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General Form of Nondegenerate Darboux Matrices of First Order for 1+1 Dimensional Unreduced Lax Pairs*

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Abstract

In this paper, we determine all the nondegenerate Darboux matrices of first order for quite general $1+1$ dimensional unreduced Lax pairs, in which the potentials can be arbitrary rational functions of the spectral parameter. An explicit way to get Darboux matrix in terms of its initial value and the fundamental solution of the Lax pair is provided. Also, the permutability property of the Darboux matrices is obtained by twice Darboux transformations.

1 Introduction

Darboux matrix method is an effective method to get explicit solutions of integrable nonlinear partial differential equations (NPDEs). The main task is to construct the Darboux matrices (DMs). For a 1+1 dimensional NPDE possessing a Lax pair, if one solution of the NPDE is known, and if we can construct a DM from the solutions of the Lax pair, the problem to get special solutions of the nonlinear equation can be reduced to a linear problem to solve the Lax pair. Furthermore, we can get a series of solutions of the original NPDE by a purely algebraic algorithm so long as we know a solution of the Lax pair.

The Darboux transformations (DTs) for some specific equations have been studied since 1975 (see, for example, [6,11,13]). For the 2×2 ZS-AKNS system, DMs of first order as well as higher order in a quite

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general form were also known in 1984 [7,10], and they transform a solution of an equation in the system to a solution of the same equation [2,4,5,8] . As for the ZS-AKNS system of arbitrary order, the DMs in a quite general form were constructed by [3,12].

The ZS-AKNS system is

$$\begin{cases} \Phi_x = (\lambda J + P)\Phi, \\ \Phi_t = \sum_{j=0}^{n} V_{n-j}\lambda^j \Phi. \end{cases} \tag{1}$$

Here J is a constant $N \times N$ diagonal matrix with mutually different diagonal entries, P is a valued in $N \times N$ off-diagonal matrices, V_j's are actually differential polynomials of P. According to [3], the following conclusion is true.

Given complex numbers $\lambda_1, \cdots, \lambda_N$, let h_i be a column solution of (1) with $\lambda = \lambda_i$, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, $H = (h_1, \cdots, h_N)$. If $\det H \neq 0$, then $G = \lambda - H\Lambda H^{-1}$ is a DM of (1).

This kind of DMs have interesting physical meanings. They are closely related to the increasing and decreasing of solitons [1,9,12].

[3] also proved that any DM in this form transforms a solution of any equation generated by (1) to a solution of the same equation. On the other hand, [12] showed that these are all the possible DMs of form $\lambda - S(x,t)$ if P is integrable with respect to x in the whole real line and G is bounded for fixed λ, t.

The natural questions are, what are the DMs for other integrable systems, and what happens for the solution which does not decay at infinity? This paper is devoted to solving these problems, and constructing all the nondegenerate DMs of first order for quite general unreduced Lax pairs.

2 Some definitions and main theorems

In this paper, m_N denoted the set of all $N \times N$ complex matrices. Ω is a simple connected domain in $\Re^2$, $\nu_1, \cdots, \nu_l$ are complex numbers. Every function is assumed to be infinitely differentiable.

An equation (or a system of equations)

$$F(x, t, u, u_x, u_t, u_{xx}, \cdots) = 0 \tag{2}$$

of unknowns $u = (u_1, \cdots, u_s)$ (defined in Ω) is integrable if there exist two m_N-valued functions $U[u,x,t,\lambda]$, $V[u,x,t,\lambda]$ which are differential polynomials of u, such that (2) is equivalent to

$$U_t - V_x + [U, V] = 0. \tag{3}$$

Here λ is a parameter.

(3) is the integrability condition of the linear equations

$$\begin{cases} \Phi_x = U[u,x,t,\lambda]\Phi, \\ \Phi_t = V[u,x,t,\lambda]\Phi. \end{cases} \tag{4}$$

This is called a Lax pair of (2).

