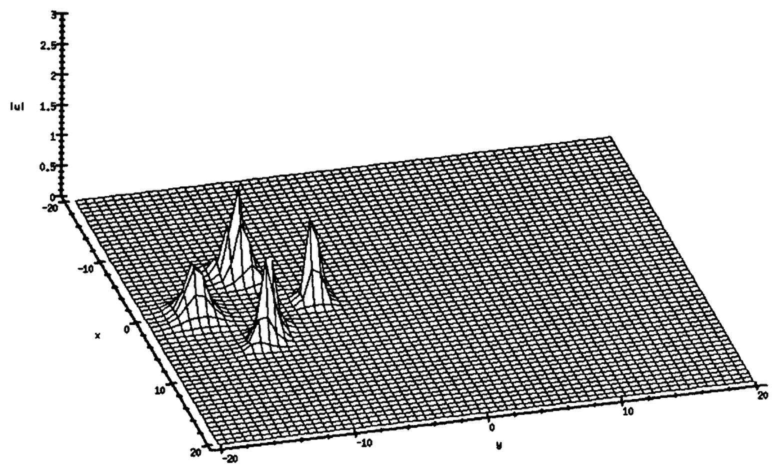

Figure 8. $t = -2$.

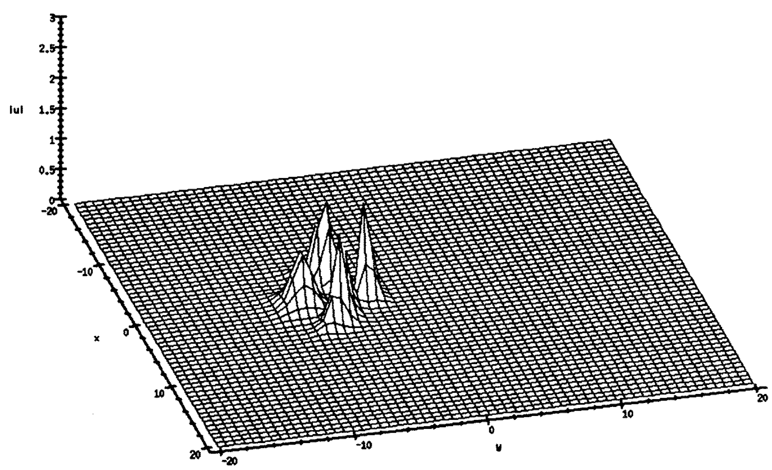

Figure 9. $t = -1$.

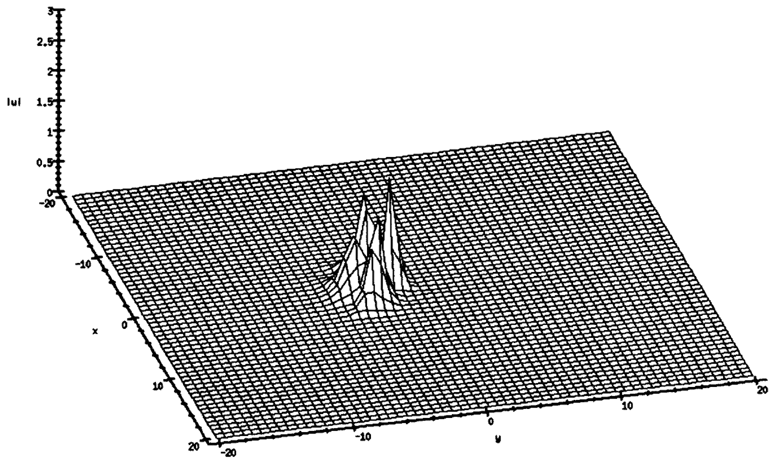

Figure 10. $t = -0.5$.

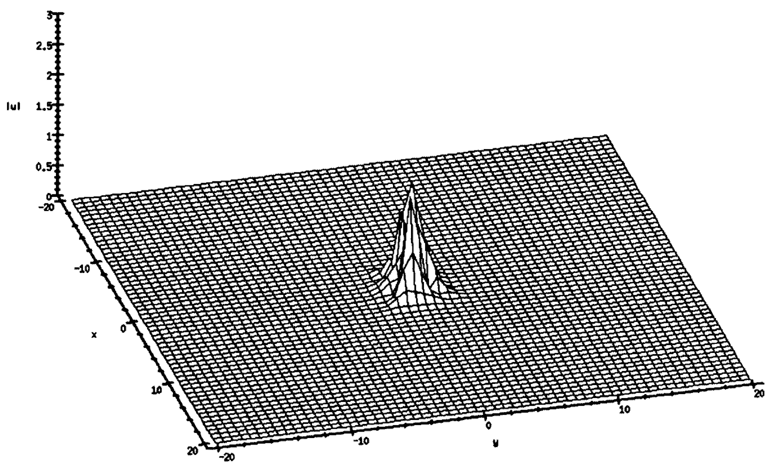

Figure 11. $t = 0$.

