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Research paper

Darboux transformations and global solutions for a nonlocal derivative nonlinear Schrödinger equation

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ABSTRACT

A nonlocal derivative nonlinear Schrödinger equation is discussed. By constructing its Darboux transformations of degree $2n$, the explicit expressions of new solutions are derived from zero seed solutions. Usually the derived solutions of this nonlocal equation may have singularities. However, it is shown that the solutions of the nonlocal derivative nonlinear Schrödinger equation can be globally defined and bounded for all (x, t) if the eigenvalues and the parameters characterizing the ratio of the two entries of the solutions of the Lax pair are chosen properly.

1. Introduction

In [1], Ablowitz and Musslimani introduced the nonlocal nonlinear Schrödinger equation and got its explicit solutions by inverse scattering. Quite a lot of work were done after that [2–6]. Various other nonlocal integrable equations are also considered, including high dimensional equations and discrete equations [2,7–13].

The derivative nonlinear Schrödinger equation [14,15]

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) + \mathrm{i}\varepsilon(q(x,t)^2 q^*(x,t))_x \quad (\varepsilon = \pm 1) \tag{1}$$

is an important integrable equation which describes the Alfvén wave in plasma physics. In this paper, we propose an integrable equation — a nonlocal derivative nonlinear Schrödinger equation

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) + \varepsilon(q(x,t)^2 q^*(-x,t))_x \quad (\varepsilon = \pm 1). \tag{2}$$

Here * means complex conjugation. From Eq. (2), we have

$$i(q^*(-x,t)q(x,t))_t = \Big(q^*(-x,t)q_x(x,t) - (q^*(-x,t))_x q(x,t)\Big)_x + \frac{3\varepsilon}{2}\Big(q^{*2}(-x,t)q^2(x,t)\Big)_x. \tag{3}$$

Hence Eq. (2) has a $\mathcal{PT}$ symmetric conserved density $q^*(-x,t)q(x,t)$, which is invariant under spacial reversion together with complex conjugation. Note that the coefficient of the nonlinear term in Eq. (2) is real, rather than purely imaginary in the usual derivative nonlinear Schrödinger Eq. (1). For analytic solutions, Eq. (2) can be derived formally from Eq. (1) by substituting $x \to -\mathrm{i}x$, $t \to -t$, as pointed by Yang and Yang [16]. In [17], a real space-time reversal derivative nonlinear Schrödinger equation was also presented.

In Section 2 of this paper, the Lax pair for the nonlocal derivative nonlinear Schrödinger equation is presented and its symmetries are considered. In Sections 3 and 4, the Darboux transformations of degree one, two and $2n$ are discussed

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respectively and explicit expressions of the new solutions are derived. In general, the derived solutions may have singularities. In Section 5, global solutions are obtained from zero seed solution by Darboux transformations of degree two, four and eight respectively with suitable choice of parameters. In Section 6, we prove that the solutions given by Darboux transformations of arbitrary degree $2n$ from zero seed solution are globally defined and bounded for $(x,t)\in\mathbf{R}^2$ if the arguments of all eigenvalues are $\pi/4$ and the parameters describing the ratio of the two entries of the solutions of the Lax pair are small enough.

2. Lax pair and its symmetries

Consider the Lax pair

$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi \tag{4}$$

where

$$\begin{aligned} U &= \lambda^2 J + \lambda P = \begin{pmatrix} \lambda^2 & \lambda q \\ \lambda r & -\lambda^2 \end{pmatrix}, \\ V &= \begin{pmatrix} -2\mathrm{i}\lambda^4 + \mathrm{i}qr\lambda^2 & -2\mathrm{i}q\lambda^3 + (-\mathrm{i}q_x + \mathrm{i}q^2 r)\lambda \\ -2\mathrm{i}r\lambda^3 + (\mathrm{i}r_x + \mathrm{i}qr^2)\lambda & 2\mathrm{i}\lambda^4 - \mathrm{i}qr\lambda^2 \end{pmatrix}, \end{aligned} \tag{5}$$

q, r are functions of (x, t), and λ is a spectral parameter.

The integrability condition $U_t - V_x + [U, V] = 0$ gives the evolution equations

$$\begin{aligned} \mathrm{i}q_t &= q_{xx} - (q^2 r)_x, \\ -\mathrm{i}r_t &= r_{xx} + (qr^2)_x. \end{aligned} \tag{6}$$

For simplicity, for a function $f(x, t)$, denote $\bar f(x,t) = f(-x,t)$. Note that $\frac{\partial}{\partial x}\bar f(x,t) = -f_1(-x,t)$. Here f_1 refers to the partial derivative of f with respect to the first variable.

With these notations, we impose a relation $r = -\varepsilon\bar q^*$ where $\varepsilon = \pm 1$. Then the system (6) is reduced to one equation — the nonlocal derivative nonlinear Schrödinger Eq. (2).

With the reduction $r = -\varepsilon\bar q^*$, the coefficients of the Lax pair satisfy

$$\begin{aligned} &JU(\lambda)J^{-1} = U(-\lambda), \quad JV(\lambda)J^{-1} = V(-\lambda), \\ &KU(\lambda)K^{-1} = -(\bar U(-\lambda^*))^*, \quad KV(\lambda)K^{-1} = (\bar V(-\lambda^*))^* \end{aligned} \tag{7}$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $K = \begin{pmatrix} 0 & -\varepsilon \\ 1 & 0 \end{pmatrix}$. Here U^* is the complex conjugation of the matrix U without transpose. We will use $U^\dagger = (U^*)^T$ in this paper.

Considering these symmetries of the Lax pair, we have immediately the symmetries of the solutions of the Lax pair as follows.