For the equation (2) with Lax pair (4), a matrix $G(x,t,\lambda)$ is called a Darboux matrix if there exists $\tilde{u}$ such that for any fundamental solution Φ of (4), $\tilde{\Phi} = G\Phi$ satisfies

$$\begin{cases} \tilde{\Phi}_x = \tilde{U}\tilde{\Phi}, \\ \tilde{\Phi}_t = \tilde{V}\tilde{\Phi}, \end{cases} \tag{5}$$

Where $\tilde{U} = U[\tilde{u}, x, t, \lambda]$, $\tilde{V} = V[\tilde{u}, x, t, \lambda]$.

After the transformation,

$$\begin{cases} \tilde{U} = GUG^{-1} + G_x G^{-1}, \\ \tilde{V} = GVG^{-1} + G_t G^{-1} \end{cases} \tag{6}$$

and

$$\tilde{U}_t - \tilde{V}_x + [\tilde{U}, \tilde{V}] = 0. \tag{7}$$

Thus, $\tilde{u}$ is another solution of (2).

In this paper, we consider the Lax pair (4). Here U, V are rational functions of the spectral parameter. They are written as

$$\begin{cases} U(x,t,\lambda) = \sum_{j=0}^{m} U_j(x,t)\lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)}(x,t)(\lambda - \nu_k)^{-j}, \\ V(x,t,\lambda) = \sum_{j=0}^{n} V_j(x,t)\lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{n_k} V_j^{(k)}(x,t)(\lambda - \nu_k)^{-j}. \end{cases} \tag{8}$$

Most of integrable equations have Lax pairs in this form. Moreover, we do not consider reduction, i.e. the entries of $U_j, V_j, U_j^{(k)}, V_j^{(k)}$ are assumed to be independent unknowns.

In this case, the general definition of DM is equivalent to that there exist

$$\begin{cases} \tilde{U}(x,t,\lambda) = \sum_{j=0}^{m} \tilde{U}_j(x,t)\lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} \tilde{U}_j^{(k)}(x,t)(\lambda-\nu_k)^{-j}, \\ \tilde{V}(x,t,\lambda) = \sum_{j=0}^{n} \tilde{V}_j(x,t)\lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{n_k} \tilde{V}_j^{(k)}(x,t)(\lambda-\nu_k)^{-j}. \end{cases}$$

such that $\tilde{\Phi} = G\Phi$ satisfies

$$\begin{cases} \tilde{\Phi}_x = \tilde{U}\tilde{\Phi}, \\ \tilde{\Phi}_t = \tilde{V}\tilde{\Phi} \end{cases}$$

for any fundamental solution Φ of (4).

A nondegenerate DM of first order is the DM of form $\lambda R(x,t) - T(x,t)$ such that R is invertible and $\nu_1,\cdots,\nu_l$ are not the eigenvalues of $S = R^{-1}T$. In this case, $G = R(\lambda - S)$, and we may set $R = I$ without loss of generality since R is a trivial DM if there is no reduction. Hence, we always choose $R = I$ in this paper. Then, a DM $\lambda - S$ is nondegenerate if $\nu_1,\cdots,\nu_l$ are not the eigenvalues of S.

The following theorem provides all the possible nondegenerate DMs of first order.

Theorem 1. *If $\nu_1,\cdots,\nu_l$ are not the eigenvalues of S, then $G(x,t,\lambda) = \lambda - S(x,t)$ is a DM of (4) if and only if $S = K\Gamma K^{-1}$, where Γ is a constant matrix, and K is an m_N-valued nondegenerate solution of the integrable equations*

$$\begin{cases} K_x = \sum_{j=0}^{m} U_j K\Gamma^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)} K(\Gamma-\nu_k)^{-j}, \\ K_t = \sum_{j=0}^{n} V_j K\Gamma^j + \sum_{k=1}^{l}\sum_{j=1}^{n_k} V_j^{(k)} K(\Gamma-\nu_k)^{-j}. \end{cases} \tag{9}$$

It is required to obtain the explicit solutions of (3) and the Lax pair (4). However, it is not easy to get the explicit solution of (9). In order to derive a series of solutions of (3) by successive DTs, we hope to express the DM only through the solutions of the Lax pair. This is possible due to the following theorem.

