NONLINEAR EVOLUTION EQUATIONS: INTEGRABILITY AND SPECTRAL METHODS

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12. EXPLICIT FORM OF BÄCKLUND TRANSFORMATIONS FOR GL(N), U(N) AND O(2N) PRINCIPAL CHIRAL FIELDS

The explicit form of Darboux matrices of Bäcklund transformations for GL(N), U(N) and O(2N) principal chiral fields is obtained. The algorithm is purely algebraic and can be continued successively, provided that a seed solution and a fundamental solution of the Lax pair is given. It is noted that a Bäcklund transformation does not create or kill solitons in the case of principal chiral fields.

1 Darboux matrix method for GL(N) principal chiral field

Let $g(t,x)$ be a GL(N)-valued function and

$$S[g] = \int tr(g_\xi g^{-1} g_\eta g^{-1}) d\xi d\eta \qquad (1)$$

an action functional. Here ξ, η are light-cone coordinates

$$\xi = x+t, \qquad \eta = x-t \qquad (2)$$

A principal chiral field is a solution of the variational problem

$$\delta S[g] = 0 \qquad (3)$$

and is called a harmonic map mathematically.
The Euler equation is

$$A_\xi + B_\eta = 0 \qquad (4)$$

Here

$$A = g_\eta g^{-1}, \qquad B = g_\xi g^{-1} \qquad (5)$$

Hence we have

$$A_\xi - B_\eta + [A,B] = 0 \tag{6}$$

(4) and (6) are equations of GL(N) principal chiral field.

Equations (4) and (6) admit a Lax pair

$$\phi_\eta = (1+\mu)^{-1}A\phi, \qquad \phi_\xi = (1-\mu)^{-1}B\phi \tag{7}$$

or

$$\phi_\eta = \lambda A\phi, \qquad \phi_\xi = \lambda(2\lambda-1)^{-1}B\phi \tag{8}$$

The zero-curvature condition

$$\lambda A_\xi - \lambda(2\lambda-1)^{-1}B_\eta + \lambda^2(2\lambda-1)^{-1}[A,B] = 0 \tag{9}$$

is exactly (4) and (6) [8].

The Bäcklund transformations for principal chiral fields were considered in [9]. We are going to construct the Darboux matrix explicitly. It should be noted that the Darboux matrix method is very powerful for AKNS system (e.g. [1,2,4,5,6,7,10]) and the Cauchy problem for the principal chiral fields of compact group in $R^{1,1}$ was first solved in [3].

Suppose that a solution (A,B) and a fundamental solution $\phi=\phi(\lambda,\eta,\xi)$ of (8) are given. The Darboux matrix of first degree has the form

$$S = I + \alpha\lambda \tag{10}$$

where α is an N x N matrix. Let $\phi_1 = S\phi$. From

$$\phi_{1\eta} = \lambda A_1\phi_1, \qquad \phi_{1\xi} = \lambda(2\lambda-1)^{-1}B_1\phi_1 \tag{11}$$

we obtain

$$\alpha_\eta\alpha = \alpha A - A\alpha$$

$$\alpha_\xi\alpha + 2\alpha_\xi = B\alpha-\alpha B \tag{12}$$

and

$$A_1 = \alpha A\alpha^{-1} = A + \alpha_\eta$$

$$B_1 = (\alpha+2)B(\alpha+2)^{-1} = B - \alpha_\xi \tag{13}$$

It is seen that (12) is a completely integrable system for α and (13) is an explicit form of the Bäcklund transformation provided α is known.

The general solution α can be constructed in the following way.

Choose N numbers λ_i $(i=1,2,\ldots,N)(\lambda_i\neq 0,1/2)$, and let $h_1(\lambda_1),\ldots\ldots,h_N(\lambda_N)$ be N column solutions of the Lax pair which can be obtained from the fundamental solution $\phi(\lambda)$. Moreover, construct a matrix

$$H = [h_1(\lambda_1),\ldots\ldots,h_N(\lambda_N)]$$

We choose $h_i(\lambda_i)$ such that det $H \neq 0$.