Lemma 1.

(1) *If Φ is a solution of the Lax pair (4) with $\lambda = \mu$, then $J\Phi$, $JK\bar\Phi^*$ and $K\bar\Phi^*$ are solutions of Eq. (4) with $\lambda = -\mu$, $\lambda = \mu^*$ and $\lambda = -\mu^*$ respectively. Equivalently, if $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ is a solution of Eq. (4) with $\lambda = \mu$, then $\begin{pmatrix} \xi \\ -\eta \end{pmatrix}$, $\begin{pmatrix} \varepsilon\bar\eta^* \\ \bar\xi^* \end{pmatrix}$ and $\begin{pmatrix} -\varepsilon\bar\eta^* \\ \bar\xi^* \end{pmatrix}$ are solutions of Eq. (4) with $\lambda = -\mu$, $\lambda = \mu^*$ and $\lambda = -\mu^*$ respectively.*

(2) *For any solution Φ of the Lax pair (4), $\Phi^T KL\Phi = 0$ holds where $L = \begin{pmatrix} \varepsilon & \\ & 1 \end{pmatrix}$.*

3. Darboux transformation of degree one and two

3.1. Darboux transformation of degree one for unreduced system

We do not consider the reduction $r = -\varepsilon\bar q^*$ temporarily. Like that for the derivative nonlinear Schrödinger equation [18–20], a Darboux transformation of degree one can be constructed as follows.

Lemma 2. *Suppose $G(x,t,\lambda) = R(x,t)(\lambda - S(x,t))$ is a Darboux matrix for Eq. (4), which transforms $U = \lambda^2 J + \lambda P = \begin{pmatrix} \lambda^2 & \lambda q \\ \lambda r & -\lambda^2 \end{pmatrix}$ to $\widetilde U = \lambda^2 J + \lambda\widetilde P = \begin{pmatrix} \lambda^2 & \lambda\widetilde q \\ \lambda\widetilde r & -\lambda^2 \end{pmatrix}$ and transforms V in Eq. (5) to $\widetilde V$ which has the same form as V where (q, r) are replaced by $(\widetilde q, \widetilde r)$, then R is a diagonal matrix and RS is a constant matrix. Moreover, the transformation of P is*

$$\widetilde P = RPR^{-1} + [J, RS]R^{-1}. \tag{8}$$

Proof. For the x-part, the condition $GU + G_x = \widetilde U G$, which means that G is a Darboux matrix, is

$$R(\lambda - S)(\lambda^2 J + \lambda P) + \lambda R_x - (RS)_x = (\lambda^2 J + \lambda\widetilde P)R(\lambda - S). \tag{9}$$

Compare the coefficients of the powers of λ. The coefficient of λ^3 implies that R is a diagonal matrix. The term without λ implies that RS is independent of x. The coefficient of λ^2 gives the transformation (8). On the other hand, RS is also independent of t by considering the t equation. The lemma is proved. □

By Lemma 1, if $\begin{pmatrix}\xi\\ \eta\end{pmatrix}$ is a solution of Eq. (1) with $\lambda=\mu$, then $\begin{pmatrix}\xi\\ -\eta\end{pmatrix}$ is a solution of Eq. (1) with $\lambda=-\mu$. Following the idea of [21] and [22] (see also [23]), let

$$\Lambda=\begin{pmatrix}\mu & \\ & -\mu\end{pmatrix},\quad H=\begin{pmatrix}\xi & \xi\\ \eta & -\eta\end{pmatrix},\tag{10}$$

then

$$S\triangleq H\Lambda H^{-1}=\mu\begin{pmatrix}0 & 1/\sigma\\ \sigma & 0\end{pmatrix}\tag{11}$$

gives a Darboux matrix $G(\lambda)=R(\lambda I-S)$ where $\sigma=\eta/\xi$, and R is a suitable invertible matrix.

To make R diagonal and RS constant, choose

$$R=\begin{pmatrix}\sigma & 0\\ 0 & 1/\sigma\end{pmatrix},\tag{12}$$

then $RS=\mu\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. The Darboux matrix is

$$G(\lambda)=R(\lambda-S)=\begin{pmatrix}\sigma\lambda & -\mu\\ -\mu & \lambda/\sigma\end{pmatrix}\tag{13}$$

and the transformation (8) becomes

$$\widetilde{q}=\sigma^2q+2\mu\sigma,\quad \widetilde{r}=\frac{r}{\sigma^2}-\frac{2\mu}{\sigma}.\tag{14}$$

3.2. Darboux transformation of degree two for unreduced system

Like Eqs. (10)–(12), take

$$\begin{aligned}&\Lambda_\alpha=\begin{pmatrix}\mu_\alpha & \\ & -\mu_\alpha\end{pmatrix},\quad H_\alpha=\begin{pmatrix}\xi_\alpha & \xi_\alpha\\ \eta_\alpha & -\eta_\alpha\end{pmatrix},\quad \sigma_\alpha=\frac{\eta_\alpha}{\xi_\alpha},\\ &S_\alpha=H_\alpha\Lambda_\alpha H_\alpha^{-1}=\mu\begin{pmatrix}0 & 1/\sigma_\alpha\\ \sigma_\alpha & 0\end{pmatrix},\quad R_\alpha=\begin{pmatrix}\sigma_\alpha & 0\\ 0 & 1/\sigma_\alpha\end{pmatrix}\quad(\alpha=1,2).\end{aligned}\tag{15}$$

Following Eq. (13), the Darboux matrix with respect to (Λ_1,H_1) is $G_1(\lambda)=\begin{pmatrix}\sigma_1\lambda & -\mu_1\\ -\mu_1 & \dfrac{\lambda}{\sigma_1}\end{pmatrix}$.

