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Construction of Explicit Solutions of Modified Principal Chiral Field in 1+2 Dimensions via Darboux Transformations

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Dedicated to Professor Su Buchin for his 90th birthday

ABSTRACT. The Darboux matrices for an integrable modified principal chiral field is constructed, and a proceedure to generate infinite number of explicit solutions is proposed. New solutions are obtained by Darboux transformation with variable spectral parameters.

1. Introduction.

It is known that the equation for principal chiral field in 1+1 dimensions is Lorentz invariant and integrable. The expression of many explicit solutions are known. See eg. [1,2,5].

In 1+2 dimensions, the equation for principal chiral field is in the same form

$$\eta^{\mu\nu}(J_\nu J^{-1})_\mu = 0 \tag{1.1}$$

where $\eta^{\mu\nu}$ is the Lorentzian metric, $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1)$. The subscript μ denotes derivative with respect to μ ($\mu = 0,1,2$). J is the chiral field which is valued in some Lie group. In 1+2 dimensions, (1.1) is still Lorentz invariant, but it is not known if it is integrable or not. In [4,6], a modified chiral model was proposed, i.e.

$$\eta^{\mu\nu}(J_\nu J^{-1})_\mu + v_\alpha \varepsilon^{\alpha\mu\nu}(J_\nu J^{-1})_\mu = 0. \tag{1.2}$$

Here $\varepsilon^{\alpha\mu\nu}$ is the permutative tensor, $v = (v_0, v_1, v_2)$ is a given vector, which is chosen as $(0,1,0)$ in this paper (as explained in [6]). This equation is not Lorentz invariant. However, it is integrable in the sense that it has a Lax pair. Therefore, we can try to obtain explicit solutions of the equation (1.2).

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It was pointed out in [6] that as $v = (0,1,0)$, (1.2) has a conserved energy-momentum vector as the unmodified chiral field (1.1), i.e.

$$p_\mu = (\frac{1}{2}\eta_{\mu 0}\eta^{\alpha\beta} - \delta_\mu^\alpha \delta_0^\beta)\,\mathrm{tr}(J_\alpha J^{-1} J_\beta J^{-1}) \tag{1.3}$$

where δ_μ^α is the Kronecker delta. Specifically, the energy density is

$$p_0 = -\frac{1}{2}\,\mathrm{tr}((J_0 J^{-1})^2 + (J_1 J^{-1})^2 + (J_2 J^{-1})^2). \tag{1.4}$$

2. Lax pair and Darboux matrices.

From [6], we know that (1.2) possesses a Lax pair

$$\begin{aligned} \Psi_y &= \lambda \Psi_x + P\Psi \\ \Psi_y &= \lambda^2 \Psi_x + \lambda P \Psi + Q\Psi \end{aligned} \tag{2.1}$$

where

$$x = \frac{x_0 + x_2}{2}, \quad y = x_1, \quad z = \frac{x_0 - x_2}{2}. \tag{2.2}$$

$$P = J_y J^{-1}, \qquad Q = J_z J^{-1}. \tag{2.3}$$

The integrability condition of (2.1) is

$$\begin{aligned} Q_x &= P_y \\ P_z - Q_y &+ [P, Q] = 0, \end{aligned} \tag{2.4}$$

which is equivalent to (1.2).

The Lax pair (2.1) is useful to obtain explicit solutions of (2.4). In [3,6], the pure soliton solutions and extended wave solutions are constructed using this Lax pair.

There are some special cases of (2.4). For instance, if P, Q are independent of x_2, (2.4) becomes the equation for principal chiral field in 1+1 dimensions:

$$\begin{aligned} Q_x &= P_y \\ P_x - Q_y &+ [P, Q] = 0. \end{aligned} \tag{2.5}$$

If

$$P = \begin{pmatrix} 0 & e^{2ix} u(y,z) \\ -e^{-2ix} u^*(y,z) & 0 \end{pmatrix}, \quad Q = \frac{1}{2i} \begin{pmatrix} |u|^2 & e^{2ix} u_y \\ e^{-2ix} u_y^* & -|u|^2 \end{pmatrix} \tag{2.6}$$

(the superscript * means complex conjugate), then (2.4) is reduced to the non-linear Schrödinger equation

$$2iu_z = u_{yy} + 2|u|^2 u. \tag{2.7}$$

If

$$Q = \frac{1}{2} \begin{pmatrix} 0 & e^{2y} u_z(x,z) \\ -e^{-2y} u_z(x,z) & 0 \end{pmatrix}, \quad P = \frac{1}{2} \begin{pmatrix} 2 - \cos u & e^{2y} \sin u \\ e^{-2y} \sin u & -2 + \cos u \end{pmatrix}, \tag{2.8}$$

(2.4) gives the sine-Gordon equation [3]

$$u_{xz} = \sin u. \tag{2.9}$$

In all the above cases, the Darboux matrices are well-known. Here we shall obtain the Darboux matrices for the 1+2 dimensional problem (2.4), and give some new explicit solutions for this equation.