For any open subset D of Ω, let

$\mathcal{S}(D) = \{H(x,t)\Lambda H(x,t)^{-1} \mid \Lambda = \mathrm{diag}(\lambda_1,\cdots,\lambda_N)$ is a constant diagonal matrix, $H = (h_1,\cdots,h_N)$ is nondegenerate, where h_i is a solution of (4) in D with $\lambda = \lambda_i\}$,

then we have

Theorem 2. *If $\nu_1,\cdots,\nu_l$ are not the eigenvalues of $S(x,t)$, then $G(x,t,\lambda) = \lambda - S(x,t)$ is a DM of (4) if and only if (i) $S \in \mathcal{S}(\Omega)$, or (ii) there exist open sets $\Omega_k \subset \Omega$ $(k = 1,2,\cdots)$ satisfying $\Omega_1 \subset \Omega_2 \subset \cdots$, $\bigcup_{k=1}^{\infty}\Omega_k = \Omega$, and $S_k \in \mathcal{S}(\Omega_k)$ such that for any point $(x,t) \in \Omega_k$, the sequences $\{S_i(x,t)\}$, $\{S_{i,x}(x,t)\}$, $\{S_{i,t}(x,t)\}$ converge to $S(x,t)$, $S_x(x,t)$, $S_t(x,t)$ respectively for $i \geq k$.*

3 Proof of the theorems

Before the proof of these two theorems, we first derive an equation which a DM should satisfy.

For $M \in m_N$ with eigenvalues different from $\nu_1,\cdots,\nu_l$, define

$$U(M) = \sum_{j=0}^{m} U_j M^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)}(M-\nu_k)^{-j}, \tag{10}$$

then we have

Lemma 1. *Under the action of the nondegenerate DM $G = \lambda - S$,*

$$GUG^{-1} + G_x G^{-1} = \sum_{j=0}^{m} \tilde{U}_j\lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} \tilde{U}_j^{(k)}(\lambda-\nu_k)^{-j} - \Delta(\lambda-S)^{-1}. \tag{11}$$

Here

$$\begin{cases} \tilde{U}_j = U_j + \sum_{i=1}^{m-j} [U_{j+i}, S]S^{i-1}, \\ \tilde{U}_j^{(k)} = (S-\nu_k)U_j^{(k)}(S-\lambda_k)^{-1} - \sum_{i=1}^{m_k-j} [U_{j+i}^{(k)}, S](S-\nu_k)^{-i-1}, \\ \Delta = S_x + [S, U(S)]. \end{cases} \tag{12}$$

Therefore, G is a DM if and only if

$$\begin{cases} S_x + [S, U(S)] = 0, \\ S_t + [S, V(S)] = 0. \end{cases} \tag{13}$$

Proof.

$$
\begin{aligned}
&GUG^{-1}+G_xG^{-1}\\
&\quad=(\lambda-S)U(\lambda)(\lambda-S)^{-1}-S_x(\lambda-S)^{-1}\\
&\quad=U(\lambda)-[S,(U(\lambda)-U(S))(\lambda-S)^{-1}]-\Delta(\lambda-S)^{-1}. \qquad (14)
\end{aligned}
$$