After the action of $G_1(\lambda)$, the two columns of H_2 are transformed to

$$\begin{aligned}&G_1(\mu_2)\begin{pmatrix}\xi_2\\ \eta_2\end{pmatrix}=\xi_2\begin{pmatrix}\mu_2\sigma_1-\mu_1\sigma_2\\ -\mu_1+\dfrac{\mu_2\sigma_2}{\sigma_1}\end{pmatrix}\triangleq\begin{pmatrix}\widetilde{\xi}_2\\ \widetilde{\eta}_2\end{pmatrix},\\ &G_1(-\mu_2)\begin{pmatrix}\xi_2\\ -\eta_2\end{pmatrix}=\xi_2\begin{pmatrix}-\mu_2\sigma_1+\mu_1\sigma_2\\ -\mu_1+\dfrac{\mu_2\sigma_2}{\sigma_1}\end{pmatrix}=-\begin{pmatrix}\widetilde{\xi}_2\\ -\widetilde{\eta}_2\end{pmatrix},\end{aligned}\tag{16}$$

which are solutions of Eq. (4) with potentials $\widetilde{q},\widetilde{r}$ where the eigenvalues are taken as $\lambda=\mu_2$ and $\lambda=-\mu_2$ respectively. Define

$$\widetilde{\sigma}_2\triangleq\frac{\widetilde{\eta}_2}{\widetilde{\xi}_2}=\frac{1}{\sigma_1}\frac{\mu_1\sigma_1-\mu_2\sigma_2}{\mu_1\sigma_2-\mu_2\sigma_1},\tag{17}$$

then the Darboux matrix given by Eq. (13) for $(\tilde{q}, \tilde{r})$ is $\tilde{G}_2(\lambda) = \begin{pmatrix} \tilde{\sigma}_2\lambda & -\mu_2 \\ -\mu_2 & \dfrac{\lambda}{\tilde{\sigma}_2} \end{pmatrix}$. The Darboux matrix of degree two for (q, r) is

$$\begin{aligned} G(\lambda) = \tilde{G}_2(\lambda)G_1(\lambda) &= \begin{pmatrix} \sigma_1\tilde{\sigma}_2\lambda^2 + \mu_1\mu_2 & -\left(\mu_1\tilde{\sigma}_2 + \dfrac{\mu_2}{\sigma_1}\right)\lambda \\ -\left(\mu_2\sigma_1 + \dfrac{\mu_1}{\tilde{\sigma}_2}\right)\lambda & \dfrac{1}{\sigma_1\tilde{\sigma}_2}\lambda^2 + \mu_1\mu_2 \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\mu_1\sigma_1 - \mu_2\sigma_2}{\mu_1\sigma_2 - \mu_2\sigma_1}\lambda^2 + \mu_1\mu_2 & \dfrac{\mu_2^2 - \mu_1^2}{\mu_1\sigma_2 - \mu_2\sigma_1}\lambda \\ \dfrac{(\mu_2^2 - \mu_1^2)\sigma_1\sigma_2}{\mu_1\sigma_1 - \mu_2\sigma_2}\lambda & \dfrac{\mu_1\sigma_2 - \mu_2\sigma_1}{\mu_1\sigma_1 - \mu_2\sigma_2}\lambda^2 + \mu_1\mu_2 \end{pmatrix}. \end{aligned} \tag{18}$$

By Eq. (14), the solution of Eq. (2) derived from this $G(\lambda)$ is

$$\begin{aligned} \tilde{\tilde{q}} &= \tilde{\sigma}_2^2(\sigma_1^2 q + 2\mu_1\sigma_1) + 2\mu_2\tilde{\sigma}_2 = \left(\frac{\mu_1\sigma_1 - \mu_2\sigma_2}{\mu_1\sigma_2 - \mu_2\sigma_1}\right)^2\left(q + \frac{2(\mu_1^2 - \mu_2^2)}{\mu_1\sigma_1 - \mu_2\sigma_2}\right), \\ \tilde{\tilde{r}} &= \frac{1}{\tilde{\sigma}_2^2}\left(\frac{r}{\sigma_1^2} - \frac{2\mu_1}{\sigma_1}\right) - \frac{2\mu_2}{\tilde{\sigma}_2} = \left(\frac{\mu_1\sigma_2 - \mu_2\sigma_1}{\mu_1\sigma_1 - \mu_2\sigma_2}\right)^2\left(r - \frac{2(\mu_1^2 - \mu_2^2)\sigma_1\sigma_2}{\mu_1\sigma_2 - \mu_2\sigma_1}\right). \end{aligned} \tag{19}$$

This result is similar to that in [24] for usual Kaup–Newell system.

3.3. Darboux transformation of degree two for nonlocal derivative nonlinear Schrödinger equation

According to Lemma 1, all the eigenvalues $\mu, -\mu, \mu^*, -\mu^*$ should be considered in constructing Darboux matrix when μ^2 is not real. Therefore, a Darboux matrix of degree two is necessary in this case.

