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Explicit solutions of the N-wave equation in $1+2$ dimensions ☆

Zixiang Zhou
Institute of Mathematics, Fudan University, Shanghai 200433, China

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A linear system in $1+1$ dimensions is found to be related to the Lax pair of the N-wave equation in $1+2$ dimensions. This relation gives a method to obtain explicit solutions of the N-wave equation in $1+2$ dimensions by the Darboux transformation.

The N-wave equation $1+2$ dimensions is [1]

$$u_{ij,t}=c_{ij}u_{ij,y}+\frac{a_ib_j-a_jb_i}{a_i-a_j}u_{ij,x}+\sum_{k=1}^{n}(c_{ik}-c_{kj})u_{ik}u_{kj}\qquad(i,j=1,...,n)\,, \tag{1}$$

where

$$u_{ji}=-u_{ij}^{*}\qquad(i\neq j)\,,\qquad u_{ii}=0\,, \tag{2}$$

$$c_{ij}=\frac{b_i-b_j}{a_i-a_j}\,, \tag{3}$$

and a_i, b_i are real constants with $a_i\neq a_j$, $b_i\neq b_j$ for $i\neq j$.

The simplest example is the three-wave equation, which can be written as

$$w_{1,t}=\alpha_1w_{1,y}+\beta_1w_{1,x}+(\alpha_3-\alpha_2)w_2w_3^{*}\,,\qquad w_{2,t}=\alpha_2w_{2,y}+\beta_2w_{2,x}+(\alpha_1-\alpha_3)w_1w_3\,,$$

$$w_{3,t}=\alpha_3w_{3,y}+\beta_3w_{3,x}+(\alpha_2-\alpha_1)w_1^{*}w_2\,,$$

where $u_{12}=w_1$, $u_{13}=w_2$, $u_{23}=w_3$ comparing with (1) and

$$\alpha_1=c_{12}\,,\qquad\alpha_2=c_{13}\,,\qquad\alpha_3=c_{23}\,,\qquad\beta_1=b_1-a_1c_{12}\,,\qquad\beta_2=b_1-a_1c_{13}\,,\qquad\beta_3=b_2-a_2c_{23}\,.$$

It is well-known that eq. (1) has a Lax pair

$$\Psi_y=A\Psi_x+U\Psi\,,\qquad\Psi_t=B\Psi_x+V\Psi\,, \tag{4}$$

where $A=\mathrm{diag}(a_1,...,a_n)$, $B=\mathrm{diag}(b_1,...,b_n)$ are $n\times n$ real diagonal matrices whose diagonal entries are mutually different respectively, $U=(u_{ij})$, and U, $V\in\mathrm{su}(n)$ (i.e., U, V are anti-Hermitian).

The integrability condition of (4) is

$$[A,V]=[B,U]\,,\qquad U_t-V_y+AV_x-BU_x+[U,V]=0\,. \tag{5}$$

This system of equations is the same as (1).

If we suppose $U\in\mathrm{gl}(n,\mathbb{C})$, i.e. there are no relations among all the entries of U, then using the method in ref. [2], we can construct the Darboux transformation for (4) and get an infinite number of explicit solutions

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of (5) (or equivalently (1)) by successive Darboux transformations. However, if we want U, $V\in \mathrm{su}(n)$, it is not so easy to get the Darboux transformation using that method. In other approaches, this equation has been studied by the inverse scattering method [3], the Bäcklund transformation method [4] etc. Here, we hope to get a procedure to obtain new solutions explicitly from a known simple solution.

In ref. [5], the connection between the KP equation and the second and third equations in the AKNS system is found. This gives a method to obtain the solutions of the KP equation from the solutions of the equations in the AKNS system. Hence it reduces the problem of solving a nonlinear equation in 1+2 dimensions to the problem of solving some nonlinear equations in 1+1 dimensions. In this paper, we have show that this method can also be applied to the N-wave equation in 1+2 dimensions. We construct a 1+1 dimensional system (6) which has a close relation with (4). Using the idea of nonlinearization (see, e.g., ref. [6]), we add some nonlinear restrictions on U and Ψ in (4) so that U is expressed by a nonlinear function of Ψ. Then (4) becomes a nonlinear equation for Ψ and (6) can be its Lax pair. The explicit solutions of (5) can be obtained by the Darboux transformation of (6).