From the identities

$$
\begin{aligned}
&(\lambda^j-S^j)(\lambda-S)^{-1}=\textstyle\sum_{i=0}^{j-1}S^{j-i-1}\lambda^i,\\
&\{(\lambda-\nu_k)^{-j}-(S-\nu_k)^{-j}\}(\lambda-S)^{-1}\\
&\qquad=-\textstyle\sum_{i=1}^{j}(S-\lambda_k)^{-j+i-1}(\lambda-\nu_k)^{-i},
\end{aligned}
\qquad (15)
$$

we get (12) by direct calculation. □

Proof of Theorem 1. Suppose K is a nondegenerate solution of (9), and let $S=K\Gamma K^{-1}$, then

$$K_xK^{-1}=U(S). \qquad (16)$$

Hence

$$S_x=K_x\Gamma K^{-1}-K\Gamma K^{-1}K_xK^{-1}=[U(S),S].$$

This implies $G=\lambda-S$ is a DM by the lemma.

Conversely, suppose $\lambda-S$ is a DM, then

$$S_x=[U(S),S],\quad S_t=[V(S),S].$$

Since U_t, V_x, UV, VU are all m_N-valued rational functions of λ, we can define $U_t(S)$, $V_x(S)$, $(UV)(S)$, $(VU)(S)$ as (10). Thus, the identity

$$U_t(\lambda)-V_x(\lambda)+[U(\lambda),V(\lambda)]=0$$

implies

$$U_t(S)-V_x(S)+(UV)(S)-(VU)(S)=0. \qquad (17)$$

However,

$$
\begin{aligned}
(U(S))_t &= \left(\sum_{j=0}^{m}U_jS^j\right)_t+\left(\sum_{k=1}^{l}\sum_{j=1}^{m_k}U_j^{(k)}(S-\nu_k)^{-j}\right)_t\\
&= \sum_{j=0}^{m}U_{j,t}S^j+\sum_{k=1}^{l}\sum_{j=1}^{m_k}U_{j,t}^{(k)}(S-\nu_k)^{-j}+\sum_{j=0}^{m}\sum_{i=0}^{j-1}U_jS^iS_tS^{j-i-1}\\
&\quad -\sum_{k=1}^{l}\sum_{j=1}^{m_k}\sum_{i=0}^{j-1}U_j^{(k)}(S-\nu_k)^{-i-1}S_t(S-\nu_k)^{-j+i}\\
&= U_t(S)+\sum_{j=0}^{m}\sum_{i=0}^{j-1}U_jS^i[V(S),S]S^{j-i-1}\\
&\quad -\sum_{k=1}^{l}\sum_{j=1}^{m_k}\sum_{i=0}^{j-1}U_j^{(k)}(S-\nu_k)^{-i-1}[V(S),S](S-\nu_k)^{-j+i}\\
&= U_t(S)+\sum_{j=0}^{m}U_j[V(S),S^j]+\sum_{k=1}^{l}\sum_{j=1}^{m_k}U_j^{(k)}[V(S),(S-\nu_k)^{-j}]\\
&= U_t(S)+\sum_{j=0}^{m}U_jV(S)S^j+\sum_{k=1}^{l}\sum_{j=1}^{m_k}U_j^{(k)}V(S)(S-\nu_k)^{-j}\\
&\quad -\sum_{j=0}^{m}U_jS^jV(S)-\sum_{k=1}^{l}\sum_{j=1}^{m_k}U_j^{(k)}(S-\nu_k)^{-j}V(S)\\
&= U_t(S)+(UV)(S)-U(S)V(S),
\end{aligned}
$$

likewise,

$$(V(S))_x=V_x(S)+(VU)(S)-V(S)U(S).$$

Combining these with (17), we have

$$(U(S))_t-(V(S))_x+[U(S),V(S)]=0. \qquad (18)$$

Choose any point $(x_0,t_0)\in\Omega$, (18) implies that the equation

$$
\begin{cases}
K_x=U(S)K,\\
K_t=V(S)K,\\
K(x_0,t_0)=I
\end{cases}
\qquad (19)
$$

has a unique solution K with $\det K\neq 0$ in Ω.