Following Lemma 1, those in Eq. (15) are now

$$\begin{aligned} &\mu_1 = \mu, \quad \mu_2 = \mu^*, \quad \xi_1 = \xi, \quad \eta_1 = \eta, \quad \xi_2 = \varepsilon\bar{\eta}^*, \quad \eta_2 = \bar{\xi}^*, \\ &\sigma_1 = \sigma, \quad \sigma_2 = \varepsilon/\bar{\sigma}^*. \end{aligned} \tag{20}$$

The Darboux matrix is given by Eq. (18), that is

$$G(\lambda) = \begin{pmatrix} -\dfrac{\mu^* - \varepsilon\mu\sigma\bar{\sigma}^*}{\mu - \varepsilon\mu^*\sigma\bar{\sigma}^*}\lambda^2 + |\mu|^2 & \dfrac{\varepsilon(\mu^{*2} - \mu^2)\bar{\sigma}^*}{\mu - \varepsilon\mu^*\sigma\bar{\sigma}^*}\lambda \\ -\dfrac{(\mu^{*2} - \mu^2)\sigma}{\mu^* - \varepsilon\mu\sigma\bar{\sigma}^*}\lambda & -\dfrac{\mu - \varepsilon\mu^*\sigma\bar{\sigma}^*}{\mu^* - \varepsilon\mu\sigma\bar{\sigma}^*}\lambda^2 + |\mu|^2 \end{pmatrix}. \tag{21}$$

It can be checked that $G(\lambda)$ satisfies the reductions $G(-\lambda) = J^{-1}G(\lambda)J$, $(\bar{G}(-\lambda^*))^* = K^{-1}G(\lambda)K$, which are compatible with Eq. (7).

The new solution is given by Eq. (19). That is

Theorem 1. *Suppose q is a solution of Eq. (2). Let μ be a nonzero complex constant which is neither real nor purely imaginary. Let $(\xi, \eta)^T$ be a solution of Eq. (4) with $\lambda = \mu$. Then*

$$\tilde{q} = \left(\frac{\mu^* - \varepsilon\mu\sigma\bar{\sigma}^*}{\mu - \varepsilon\mu^*\sigma\bar{\sigma}^*}\right)^2\left(q - \frac{2(\mu^2 - \mu^{*2})}{\mu^* - \varepsilon\mu\sigma\bar{\sigma}^*}\varepsilon\bar{\sigma}^*\right) \tag{22}$$

is a new solution of Eq. (2) where $\sigma = \eta/\xi$. The corresponding Darboux matrix is given by Eq. (21).

4. Darboux transformation of degree 2*n*

The Darboux transformation of degree two is the Darboux transformation of lowest degree that keeps all the reductions of nonlocal derivative nonlinear Schrödinger equation if μ^2 is not real. The composition of these Darboux transformations leads to a Darboux transformation of higher degree. However, this can be constructed equivalently and more compactly following the idea of [25]. The following construction is based on Lemma 1.

Theorem 2. *Take n complex numbers μ_j so that μ_j^2's are not real and take the solution $(\xi_j, \eta_j)^T$ of the Lax pair (4) with $\lambda = \mu_j$ $(j = 1, 2, \ldots, n)$. Let $\lambda_{2j-1} = \mu_j$, $\lambda_{2j} = -\mu_j$, $h_{2j-1} = (\xi_j, \eta_j)^T$, $h_{2j} = (\xi_j, -\eta_j)^T (j = 1, 2, \ldots, n)$. Let*

$$\Gamma_{\alpha\beta} = \frac{\bar{h}_\alpha^\dagger L h_\beta}{\lambda_\alpha^* + \lambda_\beta} \quad (\alpha, \beta = 1, \ldots, 2n), \tag{23}$$

$$G(\lambda) = \prod_{\gamma=1}^{2n}(\lambda + \lambda_\gamma^*) \cdot F^{-1}\left(I - \sum_{\alpha,\beta=1}^{2n}\frac{(\Gamma^{-1})_{\alpha\beta}h_\alpha\bar{h}_\beta^\dagger L}{\lambda + \lambda_\beta^*}\right) \tag{24}$$

where

$$F = I - \sum_{\alpha,\beta=1}^{2n} \frac{(\Gamma^{-1})_{\alpha\beta} h_\alpha \bar{h}_\beta^\dagger L}{\lambda_\beta^*} \tag{25}$$

is a diagonal matrix. Then $G(\lambda)$ *is a Darboux matrix for the Lax pair (4), and the transformation of* q *is given by*

$$\tilde{q} = \frac{I - \displaystyle\sum_{\alpha,\beta=1}^{2n} \frac{(\Gamma^{-1})_{\alpha\beta} \eta_\alpha \bar{\eta}_\beta^*}{\lambda_\beta^*}}{I - \varepsilon \displaystyle\sum_{\alpha,\beta=1}^{2n} \frac{(\Gamma^{-1})_{\alpha\beta} \xi_\alpha \bar{\xi}_\beta^*}{\lambda_\beta^*}} \left(q + 2 \sum_{\alpha,\beta=1}^{2n} (\Gamma^{-1})_{\alpha\beta} \xi_\alpha \bar{\eta}_\beta^* \right). \tag{26}$$

Remark 1.

(i) $G(\lambda)$ is a polynomial of λ of degree $2n$. Moreover, $G(0) = (-1)^n \prod_{l=1}^{n} \mu_l^{*2} I$ is a constant scalar matrix. This is necessary since $G(0)$ of a Darboux matrix (21) of degree two is constant, so is their product.
(ii) Eq. (24) gives the same Darboux matrix as Eq. (21) up to a constant scalar multiplier when $n = 1$.