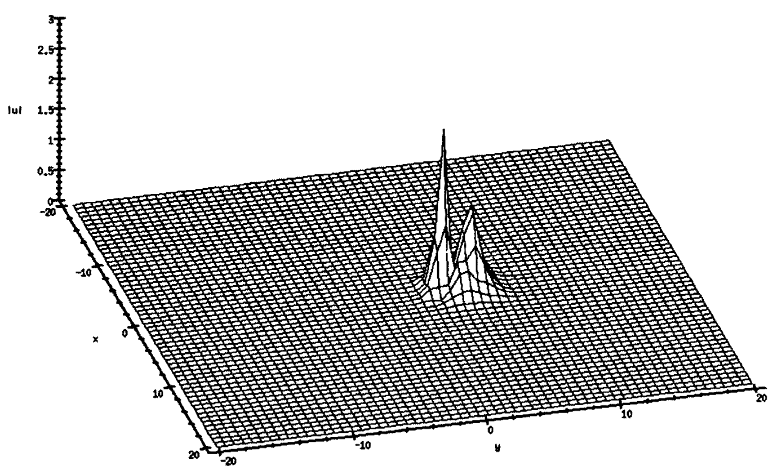

Figure 12. $t = 0.5$.

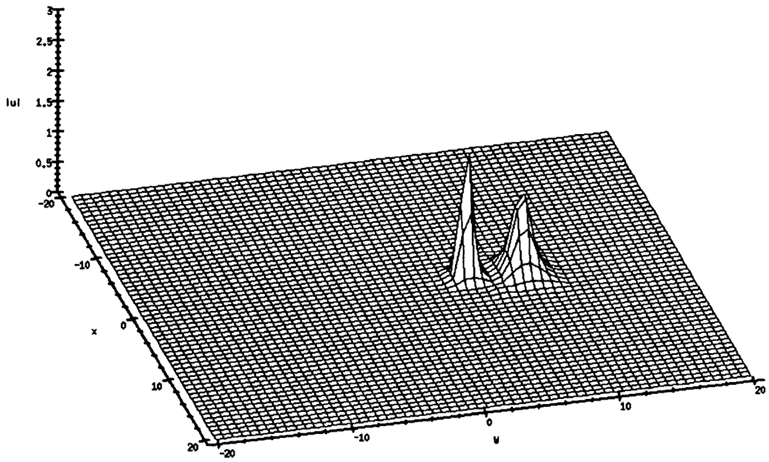

Figure 13. $t = 1$.

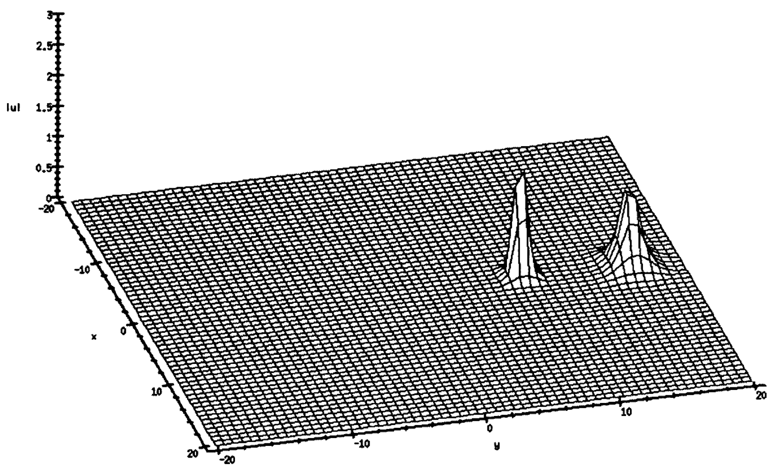

Figure 14. $t = 2$.