It is seen that

$$\alpha = -H\Lambda^{-1}H^{-1} \tag{14}$$

with

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & & \\ 0 & & \lambda_N \end{pmatrix} \tag{15}$$

is a solution of the system (12). In fact, we have

$$H_\eta = AH\Lambda$$

$$H_\xi = BH\Lambda(2\Lambda-1)^{-1} \tag{16}$$

Hence

$$\alpha_\eta\alpha = H_\eta\Lambda^{-2}H^{-1} - H\Lambda^{-1}H^{-1}H_\eta\Lambda^{-1}H^{-1}$$

$$= \alpha A - A\alpha$$

$$\alpha_\xi\alpha + 2\alpha_\xi = H_\xi\Lambda^{-2}H^{-1} - H\Lambda^{-1}H^{-1}{}_\xi\Lambda^{-1}H^{-1} - 2H_\xi\Lambda^{-1}H^{-1} + 2H\Lambda^{-1}H^{-1}H_\xi H^{-1}$$

$$= B\alpha-\alpha B$$

Thus (12) is satisfied.

It is also noted that all solutions of (12) which are similar to diagonal matrices at a point (hence at all points) are obtained if the diagonal elements are neither zero nor (-2).

Moreover, from (11), we know that the corresponding principal chiral field is

$$g_I = (I + \alpha)g \tag{17}$$

if $\lambda_i \neq 1$ $(i=1,2,\ldots\ldots,N)$.

2 Example

For simplicity, we choose $N = 2$. Take a seed solution (A,B) such that A,B are diagonal,

$$A = \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix}, \quad B = \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \tag{18}$$

Since A and B commute each other, the seed solution consists of linear waves

$$A = \begin{bmatrix} \varphi_1(\eta) & 0 \\ 0 & \varphi_2(\eta) \end{bmatrix}, \quad B = \begin{bmatrix} \psi_1(\xi) & 0 \\ 0 & \psi_2(\xi) \end{bmatrix} \tag{19}$$

where φ_1, φ_2, ψ_1, ψ_2 are arbitrary functions. Moreover, $\varphi_1(\eta)$, $\varphi_2(\eta)$ are waves going to the right and $\psi_1(\xi)$, $\psi_2(\xi)$ are waves going to the left. If these functions contain several impulses, we have "linear multi-solitons".

We assume that $\varphi_a(\eta)$, $\psi_a(\xi)$ $(a=1,2)$ approach zero rapidly as $\eta,\xi\to-\infty$, then a fundamental solution to the Lax pair is

$$\phi(\lambda) = \begin{bmatrix} P_1Q_1 & 0 \\ 0 & P_2Q_2 \end{bmatrix} \tag{20}$$

where

$$P_a = \exp(\lambda\int_{-\infty}^{\eta}\varphi_a(\eta)d\eta)$$
$$Q_a = \exp(\lambda(2\lambda-1)^{-1}\int_{-\infty}^{\xi}\psi_a(\xi)d\xi) \qquad (a=1,2) \tag{21}$$

Let

$$h_1(\lambda_1) = \begin{bmatrix} aP_1(\lambda_1)Q_1(\lambda_1) \\ bP_2(\lambda1)Q_2(\lambda_1) \end{bmatrix} \quad h_2(\lambda_2) = \begin{bmatrix} cP_1(\lambda_2)Q_1(\lambda_2) \\ dP_2(\lambda_2)Q_2(\lambda_2) \end{bmatrix} \tag{22}$$

where a,b,c,d are constants such that

$$\det H = \det [h_1(\lambda_1),\ h_2(\lambda_2)] \neq 0$$

Then

$$\alpha = -(adP_1(\lambda_1)Q_1(\lambda_1)P_2(\lambda_2)Q_2(\lambda_2) - bcP_1(\lambda_2)Q_1(\lambda_2)P_2(\lambda_1)Q_2(\lambda_1))^{-1}.$$
$$\cdot \begin{bmatrix} aP_1(\lambda_1)Q_1(\lambda_1)\lambda_1^{-1} & cP_1(\lambda_2)Q_1(\lambda_2)\lambda_2^{-1} \\ bP_2(\lambda_1)Q_2(\lambda_1)\lambda_1^{-1} & dP_2(\lambda_2)Q_2(\lambda_2)\lambda_2^{-1} \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} dP_2(\lambda_2)Q_2(\lambda_2) & -cp_1(\lambda_2)Q_1(\lambda_2) \\ -bP_2(\lambda_1)Q_2(\lambda_1) & aP_1(\lambda_1)Q_1(\lambda_1) \end{bmatrix}$$