Proof. We have

$$\begin{aligned} &\left(I - \sum_{\alpha,\beta=1}^{2n} \frac{(\Gamma^{-1})_{\alpha\beta} h_\alpha \bar{h}_\beta^\dagger L}{\lambda + \lambda_\beta^*} \right)\Bigg|_{\lambda=\lambda_\gamma} h_\gamma = h_\gamma - \sum_{\alpha,\beta=1}^{2n} \frac{(\Gamma^{-1})_{\alpha\beta} h_\alpha \bar{h}_\beta^\dagger L h_\gamma}{\lambda_\gamma + \lambda_\beta^*} = 0, \\ &(\lambda + \lambda_\gamma^*) \left(I - \sum_{\alpha,\beta=1}^{2n} \frac{(\Gamma^{-1})_{\alpha\beta} h_\alpha \bar{h}_\beta^\dagger L}{\lambda + \lambda_\beta^*} \right)\Bigg|_{\lambda=-\lambda_\gamma^*} K \bar{h}_\gamma^* = 0 \end{aligned} \tag{27}$$

since

$$\bar{h}_\gamma^\dagger L K \bar{h}_\gamma^* = \begin{pmatrix} \bar{\xi}_\gamma^* & \bar{\eta}_\gamma^* \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi}_\gamma^* \\ \bar{\eta}_\gamma^* \end{pmatrix} = 0. \tag{28}$$

Hence

$$G(\lambda_\gamma) h_\gamma = 0, \quad G(-\lambda_\gamma^*) K \bar{h}_\gamma^* = 0 \quad (\gamma = 1, 2. \ldots, 2n). \tag{29}$$

Since h_γ is a solution of the Lax pair (4) with $\lambda = \lambda_\gamma$, Lemma 1 implies that $K\bar{h}_\gamma^*$ is a solution of Eq. (4) with $\lambda = -\lambda_\gamma^*$. According to Lemma 4 of [26], $G(\lambda)$ is a Darboux matrix without considering the reductions. What is more, by the standard construction of Darboux transformation [23], $G(\lambda)$ is a composition of $2n$ Darboux matrices in the form (13). From the choice of λ_j's and h_j's, $G(\lambda)$ is a composition of n Darboux matrices in the form (18). That is,

$$G(\lambda) = (-1)^n G_n(\lambda) \ldots G_1(\lambda) \prod_{l=1}^{n} \frac{\mu_l^*}{\mu_l} \tag{30}$$

where each $G_l(\lambda)$ $(l = 1, \ldots, n)$ is a Darboux matrix in the form (18). Hence $G(\lambda)$ keeps the reduction $r = -\varepsilon \bar{q}^*$, i.e. it is a Darboux matrix for the nonlocal derivative nonlinear Schrödinger equation.

Write

$$G(\lambda) = M_0 \lambda^{2n} + M_1 \lambda^{2n-1} + \ldots + M_{2n}, \tag{31}$$

then the coefficient of λ^{2n+1} in $G(\lambda)U(\lambda) + G_x(\lambda) = \widetilde{U}(\lambda)G(\lambda)$ gives the transformation

$$\widetilde{P} = M_0 P M_0^{-1} - [J, M_1] M_0^{-1}. \tag{32}$$

Expanding Eq. (24) as a polynomial of λ and using $\sum_{\gamma=1}^{2n} \lambda_\gamma^* = 0$, we get

$$M_0 = F^{-1}, \quad M_1 = -F^{-1} \sum_{\alpha,\beta=1}^{2n} (\Gamma^{-1})_{\alpha\beta} h_\alpha \bar{h}_\beta^* L. \tag{33}$$

Since each $G_k(\lambda)$ $(k = 1, 2, \ldots, n)$ in Eq. (30) is in the form (18), $F = M_0^{-1}$ must be diagonal. (This can also be verified algebraically from the expression (25) by exchanging each pair of $(2j-1, 2j)(j = 1, \ldots, n)$ in the subscripts.) The (1, 2) entry of Eq. (32) gives Eq. (26). The theorem is proved. □

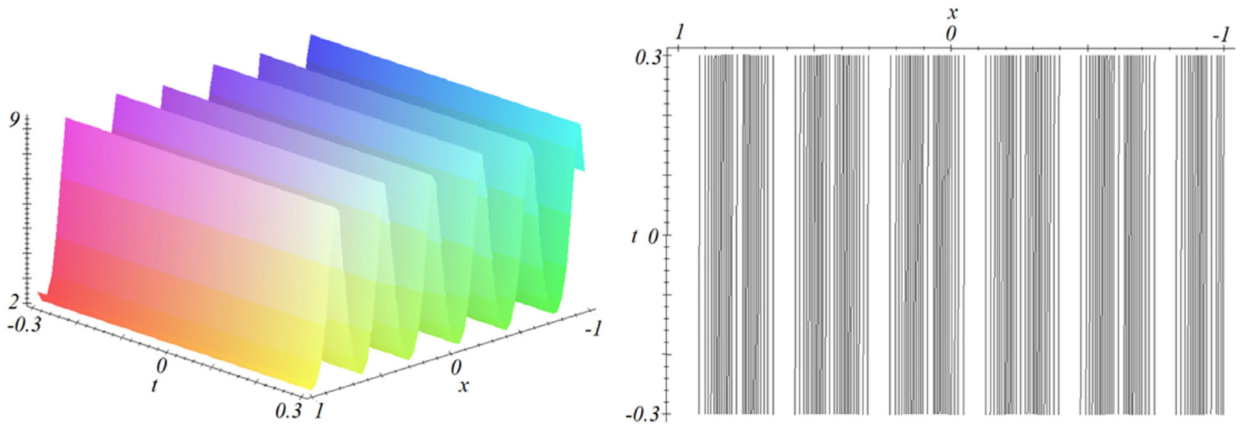

Fig. 1. $|\tilde{q}|$: $\tilde{q}$ is given by a Darboux transformation of degree 2.