We introduce the linear system

$$\Phi_x=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}\lambda\Phi+\begin{pmatrix} 0 & p\\ -p^* & 0\end{pmatrix}\Phi, \qquad \Phi_y=\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}\lambda\Phi+\begin{pmatrix} U & p^{(1)}\\ q^{(1)*} & 0\end{pmatrix}\Phi,$$

$$\Phi_t=\begin{pmatrix} B & 0\\ 0 & 0\end{pmatrix}\lambda\Phi+\begin{pmatrix} V & p^{(2)}\\ q^{(2)*} & 0\end{pmatrix}\Phi, \tag{6}$$

where $p=(p_1, ..., p_n)^{\mathrm{T}}$ is a vector function of x, y and t, $p^{(i)}=(p_1^{(i)}, ..., p_n^{(i)})^{\mathrm{T}}$, $q^{(i)}=(q_1^{(i)}, ..., q_n^{(i)})^{\mathrm{T}}$ $(i=1, 2)$, $U=(u_{jk})_{1\leqslant i,j\leqslant n}$ and $V=(v_{jk})_{1\leqslant i,j\leqslant n}$ will be determined by the integrability condition of (6).

From the integrability condition $\Phi_{xy}=\Phi_{yx}$ and $\Phi_{xt}=\Phi_{tx}$, we have

$$p^{(1)}=Ap, \qquad p^{(2)}=Bp, \qquad q^{(1)}=-Ap, \qquad q^{(2)}=-Bp, \tag{7}$$

$$U_x=[A, pp^*], \qquad V_x=[B, pp^*] \tag{8}$$

and

$$p_y=Ap_x+Up, \qquad p_t=Bp_x+Vp. \tag{9}$$

The equality $\Phi_{yt}=\Phi_{ty}$ gives

$$U_t-V_y+[U, V]-App^*B+Bpp^*A=0. \tag{10}$$

Hence (6) is completely integrable if and only if (7)–(10) hold.

(10) is indeed the integrability condition of (9) if we regard U, V as independent of p. Hence (6) is completely integrable if and only if (7)–(9) hold. In this case, U, V satisfy (5).

Therefore, we have

Theorem. The system

$$\Phi_x=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}\lambda\Phi+\begin{pmatrix} 0 & p\\ -p^* & 0\end{pmatrix}\Phi, \qquad \Phi_y=\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}\lambda\Phi+\begin{pmatrix} U & Ap\\ -p^*A & 0\end{pmatrix}\Phi,$$

$$\Phi_t=\begin{pmatrix} B & 0\\ 0 & 0\end{pmatrix}\lambda\Phi+\begin{pmatrix} V & Bp\\ -p^*B & 0\end{pmatrix}\Phi \tag{11}$$

is completely integrable if and only if (p, U, V) satisfies

$$p_y=Ap_x+Up, \qquad p_t=Bp_x+Vp, \tag{12}$$

$$U_x=[A, pp^*], \qquad V_x=[B, pp^*]. \tag{13}$$

In this case, $U=(u_{ij})$ gives a solution of (1).

Remark. If we add a certain boundary condition on p, eqs. (12), (13) will be simplified. For instance, suppose $p\to 0$ sufficiently fast together with its derivatives as $x\to -\infty$, then we can define

$$\sigma(x,y,t)=\int_{-\infty}^{x}(pp^*)(\zeta,y,t)\,\mathrm{d}\zeta \tag{14}$$

and

$$U=[A,\sigma]\,,\qquad V=[B,\sigma]\,. \tag{15}$$

(12), (13) become

$$p_y=Ap_x+[A,\sigma]p\,,\qquad p_t=Bp_x+[B,\sigma]p\,. \tag{16}$$

It can checked directly that (16) is completely integrable. The theorem implies that (11) is completely integrable if and only if p satisfies (16).