Thus, from (13), we obtain a new solution for which A_1,B_1 do not commute each other. Moreover, if $\varphi_a(\eta)$, $\varphi_a(\xi)$ approach zero rapidly as $\eta\to\infty$ and $\xi\to\infty$, then derivatives of

$$\int_{-\infty}^{\eta}\varphi_a(\eta)d\eta, \quad \int_{-\infty}^{\xi}\psi_a(\xi)d\xi$$

tend to zero as η and $\xi\to\infty$. The nonlinear solution approaches the original one as $t\to\infty$. In this sense, we see that the Bäcklund transformation may not create or kill solutions for the principal chiral fields, while, as is well-known, the Bäcklund transformations do create or kill solitons for KdV, MKdV, NLS equations, etc.

3 U(N) principal chiral field

In this section, we consider U(N) principal chiral field, i.e. g takes value in U(N), while A,B satisfy $A^+ = -A, B^+ = -B$. Here + denotes complex conjugate transpose.

The Darboux matrix S is constructed as in Section 2, with two additional requirements.

(i) λ_i can be either λ_0 or $\bar{\lambda}_0$, where λ_0 is a complex number which is not real.

(ii) At some point (η_0,ξ_0), $h_1,\dots,h_N$ are linearly independent, and $(h_i^+h_j)|_{(\eta_0\xi_0)} = 0$ if $\lambda_i \neq \lambda_j$ (i.e. $\lambda_i = \bar{\lambda}_j$).

We know from (8) that $(h_i^+h_j)_\eta=(h_i^+h_j)_\xi=0$ if $\lambda_i=\bar{\lambda}_j$, and hence $h_i^+h_j=0$ everywhere. Thus, the linear independence of $\{h_i|\lambda_i=\lambda_0\}$ and $\{h_i|\lambda_i=\bar{\lambda}_0\}$ implies the linear independence of all $\{h_i\}$. Therefore, H is nondegenerate everywhere, and $\alpha = -H\Lambda^{-1}H^{-1}$ is globally defined.

From

$$\alpha[h_1,\dots,h_N] = -[\lambda_1^{-1}h_1,\dots,\lambda_N^{-1}h_N] \tag{23}$$

we have

$$h_i^+(\alpha^+ + \alpha)h_j = -(\bar{\lambda}_i^{-1} + \lambda_j^{-1})h_i^+h_j = -(\lambda_0^{-1} + \bar{\lambda}_0^{-1})h_i^+h_j \tag{24}$$

which implies

$$\alpha^+ + \alpha = -(\lambda_0^{-1} + \bar{\lambda}_0^{-1})I \tag{25}$$

since $\{h_i\}$ is a basis of R^N. Hence

$$\begin{aligned}\alpha^+\alpha &= (-(\lambda_0^{-1} + \bar{\lambda}_0^{-1})I-\alpha)\alpha \\ &= H((\lambda_0^{-1} + \bar{\lambda}_0^{-1})I-\Lambda^{-1})\Lambda^{-1}H^{-1} \\ &= |\lambda_0|^{-2}I\end{aligned} \tag{26}$$

and

$$(I + \alpha)^+(I+\alpha) = |1-\lambda_0^{-1}|^2 I \tag{27}$$

Considering the transformation (13), we have

$$A_1^+ = -A_1, \quad B_1^+ = -B_1 \tag{28}$$

since

$$\alpha_\eta^+ = -\alpha_\eta, \quad \alpha_\xi^+ = -\alpha_\xi \tag{29}$$

by (25). Therefore, the Darboux matrix given above keeps the reduction.

The transformation on g is

$$g_1 = (I + \alpha)g\Lambda(\Lambda-1)^{-1} \tag{30}$$

We take a right multiplier $\Lambda(\Lambda-1)^{-1}$ such that g_1 takes value in U(N) by (27).