5. Examples

When $q = r = 0$, the Lax pair (4) becomes

$$\Phi_x = \begin{pmatrix} \lambda^2 & 0 \\ 0 & -\lambda^2 \end{pmatrix}\Phi, \quad \Phi_t = \begin{pmatrix} -2\mathrm{i}\lambda^4 & 0 \\ 0 & 2\mathrm{i}\lambda^4 \end{pmatrix}\Phi. \tag{34}$$

The solution of this system for $\lambda = \mu$ is $\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} ae\theta \\ be^{-\theta} \end{pmatrix}$ where $\theta = \mu^2 x - 2\mathrm{i}\mu^4 t$ and a, b are complex constants. Then

$$\sigma = \frac{\eta}{\xi} = ce^{-2\theta} \tag{35}$$

where $c = b/a$ is a complex constant.

The behavior of the solutions depends on $4\arg\mu/\pi$. In our construction, we have already assumed that $4\arg\mu/\pi$ is not an even integer. In fact, the derived solution will not be bounded if $4\arg\mu/\pi$ is an even integer. When $4\arg\mu/\pi$ is not an integer, a Darboux matrix of degree two should be used as in Theorem 1. In Eq. (21), one of the denominators

$$\mu - \varepsilon\mu^*\sigma\bar{\sigma}^* = \varepsilon\mu^*|c|^2\left(\frac{\varepsilon\mu}{\mu^*|c|^2} - \mathrm{e}^{-2(\mu^2-\mu^{*2})\mathrm{x}+4\mathrm{i}(\mu^4-\mu^{*4})\mathrm{t}}\right) \tag{36}$$

can be zero if $\mu^2 \neq \mu^{*2}$. In order to get solutions without singularities, we need $\mu^2 = \mu^{*2}$, hence $4\arg\mu/\pi$ should be an odd integer. Without loss of generality, take $\arg\mu_j = \pi/4$ $(j = 1, \ldots, n)$ for a Darboux transformation of degree $2n$.

First consider a Darboux transformation of degree 2 with $\mu = a(1+\mathrm{i})$ where a is a positive constant. By Eq. (22),

$$\tilde{q} = -4(1-\mathrm{i})\varepsilon ac^* \frac{\mathrm{e}^{-4\mathrm{i}a^2\mathrm{x}}(1 - \mathrm{i}\varepsilon|c|^2\mathrm{e}^{-8\mathrm{i}a^2\mathrm{x}})}{(1+\mathrm{i}\varepsilon|c|^2\mathrm{e}^{-8\mathrm{i}a^2\mathrm{x}})^2}\mathrm{e}^{16\mathrm{i}a^4\mathrm{t}}. \tag{37}$$

This solution is global if $|c| \neq 1$. It is periodic in both x and t, and $|\tilde{q}|$ is a function of x only. Fig. 1 shows the norm of the solution with $\varepsilon = 1$, $\mu = 1.5(1+\mathrm{i})$, $c = 0.5$. Hereafter, the figure on the right shows the contour plot of the one on the left.

By using Theorem 2, the figures for the norm of the solutions given by Darboux transformations of degree 4 and 8 are plotted in Fig. 2 ($\varepsilon = 1$, $a_1 = 1.5$, $a_2 = 1.3$, $c_1 = 0.05$, $c_2 = -0.02$) and Fig. 3 ($\varepsilon = 1$, $a_1 = 1.5$, $a_2 = 1.3$, $a_3 = 1.1$, $a_4 = 0.9$, $c_1 = 0.001$, $c_2 = -0.002$, $c_3 = 0.001\mathrm{i}$, $c_4 = -0.001$).

6. Globalness of the solutions

Although the solutions (37) given by Darboux transformation of degree two are always global when $|c| \neq 1$, those given by Darboux transformation of higher degree may not. However, we can prove that the solutions have no singularities when all $|c_j|$'s are small enough.

Before proving that theorem, we need the follow algebraic lemma.

Lemma 3. *Let $\lambda_1, \ldots, \lambda_m, \omega_1, \ldots, \omega_m$ be nonzero complex constants such that $\omega_j + \lambda_k \neq 0 (j, k = 1, \ldots, m)$, and $C = \left(\frac{1}{\omega_j + \lambda_k}\right)_{1\le j,k\le m}$. Then*

$$1 - \sum_{1\le j,k\le m}(C^{-1})_{jk}\frac{1}{\omega_k} = \prod_{l=1}^{m}\left(-\frac{\lambda_l}{\omega_l}\right). \tag{38}$$

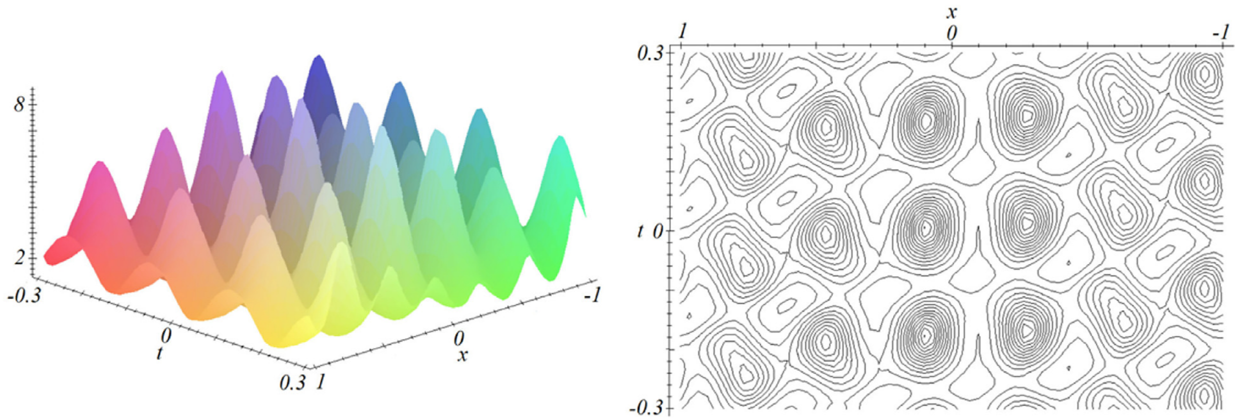

Fig. 2. $|\widetilde{q}|$: $\widetilde{q}$ is given by a Darboux transformation of degree 4.