Let

$$\Gamma=S(x_0,t_0),\quad S'=S-K\Gamma K^{-1},$$

then

$$
\begin{aligned}
S'_x&=[U(S),S]-[U(S),K\Gamma K^{-1}]=[U(S),S'],\\
S'_t&=[V(S),S']
\end{aligned}
$$

and

$$S'(x_0,t_0)=0.$$

Hence

$$S = K\Gamma K^{-1}$$

everywhere. (19) implies (9). This proves the theorem. □

Proof of theorem 2. First suppose $S = H\Lambda H^{-1} \in \mathcal{S}(\Omega)$, where Λ is a constant diagonal matrix, and $H = (h_1, \cdots, h_N)$ with h_i a solution of (4) as $\lambda = \lambda_i$. Then H is a solution of (9) with $\Gamma = \Lambda$. Hence $G = \lambda - S$ is a DM of (4) by theorem 1. On the other hand, if S satisfies (ii) of this theorem, G is also a DM by the limit of (13). This proves the necessity.

Now suppose $\lambda - S$ is a DM. From Theorem 1, $S = K\Gamma K^{-1}$ where Γ is a constant matrix and K satisfies (9).

If Γ is diagonizable, $\Gamma = T\Lambda T^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$. Let $H = KT$. Then $S = H\Lambda H^{-1}$, and

$$\begin{cases} H_x = \sum_{j=1}^{m} U_j H\Lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)} H(\Lambda - \nu_k)^{-j}, \\ H_t = \sum_{j=0}^{n} V_j H\Lambda^j + \sum_{k=1}^{l}\sum_{j=1}^{n_k} V_j^{(k)} H(\Lambda - \nu_k)^{-j}. \end{cases}$$

If we write $H = (h_1, \cdots, h_N)$, then h_i is a solution of (4) with $\lambda = \lambda_i$. Hence $S \in \mathcal{S}(\Omega)$.

Now suppose Γ is not diagonizable. Choose a constant matrix Θ such that $\Gamma^{(\varepsilon)} = \Gamma + \varepsilon\Theta$ is diagonizable with eigenvalues different from ν_k $(k = 1, \cdots, l)$ for sufficiently small ε. Let (x_0, t_0) be any point in Ω and $K^{(\varepsilon)}$ be the solution of

$$\begin{cases} K_x^{(\varepsilon)} = \sum_{j=0}^{m} U_j K^{(\varepsilon)}(\Gamma^{(\varepsilon)})^j + \sum_{k=1}^{l}\sum_{j=1}^{m_k} U_j^{(k)} K^{(\varepsilon)}(\Gamma^{(\varepsilon)} - \nu_k)^{-j}, \\ K_t^{(\varepsilon)} = \sum_{j=0}^{n} V_j K^{(\varepsilon)}(\Gamma^{(\varepsilon)})^j + \sum_{k=1}^{l}\sum_{j=1}^{n_k} V_j^{(k)} K^{(\varepsilon)}(\Gamma^{(\varepsilon)} - \nu_k)^{-j}, \\ K^{(\varepsilon)}(x_0, t_0) = K(x_0, t_0). \end{cases} \tag{20}$$

Let $K_k = K^{(1/k)}$, $\Gamma_k = \Gamma^{(1/k)}$, Ω_k be the interior of

$$\bigcap_{\varepsilon \le 1/k} \{(x,t) \in \Omega \mid \det K^{(\varepsilon)}(x,t) \neq 0\}.$$

Then, at any given point $(x', t') \in \Omega$, $\det K^{(\varepsilon)}(x', t') \neq 0$ for sufficiently small ε since $K^{(\varepsilon)}$ depends continuously on ε in Ω and $\det K(x', t') \neq 0$.

Hence there exists k_0 such that $(x', t') \in \Omega_k$ for $k \ge k_0$. This implies

$$\bigcup_{k=1}^{\infty} \Omega_k = \Omega.$$

By (20), we have

$$K_k(x,t) \to K(x,t), \quad K_{k,x}(x,t) \to K_x(x,t), \quad K_{k,t} \to K_t(x,t)$$

as $k \to \infty$ with $\det K_k(x,t) \neq 0$ in Ω_{k_0} for $k \ge k_0$.