Moreover, if we consider SU(N) principal chiral field, the construction of the Darboux matrix is the same, because the transformation (13) keeps trace zero.

4 O(2N) principal chiral field

For O(2N) model, A,B are real matrices with $A^T = -A, B^T = -B$. The Darboux matrix is dealt with in a similar way as in Section 3. However, the matrix α should be real.

Given $\lambda_0 = (\mu + \nu i)^{-1}$ $(\nu \neq 0)$, let

$$J = \begin{bmatrix} \mu I & \nu I \\ -\nu I & \mu I \end{bmatrix} \tag{31}$$

where I is N x N identity matrix. Then, choose a nondegenerate real matrix R such that R^TR has the form

$$\begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix}$$

for some N x N matrix W.

Let $\alpha_0 = -RJR^{-1}$, then

$$\begin{aligned}\alpha_0^T\alpha_0 &= R^{T-1}J^TR^TRJR^{-1} \\ &= (\mu^2 + \nu^2)\, R^{T-1}\begin{bmatrix} W & \\ & W\end{bmatrix} R^{-1} = (\mu^2 + \nu^2)I_{2N}\end{aligned} \tag{32}$$

Hence α_0 is a multiple of an orthogonal matrix, which is certainly diagonizable.

We can write $\alpha_0 = -K\Lambda^{-1}K^{-1}$ where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2N} \end{bmatrix}$$

with $\lambda_i = \lambda_0$ or $\bar{\lambda}_0$, and $K = [k_1,\ldots,k_{2N}]$ nondegenerate. Inserting this expression of α_0 into (32), we have

$$(\mu^2 + \nu^2)\Lambda^+K^+K\Lambda = K^+K \tag{33}$$

i.e.

$$(\bar{\lambda}_i\lambda_j - |\lambda_0|^2)k_i^+k_j = 0 \quad (i,j=1,\ldots 2N) \tag{34}$$

which implies $k_i^+k_j=0$ if $\lambda_i \neq \lambda_j$.

Let $h_i(\lambda_i,\eta,\xi)$ be the solution of

$$\begin{cases} h_{i\eta} = \lambda_i A h_i \\ h_{i\xi} = \lambda_i(2\lambda_i-1)^{-1}Bh_i \\ h_i(\eta_0,\xi_0) = k_i \end{cases} \tag{35}$$

and $H = [h_1,\ldots,h_{2N}]$, the $h_i^+h_j = 0$ if $\lambda_i \neq \lambda_j$, and det $H \neq 0$. From

Section 3, we only need to prove that $\alpha = -H\Lambda^{-1}H^{-1}$ takes real values.

Write $\alpha = F + Gi$, where F,G are real-matrix-valued functions. Let $Z = \{(\eta,\xi)\in R^{1,1} | G(\eta,\xi)=0\}$, then $(\eta_0,\xi_0)\in Z$. Near a point $(\eta_1,\xi_1)\in Z$, F, F+2 are nondegenerate since α has no real eigenvalues. Here 2 is the abbreviation of the matrix $2I_{2N}$.

Considering (12), we have

$$\begin{aligned} &F_\eta F - G_\eta G = [F,A] \\ &F_\eta G + G_\eta F = [G,A] \\ &F_\xi(F+2) - G_\xi G = [B,F] \\ &F_\xi G + G_\xi(F+2) = [B,G] \end{aligned} \tag{36}$$

Eliminating F_η, F_ξ, we have

$$\begin{aligned} &G_\eta(F+GF^{-1}G) = GA-FAF^{-1}G \\ &G_\xi(F+2+G(F+2)^{-1}G) = (F+2)B(F+2)^{-1}G-GB \end{aligned} \tag{37}$$

G=0 is clearly a solution of this group of equations. Moreover, since the coefficient matrices $F + GF^{-1}G$, $F + 2 + G(F + 2)^{-1}G$ are nondegenerate near (η_1,ξ_1), the uniqueness of (37) implies G=0 near (η_1,ξ_1), which means Z is open. Hence G = 0 everywhere and α has real value.

The Darboux transformation on A,B is given by (13), and

$$g_1 = (I + \alpha)g|1-1/\lambda_0|^{-1}$$

is an O(2N) principal chiral field.

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