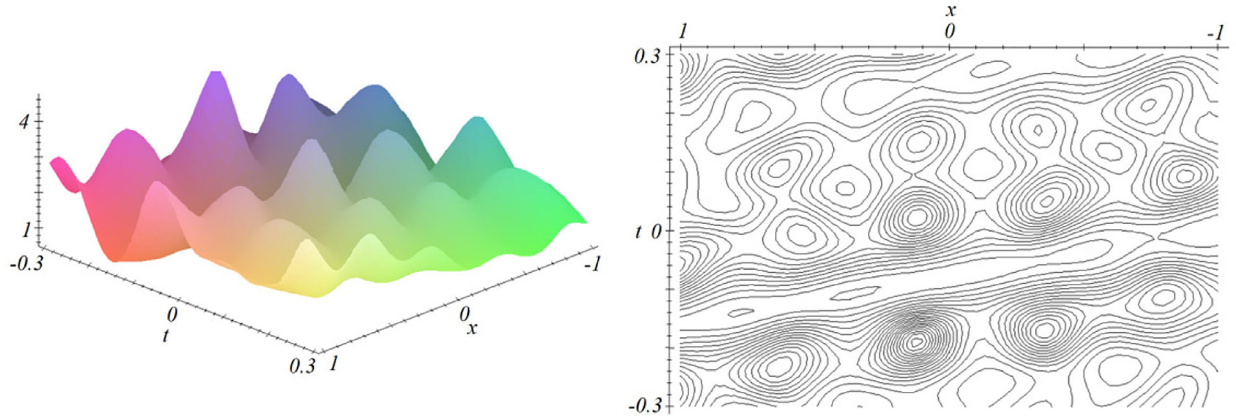

Fig. 3. $|\widetilde{q}|$: $\widetilde{q}$ is given by a Darboux transformation of degree 8.

Proof. By the formula for computing the determinant of a block matrix,

$$1-\sum_{1\le j,k\le m}(C^{-1})_{jk}\frac{1}{\omega_k}=\frac{1}{\det C}\begin{vmatrix}\dfrac{1}{\omega_1+\lambda_1} & \cdots & \dfrac{1}{\omega_1+\lambda_m} & \dfrac{1}{\omega_1}\\ \vdots & & \vdots & \vdots\\ \dfrac{1}{\omega_m+\lambda_1} & \cdots & \dfrac{1}{\omega_m+\lambda_m} & \dfrac{1}{\omega_m}\\ 1 & \cdots & 1 & 1\end{vmatrix}\triangleq\frac{\Delta}{\det C}. \tag{39}$$

Clearly,

$$\Delta=\frac{T(\lambda_1,\ldots,\lambda_m,\omega_1,\ldots,\omega_m)}{\prod_{l=1}^{m}\omega_l\prod_{1\le j,k\le m}(\omega_j+\lambda_k)} \tag{40}$$

where T is a polynomial of degree m^2. Note that $T=0$ when $\omega_j=\omega_k$ or $\lambda_j=\lambda_k$ for certain $j\neq k$ with $1\le j,\ k\le m$, or when $\lambda_l=0$ for certain l with $1\le l\le m$. Considering the degree of the polynomial, we have

$$T=\rho\prod_{l=1}^{m}\lambda_l\prod_{1\le j<k\le m}(\lambda_j-\lambda_k)(\omega_j-\omega_k) \tag{41}$$

where ρ is a constant. When $\lambda_1,\ldots,\lambda_m$ are distinct,

$$1=\lim_{\omega_1\to-\lambda_1,\ldots,\omega_m\to-\lambda_m}\Delta\prod_{j=1}^{m}(\omega_j+\lambda_j)=(-1)^m\rho \tag{42}$$

which leads to $\rho = (-1)^m$. Likewise, we have

$$\det C = \frac{\prod_{1\le j<k\le m} (\lambda_j - \lambda_k)(\omega_j - \omega_k)}{\prod_{1\le j,k\le m} (\omega_j + \lambda_k)}, \tag{43}$$

which is the standard Cauchy determinant. Therefore,

$$1 - \sum_{1\le j,k\le m} (C^{-1})_{jk} \frac{1}{\omega_k} = \prod_{l=1}^{m} \left(-\frac{\lambda_l}{\omega_l}\right). \tag{44}$$

The lemma is proved. □

Theorem 3. *Take the seed solution $q = 0$. Suppose $\mu_j = a_j e^{\pi i/4}$ where $a_1, \ldots, a_n$ are distinct positive numbers. Then there is a positive constant δ such that the solution (26) given by a Darboux transformation of degree $2n$ is globally defined and bounded for $(x, t) \in \mathbf{R}^2$ when $|c_j| < \delta (j = 1, \ldots, n)$.*

Proof. It is easy to check from Eqs. (23) and (24) that $G(\lambda)$ is invariant if each (ξ_j, η_j) is changed to $\rho_j(\xi_j, \eta_j)$ for any nonzero constants ρ_j $(j = 1, \ldots, n)$. Hence (ξ_j, η_j) can be replaced by $(1, \sigma_j)$.

It is only necessary to prove that Γ in Eq. (23) and F in Eq. (25) are both invertible for all (x, t) when $|c_j|$ $(j = 1, \ldots, n)$ are small enough.