We want to construct a Darboux matrix $G(x, y, z, \lambda)$ through the solution of the Lax pair, so that for any solution Ψ of (2.1), $\Psi' = G\Psi$ satisfies

$$\begin{aligned} \Psi'_y &= \lambda \Psi'_x + P'\Psi' \\ \Psi'_z &= \lambda^2 \Psi'_x + \lambda P'\Psi' + Q'\Psi'. \end{aligned} \tag{2.10}$$

In the simplest and most useful case when $G(x, y, z, \lambda) = \lambda - S(x, y, z)$, we have, from (2.10)

$$\begin{aligned} P' &= P + S_x \\ Q' &= Q + [P, S] + S_x S, \end{aligned} \tag{2.11}$$

and (P', Q') satisfies

$$\begin{aligned} Q'_x &= P'_y \\ P'_z - Q'_y &+ [P', Q'] = 0. \end{aligned} \tag{2.12}$$

Hence if we can know S, then (P', Q') gives a new solution of equation (2.4), or equivalently (1.2).

3. Construction of Darboux matrices.

In this section, the Lie group is $GL(n, R)$ or $GL(n, C)$.

We first introduce the auxiliary equation

$$\begin{aligned} H_y &= H_x \Lambda + PH \\ H_z &= H_x \Lambda^2 + PH\Lambda + QH \end{aligned} \tag{3.1}$$

where $\Lambda(x, y, z)$ satisfies

$$\begin{aligned} \Lambda_y &= \Lambda_x \Lambda \\ \Lambda_z &= \Lambda_x \Lambda^2. \end{aligned} \tag{3.2}$$

It can be checked that (3.1) is integrable if Λ satisfies (3.2) and (P, Q) satisfies (2.4).

Now we use the solutions of (3.1) to construct Darboux matrices. Let H be a nondegenerate $n \times n$-matrix-valued solution of (3.1). Let $S = H\Lambda H^{-1}$. Then,

$$\begin{aligned} S_y &= S_x S + [P, S] \\ S_z &= S_x S^2 + [P, S]S + [Q, S]. \end{aligned} \tag{3.3}$$

For any solution Ψ of (2.1), let $\Psi' = (\lambda - S)\Psi$, then Ψ' satisfies (2.10) with P', Q' defined in (2.11). Therefore, provided we know a solution of (3.2) and a solution of (3.1), we obtain a Darboux matrix for (2.4).

The solutions given by [3,6] correspond to the case $\Lambda = \text{const}$. Here Λ satisfies (3.2). We call it the variable spectral parameter.

An important advantage of Darboux matrix method in 1+1 dimensions is that, after the construction of the first Darboux matrix, we can obtain infinite number of explicit solutions by successive Darboux transformations in a purely algebraic way. Thus, here we also want to obtain explicitly the solutions of (3.1) with respect to P', Q' for other Λ. This is obtained directly in most 1+1 dimensional problems, since Λ is constant in that case. By diagonization of Λ, the solution of (3.1) with respect to (P', Q') comes from the solution of (2.10). However, in this problem we only know that Λ satisfies (3.2). The solution of (3.1) is not deduced from that of (2.10) automatically. In order to get the second Darboux transformation for Λ', we need the solution K' of

$$\begin{aligned} K'_y &= K'_x \Lambda' + P'K' \\ K'_z &= K'_x \Lambda'^2 + P'K'\Lambda' + Q'K'. \end{aligned} \tag{3.4}$$

To do this, we have the following transformation.

For any solution Λ' of (3.2) and the solution K of

$$\begin{aligned} K_y &= K_x \Lambda' + PK \\ K_z &= K_x \Lambda'^2 + PK\Lambda' + QK, \end{aligned} \tag{3.5}$$

let $K' = K\Lambda' - H\Lambda H^{-1}K$, where H is a solution of (3.1). Then we can check that K' satisfies (3.4) where P', Q' are given by (2.11).

Thus, we can construct a new Darboux transformation $\lambda - K'\Lambda'K'^{-1}$ for P', Q'. New solutions P'', Q'' are obtained in this way.

Continuing this process, we can obtain infinite number of solutions of (2.4) in the following way.

For given (P, Q), suppose we know n solutions Λ_i $(i = 1, 2, \cdots, n)$ of (3.2), and nondegenerate solutions H_i of (3.1) with respect to Λ_i. Also, suppose that they satisfy

$$\det(H_{\sigma(j)} \Lambda_{\sigma(j)}^{k-1})_{1 \le j \le r, 1 \le k \le r} \neq 0 \tag{3.6}$$

for all $r \le n$ and all permutations σ of $(1, \cdots, n)$.