Figures 15–21:

$$\lambda_1 = 1 + 2\mathrm{i} \qquad \lambda_2 = 2 - \mathrm{i} \qquad C_1 = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{pmatrix}.$$

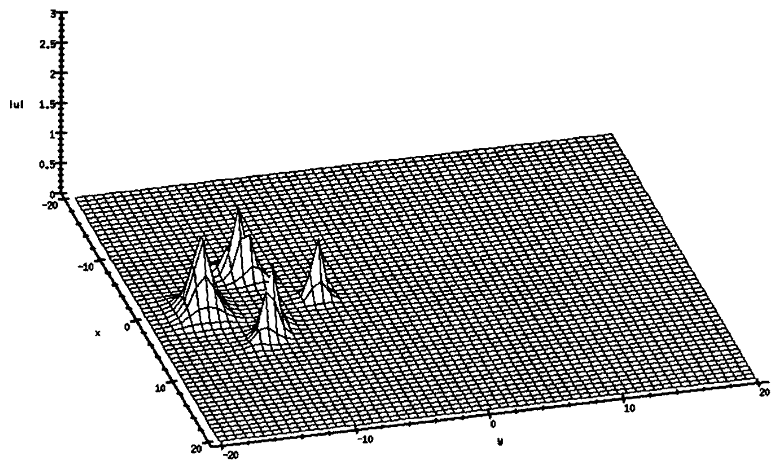

Figure 15. $t = -2$.

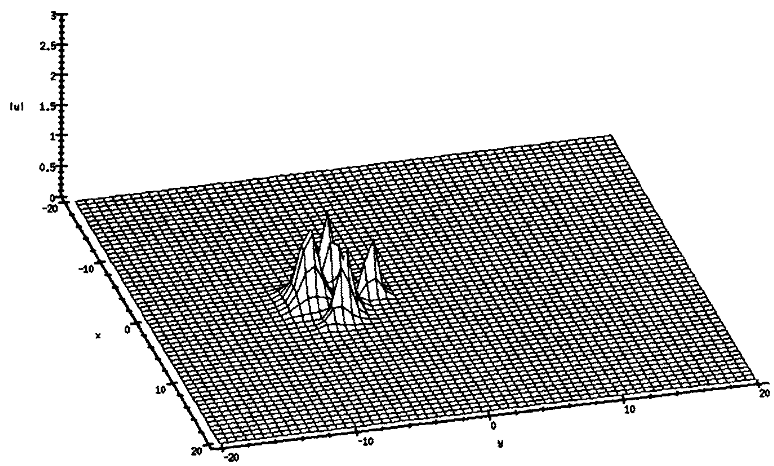

Figure 16. $t = -1$.

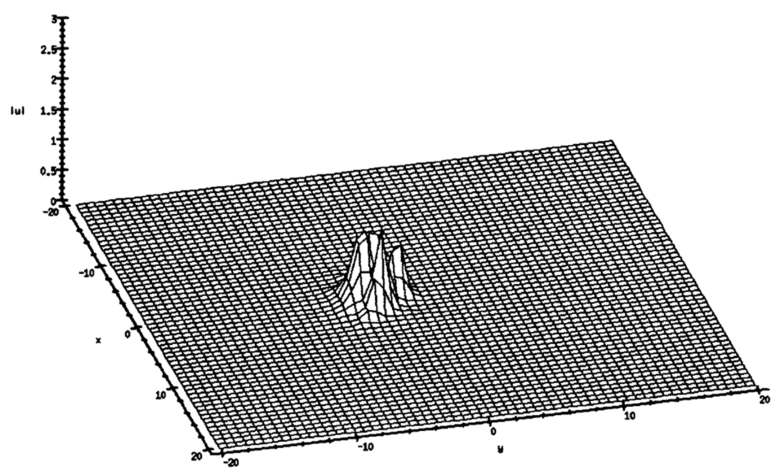

Figure 17. $t = -0.5$.

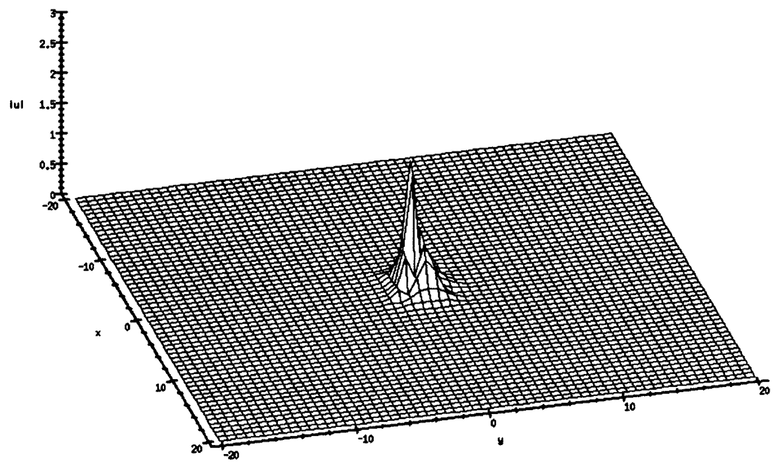

Figure 18. $t = 0$.