Now we discuss how to get the explicit solutions of (12), (13). Here we use the method of the Darboux matrix [7].

Suppose (p, U, V) satisfies (12), (13), then (11) is completely integrable. For given $\lambda_0\in\mathbb{C}$, let

$$\begin{aligned}\lambda_i&=-\lambda_0^*\,,\qquad 1\leqslant i\leqslant l\,,\\ &=\lambda_0\,,\qquad l+1\leqslant i\leqslant n+1\,,\end{aligned} \tag{17}$$

and $\Lambda=\mathrm{diag}(\lambda_1, ..., \lambda_{n+1})$. Choose solutions h_i of (11) such that $h_i^*h_j=0$ if $\lambda_j=-\lambda_i^*$ (this is possible since $(h_i^*h_j)_x=(h_i^*h_j)_y=(h_i^*h_j)_t=0$ in this case) and $h_1, ..., h_l$ are linearly independent, $h_{l+1}, ..., h_{n+1}$ are also linearly independent. Then $H=(h_1, ..., h_{n+1})$ is nondegenerate everywhere. Denote $T=H\Lambda H^{-1}$, then we can check that $T^*T=|\lambda_0|^2I$ and $T^*-T=(\lambda_0^*-\lambda_0)I$. Moreover, by direct calculation,

$$S=\lambda-T=\lambda-H\Lambda H^{-1}$$

is a Darboux matrix for (11), i.e., for any solution Φ of (11), $\tilde\Phi=S\Phi$ satisfies

$$\tilde\Phi_x=\begin{pmatrix}I&0\\0&0\end{pmatrix}\lambda\tilde\Phi+\begin{pmatrix}0&\tilde p\\-\tilde p^*&0\end{pmatrix}\tilde\Phi\,,\qquad \tilde\Phi_y=\begin{pmatrix}A&0\\0&0\end{pmatrix}\lambda\tilde\Phi+\begin{pmatrix}\tilde U&A\tilde p\\-\tilde p^*A&0\end{pmatrix}\tilde\Phi\,,$$

$$\tilde\Phi_t=\begin{pmatrix}B&0\\0&0\end{pmatrix}\lambda\tilde\Phi+\begin{pmatrix}\tilde V&B\tilde p\\-\tilde p^*B&0\end{pmatrix}\tilde\Phi\,, \tag{18}$$

where $\tilde p=(\tilde p_i)$, $\tilde U=(\tilde U_{ij})$, $\tilde V=(\tilde V_{ij})$ are given by

$$\tilde p_i=p_i+T_{i,n+1}\,,\qquad \tilde u_{ij}=u_{ij}+(a_i-a_j)T_{ij}\,,\qquad \tilde v_{ij}=v_{ij}+(b_i-b_j)T_{ij}\qquad (1\leqslant i,j\leqslant n)\,. \tag{19}$$

Therefore, we obtain new solutions $\tilde u_{ij}$ of (1).

Now we will give special solutions for the N-wave equation. Choose $\lambda_1=...=\lambda_n=-\lambda_0^*$, $\lambda_{n+1}=\lambda_0$,

$$h_{n+1}=\begin{pmatrix}\xi_1\\ \vdots\\ \xi_{n+1}\end{pmatrix}, \tag{20}$$

then

$$T_{ij}=-\lambda_0^*\delta_{ij}+\frac{(\lambda_0+\lambda_0^*)\xi_i\xi_j^*}{\sum_{k=1}^{n+1}|\xi_k|^2}\qquad (i,j=1,2,...,n+1) \tag{21}$$

is independent of the choice of h_i ($i=1,2,...,n$) if we use the previous construction. p is transformed to $\tilde{p}=(\tilde{p}_i)$ where

$$\tilde{p}_i=p_i+\frac{(\lambda_0+\lambda_0^*)\xi_i\xi_{n+1}^*}{\sum_{k=1}^{n+1}|\xi_k|^2} \tag{22}$$

and the solution of (1) is

$$\tilde{u}_{ij}=u_{ij}+\frac{(a_i-a_j)(\lambda_0+\lambda_0^*)\xi_i\xi_j^*}{\sum_{k=1}^{n+1}|\xi_k|^2}\,. \tag{23}$$