Let

$$\begin{aligned} &S_{11} = H_1 \Lambda_1 H_1^{-1}, \quad H_{j1} = H_j \Lambda_j - S_{11} H_j \quad (j \ge 2), \\ &S_{kk} = H_{k,k-1} \Lambda_k H_{k,k-1}^{-1}, \quad H_{jk} = H_{j,k-1} \Lambda_j - S_{kk} H_{j,k-1} \\ &\quad (j \ge k+1,\ 2 \le k \le n). \end{aligned} \tag{3.7}$$

Then, all the above H_{ij} $(i \ge j)$ are nondegenerate, and

$$(\lambda - S_{nn})(\lambda - S_{n-1,n-1}) \cdots (\lambda - S_{11}) \tag{3.8}$$

is a Darboux matrix for (P, Q), which gives new solutions of (2.4).

4. Solutions in $SU(2)$.

In this section, we will use the above conclusions to give $SU(2)$ solutions explicitly. For $J \in SU(2)$, $P^* = -P$, $Q^* = -Q$. For simplicity, we first discuss the solution derived from Darboux transformation of first degree.

Choose

$$\Lambda = \begin{pmatrix} \tau(x,t,t) & 0 \\ 0 & \tau^*(x,y,t) \end{pmatrix} \tag{4.1}$$

where $\tau = \mu + \sqrt{-1}\,\nu$ satisfies

$$\tau_y = \tau \tau_x, \quad \tau_z = \tau^2 \tau_x, \tag{4.2}$$

and $\tau \neq 0$. The solution of (3.1) can be chosen as

$$H = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \tag{4.3}$$

where $h = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ satisfies

$$\begin{aligned} h_y &= \tau h_x + Ph \\ h_z &= \tau h_y + Qh. \end{aligned} \tag{4.4}$$

Let $\sigma = \beta/\alpha$, then

$$S = H\Lambda H^{-1} = \mu + \sqrt{-1}\,\nu T \tag{4.5}$$

where

$$T = \frac{1}{1 + |\sigma|^2} \begin{pmatrix} 1 - |\sigma|^2 & 2\sigma^* \\ 2\sigma & -1 + |\sigma|^2 \end{pmatrix}. \tag{4.6}$$

Using (2.11), we have new solutions (P', Q') of (2.4). Moreover, let

$$\widetilde{J} = (\mu^2 + \nu^2)^{-1/2}(\mu + \sqrt{-1}\,\nu T)J \in SU(2), \tag{4.7}$$

then

$$\widetilde{P} = \widetilde{J}_y \widetilde{J}^{-1} = P' - \mu_x = P + \sqrt{-1}\,(\nu T)_x$$
$$\widetilde{Q} = \widetilde{J}_z \widetilde{J}^{-1} = Q' - \mu\mu_x + \nu\nu_x = Q - \nu^2 T_x T + \sqrt{-1}\,(\mu\nu T)_x + \sqrt{-1}\,\nu[P,T] \tag{4.8}$$

give $SU(2)$ solution of (2.4).

A special solution of (4.2) is

$$\tau = \frac{1}{2(z-c-\sqrt{-1}\,w)}\left(-y+b+\sqrt{\frac{M+U}{2}}+\sqrt{-1}\sqrt{\frac{M-U}{2}}\right) \tag{4.9}$$
$$(x_0 = x + z > 0)$$

where

$$U = (y-b)^2 - 4(x-a)(z-c) + 4w^2$$
$$V = 4w(x+z-a-c) \tag{4.10}$$
$$M = \sqrt{U^2+V^2}$$

and a, b, c, w are real constants, with $a+c<0$, $w>0$. The τ defined by (4.9) satisfies

$$(z-c)\tau^2 + (y-b)\tau + x - a = \sqrt{-1}\,w(\tau^2+1). \tag{4.11}$$

It is C^∞ and nonzero everywhere for $x_0 = x+z \geq 0$.