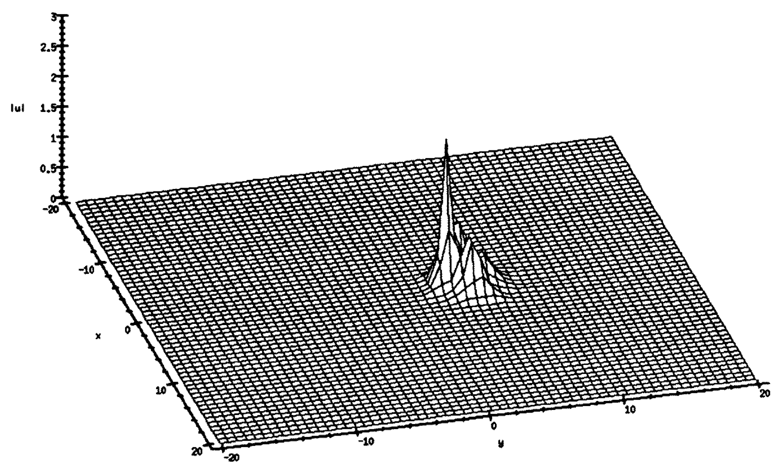

Figure 19. $t = 0.5$.

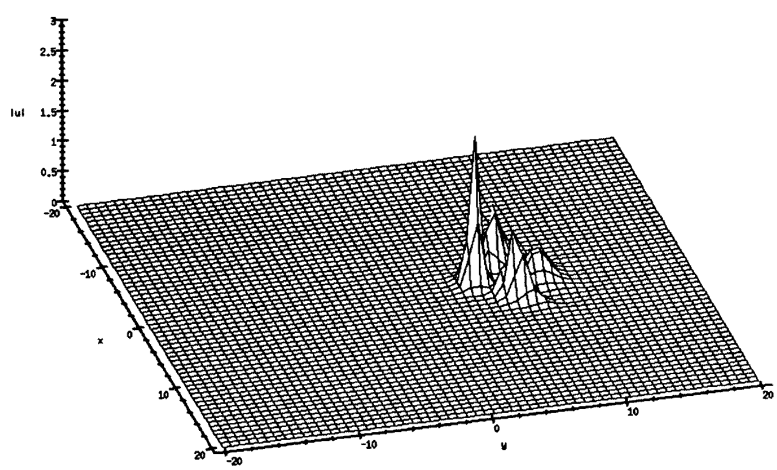

Figure 20. $t = 1$.

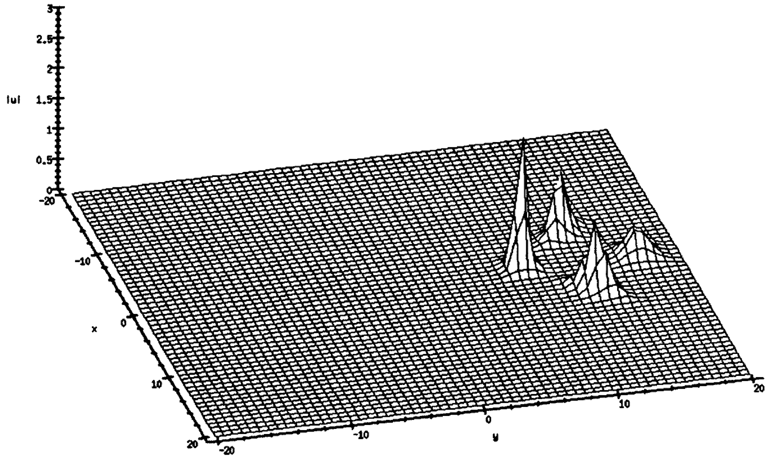

Figure 21. $t = 2$.

Acknowledgments

The author would like to thank the referees for helpful suggestions. He also would like to thank Professor Gu Chaohao and Professor Hu Hesheng for valuable advice when he was in Fudan University.

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