To obtain the "one-soliton" solution, let us take $p=0$, $U=V=0$ and $\lambda_0\in\mathbb{C}$ with $\mathrm{Re}\,\lambda_0\neq0$. Then

$$\xi_i=c_i\exp[\lambda_0(x+a_iy+b_it)]\qquad(1\leqslant i\leqslant n)\,,\qquad \xi_{n+1}=c_{n+1}\,, \tag{24}$$

where c_1, ..., c_{n+1} are constants, $c_{n+1}\neq0$.

(22), (23) give

$$\tilde{p}_i=\frac{(\lambda_0+\lambda_0^*)c_ic_{n+1}^*\exp[\lambda_0(x+a_iy+b_it)]}{\sum_{k=1}^{n}|c_k|^2\exp[(\lambda_0+\lambda_0^*)(x+a_ky+b_kt)]+|c_{n+1}|^2} \tag{25}$$

$$\tilde{u}_{ij}=\frac{(a_i-a_j)(\lambda_0+\lambda_0^*)c_ic_j^*\exp[(\lambda_0+\lambda_0^*)x+(\lambda_0a_i+\lambda_0^*a_j)y+(\lambda_0b_i+\lambda_0^*b_j)t]}{\sum_{k=1}^{n}|c_k|^2\exp[(\lambda_0+\lambda_0^*)(x+a_ky+b_kt)]+|c_{n+1}|^2}\,. \tag{26}$$

$\{\tilde{u}_{ij}\}$ is a solution of (1). (u_{ij}) decays exponentially in all the directions except in the direction given by $y=$const. In that direction, (u_{ij}) tends to a constant matrix at infinity.

"Multi-soliton" solutions can be obtained as follows. Take $p=0$, $U=V=0$. Choose λ_1, ..., $\lambda_s\in\mathbb{C}$ such that $\mathrm{Re}\,\lambda_i\neq0$ and λ_i, $-\lambda_i^*$ ($i=1,2,...,s$) are mutually different. Let h_i be the solution of (11) with $\lambda=\lambda_i$,

$$(h_j)_\alpha=A_{j\alpha}\exp[\lambda_j(x+a_\alpha y+b_\alpha t)]\qquad(\alpha=1,2,...,n)\,,\qquad (h_j)_{n+1}=A_{j,n+1}\,, \tag{27}$$

where $A_{j\alpha}$ are constants. Let

$$\Gamma_{jk}=\frac{h_j^*h_k}{\lambda_j^*+\lambda_k}\,, \tag{28}$$

then by ref. [7] or by direct calculation,

$$S(\lambda)=\sum_{j=1}^{s}(\lambda+\lambda_j^*)\left(1-\sum_{j,k=1}^{s}\frac{1}{\lambda+\lambda_k^*}(\Gamma^{-1})_{jk}h_jh_k^*\right)=\lambda^s-S_1\lambda^{s-1}+S_2\lambda^{s-2}-...+(-1)^sS_s \tag{29}$$

is a Darboux matrix for (11) with $p=0$, $U=V=0$, i.e. for any solution Φ of (11) with $p=0$, $U=V=0$, $S\Phi$ satisfies (18) where $\tilde{p}_i=(S_1)_{i,n+1}$ and $\tilde{u}_{ij}=(a_i-a_j)(S_1)_{ij}$, $\tilde{v}_{ij}=(b_i-b_j)(S_1)_{ij}$.

From (29),

$$S_1=-\sum_{j=1}^{s}\lambda_j^*+\sum_{j,k=1}^{s}(\Gamma^{-1})_{jk}h_jh_k^*\,,$$

hence

$$\tilde{u}_{\alpha\beta}=(a_\alpha-a_\beta)\left(\sum_{j,k=1}^{s}(\Gamma^{-1})_{jk}h_jh_k^*\right)_{\alpha\beta}$$

gives a solution of (1) where $\{h_j\}$ is given in (27).

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