As before, denote $h_{2j-1} = (1, \eta_j)^T$, $h_{2j} = (1, -\eta_j)^T$. Since $\sigma_j = c_j e^{-2\theta_j} = c_j e^{-2\mu_j^2 x + 4i\mu_k^4 t}$, $\sigma_k \bar{\sigma}_j^* = c_k c_j^* e^{-i\phi_{jk}}$ where $\phi_{jk} = 2(a_j^2 + a_k^2)x + 4(a_j^4 - a_k^4)t$ is real. Hence

$$\begin{aligned}
&\bar{h}_{2j-1}^\dagger L h_{2k-1} = \bar{h}_{2j}^\dagger L h_{2k} = \varepsilon + \sigma_k \bar{\sigma}_j^* = \varepsilon(1 + \varepsilon c_k c_j^* e^{-i\phi_{jk}}),\\
&\bar{h}_{2j-1}^\dagger L h_{2k} = \bar{h}_{2j}^\dagger L h_{2k-1} = \varepsilon - \sigma_k \bar{\sigma}_j^* = \varepsilon(1 - \varepsilon c_k c_j^* e^{-i\phi_{jk}}).
\end{aligned} \tag{45}$$

$\Gamma = (\Gamma_{\alpha\beta})_{2n\times 2n}$ in Eq. (23) can be written as a block matrix $\Gamma = (\widetilde{\Gamma}_{jk})_{n\times n}$ where

$$\widetilde{\Gamma}_{jk} = \begin{pmatrix} \dfrac{\bar{h}_{2j-1}^\dagger L h_{2k-1}}{\mu_j^* + \mu_k} & \dfrac{\bar{h}_{2j-1}^\dagger L h_{2k}}{\mu_j^* - \mu_k} \\ \dfrac{\bar{h}_{2j}^\dagger L h_{2k-1}}{-\mu_j^* + \mu_k} & \dfrac{\bar{h}_{2j}^\dagger L h_{2k}}{-\mu_j^* - \mu_k} \end{pmatrix} = \varepsilon \begin{pmatrix} \dfrac{1 + \varepsilon c_k c_j^* e^{-i\phi_{jk}}}{\mu_j^* + \mu_k} & \dfrac{1 - \varepsilon c_k c_j^* e^{-i\phi_{jk}}}{\mu_j^* - \mu_k} \\ \dfrac{1 - \varepsilon c_k c_j^* e^{-i\phi_{jk}}}{-\mu_j^* + \mu_k} & \dfrac{1 + \varepsilon c_k c_j^* e^{-i\phi_{jk}}}{-\mu_j^* - \mu_k} \end{pmatrix}. \tag{46}$$

Hence Γ converges to $\Gamma|_{c_1=\ldots=c_n=0}$ uniformly for $(x, t) \in \mathbf{R}^2$ when $c_1 \to 0, \ldots, c_n \to 0$. Note that

$$\Gamma|_{c_1=\ldots=c_n=0} = \varepsilon \left(\frac{1}{\lambda_j^* + \lambda_k}\right)_{1\le j,k\le 2n} \tag{47}$$

where $\lambda_{2j-1} = \mu_j$, $\lambda_{2j} = -\mu_j (j = 1, \ldots, n)$. Its determinant is

$$\det \Gamma|_{c_1=\ldots=c_n=0} = \frac{\prod_{1\le\alpha<\beta\le 2n} |\lambda_\alpha - \lambda_\beta|^2}{\prod_{1\le\alpha,\beta\le 2n} (\lambda_\alpha^* + \lambda_\beta)} \neq 0 \tag{48}$$

as Eq. (43). Hence Γ is invertible when $|c_j| (j = 1, \ldots, n)$ are small enough.

According to Lemma 3 and (47),

$$\begin{aligned}
F|_{c_1=\ldots=c_n=0} &= I - \sum_{\alpha,\beta=1}^{2n} \frac{(\Gamma|_{c_1=\ldots=c_n=0}^{-1})_{\alpha\beta} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \varepsilon & \\ & 1 \end{pmatrix}}{\lambda_\beta^*}\\
&= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \sum_{\alpha,\beta=1}^{2n} \frac{((\varepsilon\Gamma)|_{c_1=\ldots=c_n=0}^{-1})_{\alpha\beta}}{\lambda_\beta^*} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\\
&= \begin{pmatrix} \prod_{\gamma=1}^{2n} \dfrac{\lambda_\gamma}{\lambda_\gamma^*} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \prod_{l=1}^{n} \left(\dfrac{\mu_l}{\mu_l^*}\right)^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{49}$$

is invertible. Eq. (48) implies that Γ^{-1} is bounded for all (x, t) when $|c_j|$ $(j = 1, \ldots, n)$ are small enough. Hence F converges to $F|_{c_1=\ldots=c_n=0}$ uniformly for $(x, t) \in \mathbf{R}^2$ when $c_1 \to 0, \ldots, c_n \to 0$, which implies that F is invertible if $|c_j|$ $(j = 1, \ldots, n)$ are small enough.

Finally, by the expression (26), boundedness of $\widetilde{q}$ follows from Eqs. (48) and (49) and the boundedness of σ_j's. The theorem is proved. □

Remark 2. By considering $\sigma^{-1} = \xi/\eta$ instead of $\sigma = \eta/\xi$, it is easy to see that the above theorem is also true if all $|c_j|(j = 1, \ldots, n)$ are large enough.

Remark 3. For the usual derivative nonlinear Schrödinger Eq. (1) or the nonlocal nonlinear Schrödinger equation, interesting global solutions can be derived from bounded exponential seed solutions. However, there is no such kind of seed solution for Eq. (2). It is worth finding other interesting bounded seed solutions for the nonlocal derivative nonlinear Schrödinger Eq. (2).

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