If the original solution $J = I$, then σ can be chosen as a holomorphic function of τ. Using the fact

$$T^2 = I, \quad TT_\alpha + T_\alpha T = 0, \quad T_\alpha^2 = \frac{4|\sigma_\alpha|^2}{(1+|\sigma|^2)^2} I \quad (\alpha = x,y,z),$$

we have, by direct calculation,

$$-\frac{1}{2}\operatorname{tr}(\widetilde{J}_\alpha \widetilde{J}^{-1})^2 = \frac{(\mu\nu_\alpha - \nu\mu_\alpha)^2}{(\mu^2+\nu^2)^2} + \frac{4\nu^2}{\mu^2+\nu^2}\frac{|\sigma_\alpha|^2}{(1+|\sigma|^2)^2}.$$

Hence the energy density of (4.7) is

$$\begin{aligned} p_0 &= -\frac{1}{2}\operatorname{tr}((\widetilde{J}_0\widetilde{J}^{-1})^2 + (\widetilde{J}_1\widetilde{J}^{-1})^2 + (\widetilde{J}_2\widetilde{J}^{-1})^2) \\ &= -\frac{1}{4}\operatorname{tr}((\widetilde{J}_x\widetilde{J}^{-1})^2 + (\widetilde{J}_z\widetilde{J}^{-1})^2 + 2(\widetilde{J}_y\widetilde{J}^{-1})^2) \\ &= \frac{1}{2}\{(\operatorname{Im}(\tau^{-1}\tau_x))^2 + 2(\operatorname{Im}\tau_x)^2 + (\operatorname{Im}(\tau\tau_x))^2\} \\ &\quad + 2|\tau|^{-2}(1+|\sigma|^2)^{-2}(\operatorname{Im}\tau)^2(1+|\tau|^2)^2|\sigma_x|^2. \end{aligned} \tag{4.12}$$

Example. Let $\sigma(\tau) = \tau$. We consider the asymtotic behavior of p_0 when (x_1, x_2) tends to ∞ along a straight line.

(1) Along $x_2 - kx_1 = \text{const}$ $(k \neq 0)$

$$\tau = -\frac{\sqrt{1+k^2}-1}{k} + O(\frac{1}{|x_1|})$$
$$\tau_x = -\frac{1}{\sqrt{1+k^2}}\frac{1}{x_1} + O(\frac{1}{|x_1|^2}).$$

(2) Along $x_2 = \text{const}$

$$\tau = \frac{4w^2 - (x_0+x_2-2a)(x_0-x_2-2c)}{2(x_0-x_2-2c-2\sqrt{-1}\,w)}\frac{1}{x_1} + O(\frac{1}{|x_1|^2})$$
$$\tau_x = -\frac{1}{x_1} + O(\frac{1}{|x_1|^2}).$$

(3) Along $x_1 = \text{const}$

$$\tau = -1 + O(\frac{1}{|x_2|})$$
$$\tau_x = -\frac{1}{x_2} + O(\frac{1}{|x_2|^2}).$$

Therefore, along the line not parallel to x_1-axis, $p_0 \to 0$ as (x_1, x_2) tends to infinity, while along the line parallel to x_1-axis, $p_0 \to \text{const}$ as (x_1, x_2) tends to infinity. The asymtotic behavior is similar to the solutions given by [3].

The multi-soliton like solutions with variable spectral parameters can be obtained in the following way. The method is similar to that used in [6], in which the n-soliton solution with constant spectral parameters was obtained.

Given n solutions $\tau_1, \cdots, \tau_n$ with $\tau_i \neq 0$ and $\tau_i \neq \tau_j$ $(i \neq j)$. (This is true if, for example, we choose τ_i as in (4.8), with same parameters a, c, w, but different b.) Let h_i be a solution of (4.4). Let

$$\Psi' = \Psi + \sum_{j,k=1}^{n} \frac{1}{\lambda - \tau_k^*}(\Gamma^{-1})_{jk} h_j h_k^* \Psi \tag{4.13}$$

with

$$\Gamma_{jk} = \frac{1}{\tau_j^* - \tau_k} h_j^* h_k. \tag{4.14}$$

Then we can check that Ψ' is a solution of (2.10) with

$$P' = P - \sum_{j,k=1}^{n} ((\Gamma^{-1})_{jk} h_j h_k^*)_x$$
$$Q' = Q - \sum_{j,k=1}^{n} ((\Gamma^{-1})_{jk} h_j h_k^*)_y. \tag{4.15}$$

Let

$$\widetilde{J} = \prod_{k=1}^{n} |\tau_k|^{-1} \tau_k \Psi'(0), \tag{4.16}$$

then $\widetilde{J}$ is a solution of (1.2), and the corresponding $(\widetilde{P}, \widetilde{Q})$ is

$$\begin{aligned} \widetilde{P} &= P' + \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\tau_{k,x}}{\tau_k} - \frac{\tau^*_{k,x}}{\tau^*_k}\right) \\ \widetilde{Q} &= Q' + \frac{1}{2}\sum_{k=1}^{n}(\tau_{k,x} - \tau^*_{k,x}) \end{aligned} \tag{4.17}$$

which satisfy (2.4).

In the pointview of Darboux transformation, this is the solution given by n Darboux transformations of first degree from zero solution.

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