Since Γ_k is diagonizable, let

$$\Gamma_k = T_k \Lambda_k T_k^{-1}$$

with $\Lambda_k = \mathrm{diag}(\lambda_1^{(k)}, \cdots, \lambda_N^{(k)})$ and let

$$H_k = K_k T_k,$$

then $H_k = (h_1^{(k)}, \cdots, h_N^{(k)})$ where $h_j^{(k)}$ is a solution of (4) with $\lambda = \lambda_j^{(k)}$, and

$$\begin{aligned} &S_k = H_k \Lambda_k H_k^{-1} = K_k \Gamma_k K_k^{-1} \to K\Gamma K^{-1} = S, \\ &S_{k,x} = [H_{k,x} H_k^{-1}, S_k] = [K_{k,x} K_k^{-1}, S_k] \to [K_x K^{-1}, S] = S_x, \\ &S_{k,t} \to S_t \end{aligned}$$

at $(x,t) \in \Omega_k$ for $k \ge k_0$. □

Theorems 1 and 2 provide all the nondegenerate DMs of first order if we do not consider the reduction. If there is reduction, DMs will be a part of that. To determine all the nondegenerate DMs of first order which keep certain reduction is usually a much more difficult problem.

We can also show that there are DMs which are not diagonizable, so in Theorem 2, (ii) is not included in (i) in general. Furthermore, The proof of Theorem 2 provides a constructive way to get nondiagonizable DMs only through the solutions of the Lax pair. We repeat it briefly here.

For given constant matrix Γ, we can construct a DM $\lambda - S$ satisfying $S(x_0, t_0) = \Gamma$ in the following way. Choose a constant matrix Θ such that $\Gamma_k = \Gamma + \Theta/k$ $(k = 1, 2, \cdots)$ is diagonizable with eigenvalues different from ν_j $(j = 1, \cdots, l)$ for sufficiently large k. Let $\Gamma_k = T_k \Lambda_k T_k^{-1}$ with $\Lambda_k = \mathrm{diag}(\lambda_1^{(k)}, \cdots, \lambda_N^{(k)})$. Let $H_k = (h_1^{(k)}, \cdots, h_N^{(k)})$ satisfy $H_k(x_0, t_0) = T_k$ where $h_j^{(k)}$ is a solution of (4) with $\lambda = \lambda_j^{(k)}$. Then $S = \lim_{k\to\infty} S_k$ with $S_k = H_k \Lambda_k H_k^{-1}$ gives a DM $\lambda - S$. All the DMs of form $\lambda - S$ can be constructed in this way if there is no reduction.

4 Permutability

Permutability is an important property of Bäcklund transformations. The permutability theorem for diagonizable Darboux matrices for ZS-AKNS system has already been known (cf.[3,5,8,10]). Here we make a slight generalization and show that this is true universally for any two generic nondegenerate DMs of first order of Lax pair (4).

We have

Theorem 3. *Suppose* $G_\alpha(x,t,\lambda) = \lambda - S_\alpha(x,t)$ $(\alpha = 1,2)$ *are two non-degenerate* DMs *for* (U,V), $\det(S_1 - S_2) \neq 0$. *then*

$$G_{\beta\alpha}(x,t,\lambda) = \lambda - (S_\beta - S_\alpha)S_\beta(S_\beta - S_\alpha)^{-1} \tag{21}$$

is a DM *for*

$$U_\alpha = G_\alpha U G_\alpha^{-1} + G_{\alpha,x}G_\alpha^{-1}, \quad V_\alpha = G_\alpha V G_\alpha^{-1} + G_{\alpha,t}G_\alpha^{-1}$$

$(\alpha,\beta = 1,2;\ \alpha \neq \beta)$ *and the permutability*

$$G_{21}(\lambda)G_1(\lambda) = G_{12}(\lambda)G_2(\lambda) \tag{22}$$

holds.

Proof. First suppose $S_\alpha(\alpha = 1,2)$ are diagonizable. Let $S_\alpha = H_\alpha\Lambda_\alpha H_\alpha^{-1}$ where $\Lambda_\alpha = \mathrm{diag}(\lambda_1^{(\alpha)},\cdots,\lambda_N^{(\alpha)})$ with $\lambda_i^{(\alpha)} \neq \nu_j$ $(i = 1,\cdots,N;$ $j = 1,\cdots,l;\ \alpha = 1,2)$, $H_\alpha = (h_1^{(\alpha)},\cdots,h_N^{(\alpha)})$, $h_i^{(\alpha)}$ is a solution of (4) with $\lambda = \lambda_i^{(\alpha)}$. Under the action of G_α, H_β changes to

$$\begin{aligned} H_{\beta\alpha} &= \left(G_\alpha(\lambda_1^{(\beta)})h_1^{(\beta)},\cdots,G_\alpha(\lambda_N^{(\beta)})h_N^{(\beta)}\right) \\ &= H_\beta\Lambda_\beta - H_\alpha\Lambda_\alpha H_\alpha^{-1}H_\beta = (S_\beta - S_\alpha)H_\beta \end{aligned}$$

$(\alpha,\beta = 1,2;\ \alpha \neq \beta)$. Therefore,

$$S_{\beta\alpha} = H_{\beta\alpha}\Lambda_\beta H_{\beta\alpha}^{-1} = (S_\beta - S_\alpha)S_\beta(S_\beta - S_\alpha)^{-1}$$

gives a DM $G_{\beta\alpha}(\lambda) = \lambda - S_{\beta\alpha}$ for U_α, V_α. Moreover,

$$\begin{aligned} &G_{\beta\alpha}(\lambda)G_\alpha(\lambda) \\ &\quad= \{\lambda - (S_\beta - S_\alpha)S_\beta(S_\beta - S_\alpha)^{-1}\}(\lambda - S_\alpha) \\ &\quad= \lambda^2 - (S_\beta^2 - S_\alpha^2)(S_\beta - S_\alpha)^{-1}\lambda \\ &\qquad +(S_\beta - S_\alpha)S_\beta(S_\beta - S_\alpha)^{-1}S_\alpha. \end{aligned} \tag{23}$$

Noticing that

$$S_2(S_2 - S_1)^{-1}S_1 + S_1(S_1 - S_2)^{-1}S_2 = 0, \tag{24}$$

we know (23) is symmetric for S_α, S_β. Hence

$$G_{21}(\lambda)G_1(\lambda) = G_{12}(\lambda)G_2(\lambda). \tag{25}$$

For nondiagonizable $G_\alpha = \lambda - S_\alpha(x,t)$, we have nondegenerate DMs $G_\alpha^{(k)}(\lambda) = \lambda - S_\alpha^{(k)}$ such that $S_\alpha^{(k)}$ is diagonizable, $\det(S_1^{(k)} - S_2^{(k)}) \neq 0$ and $G_\alpha^{(k)} \to G_\alpha$ as $k \to \infty$. From the above discussion,

$$G_{\beta\alpha}^{(k)} = \lambda - (S_\beta^{(k)} - S_\alpha^{(k)})S_\beta^{(k)}(S_\beta^{(k)} - S_\alpha^{(K)})^{-1} \tag{26}$$

is a DM for $U_\alpha^{(k)}, V_\alpha^{(k)}$, which are transformed from U, V by $G_\alpha^{(k)}$, and

$$G_{21}^{(k)}G_1^{(k)} = G_{12}^{(k)}G_2^{(k)}. \tag{27}$$

Taking that limit in (26) and (27), we derive the conclusions in the theorem. □

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