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Determination of Nondegenerate Darboux Operators of First Order in 1+2 Dimensions

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In 1+2 dimensions, some Darboux operators have been constructed before. In this paper, all the nondegenerate Darboux operators of first order are given for quite general Lax pairs without reduction. They take the form which is already known. The Darboux operators or Darboux matrices in 1+1 dimensions are discussed as special cases.

1. Introduction

Darboux transformation method is an effective method to get explicit solutions of some nonlinear partial differential equations. For 1+1 dimensional problems, Darboux matrix has been known quite clearly (eg. [5,10]). In 1+2 dimensions, the spectral parameter in 1+1 dimensions is usually replaced by a derivative with respect to one variable. Thus the fundamental Darboux transformations are given by differential operators (Darboux operators(DOs)) rather than polynomials of the spectral parameter in 1+1 dimensions.

Let $\mathcal{M}_N$ be the set of all NxN complex matrices. Ω is a simply connected domain in $\mathbb{R}^3$ with coordinates x,y,t. Denote $\partial = \partial/\partial x$,

$$\mathcal{D}_N(\Omega) = \left\{ \sum_{j=0}^{r} A_j \partial^j \,\middle|\, A_j \in C^\infty(\Omega, \mathcal{M}_N),\ r \geq 0 \right\}.$$

All the functions are assumed to be infinitely differentiable.

Now we consider an equation (or a system of equations)

$$F(x,y,t,u,u_x,u_y,u_t,u_{xx},\dots) = 0 \tag{1}$$

of unknows $u = (u_1,\dots,u_s)$ in Ω which admits a Lax pair

$$\begin{cases} \Phi_y = U(\partial)\Phi \\ \Phi_t = V(\partial)\Phi . \end{cases} \tag{2}$$

Here

$$\begin{cases} U(\partial) = U(x,y,t,u,u_x,\dots,\partial) = \sum_{j=0}^{m} U_{m-j}(x,y,t,u,u_x,\dots)\partial^j \\ V(\partial) = V(x,y,t,u,u_x,\dots,\partial) = \sum_{j=0}^{n} V_{n-j}(x,y,t,u,u_x,\dots)\partial^j , \end{cases} \tag{3}$$

U_j, $V_j \in C^\infty(\Omega, \mathcal{M}_N)$ if u is given.

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(2) is a Lax pair of (1) implies that (1) is equivalent to

$$U_t(\partial) - V_y(\partial) + [U(\partial), V(\partial)] = 0 \qquad (4)$$

which is the integrability condition of (2). Here we assume (2) is integrable in the sense that (2) is locally solvable for any initial data defined at $x=x_0$.[12]

The Bäcklund transformation in the form of differential equations or integro-differential equations (eg.[3,6,7,8]) as well as the inverse scattering transformation (eg.[1,2,4]) for a lot of equations or systems contained in (2) have already been known. As for Darboux transformation, the DO for the equations possessing scalar Lax pair as KP equation has also been known (eg.[11]). For general unreduced Lax pair (2), [12] showed that any nondegenerate matrix solution H of (2) generated a DO $\partial - H_x H^{-1}$. The DO for Davey-Stewartson equation was obtained in this way.

In this paper, we shall show that these are all the possible DOs in the form $\partial - S(x,y,t)$. Also, by reducing to 1+1 dimensions, we shall give the corresponding conclusions for 1+1 dimensional problems.

2. Darboux operators for 1+2 dimensional Lax pairs

For equation (1) with Lax pair (2), a differential operator $G(x,y,t,\partial) \in \mathcal{D}_N(\Omega)$ is called a DO if there exists $\tilde{u}$ such that for any solution Φ of (2), $\tilde{\Phi} = G(\partial)\Phi$ satisfies

$$\begin{cases} \tilde{\Phi}_y = \tilde{U}(\partial)\tilde{\Phi} \\ \tilde{\Phi}_t = \tilde{V}(\partial)\tilde{\Phi} \end{cases} \qquad (5)$$

where $\tilde{U}(\partial) = U(x,y,t,\tilde{u},\tilde{u}_x,\ldots,\partial)$, $\tilde{V}(\partial) = V(x,y,t,\tilde{u},\tilde{u}_x,\ldots,\partial)$.

Obviously, $\tilde{U}(\partial)$, $\tilde{V}(\partial)$ satisfy

$$\begin{cases} \tilde{U}(\partial)G(\partial) = G(\partial)U(\partial) + G_y(\partial) \\ \tilde{V}(\partial)G(\partial) = G(\partial)V(\partial) + G_t(\partial) \end{cases} \qquad (6)$$

$$\tilde{U}_t(\partial) - \tilde{V}_y(\partial) + [\tilde{U}(\partial), \tilde{V}(\partial)] = 0. \qquad (7)$$

Therefore, we obtain a new solution $\tilde{u}$ of (1) by the action of the DO.

This section is devoted to the equation (4) without reduction, i.e. the entries of U_j, V_j are independent unknowns. Then $G(x,y,t,\partial) \in \mathcal{D}_N(\Omega)$ is a DO if and only if there exist $\tilde{U}(\partial)$, $\tilde{V}(\partial) \in \mathcal{D}_N(\Omega)$ such that (6) holds.

A nondegenerate DO of first order is a DO $G(x,y,t,\partial) = R(x,y,t)\cdot(\partial - S(x,y,t))$ with R nondegenerate. Since a matrix R is a trivial DO if we do not consider reduction, we can always choose R=I.

Nondegenerate DOs of first order can be constructed explicitly as follows.

Theorem 1. $\partial - S(x,y,t)$ is a DO of (2) if and only if $S=H_x H^{-1}$ for some NxN nondegenerate matrix solution H of (2).

Before the proof, we have some preparations.

For any $M \in C^\infty(\Omega, \mathcal{M}_N)$, let M_j be defined inductively by

$$\begin{cases} M_0 = I \\ M_{j+1} = M_{j,x} + M_j M \quad (j \geq 0). \end{cases} \qquad (8)$$

Let

$$U(M) = \sum_{j=0}^{m} U_{m-j} M_j . \qquad (9)$$

Then, for any Ψ satisfying $\Psi_x = M\Psi$, we have

$$U(\partial)\Psi = U(M)\Psi. \qquad (10)$$

Lemma. $\partial - S$ is a DO of (2) if and only if S satisfies

$$\begin{cases} S_y + [S, U(S)] = (U(S))_x \\ S_t + [S, V(S)] = (V(S))_x. \end{cases} \qquad (11)$$

Proof. First suppose $\partial - S$ is a DO of (2). Choose a fundamental solution matrix Ψ of $\Psi_x = S\Psi$, then (6) implies

$$S_y\Psi = (\partial - S)U(S)\Psi = (U(S))_x\Psi - [S, U(S)]\Psi,$$

which leads to (11).

Conversely, suppose S is a solution of (11). Let

$$\tilde{U}(\partial) = \sum_{j=0}^{m} \tilde{U}_{m-j}\partial^j \qquad (12)$$

where $\tilde{U}_j$'s are defined inductively by

$$\begin{cases} \tilde{U}_0 = U_0 \\ \tilde{U}_{j+1} = U_{j+1} + U_{j,x} - SU_j + \sum_{k=0}^{j} C_{m-k}^{m-j}\tilde{U}_k\,\partial^{j-k}S. \end{cases} \qquad (13)$$

Then,

$$D(\partial) \equiv S_y - (\partial - S)U(\partial) + \tilde{U}(\partial)(\partial - S) \in C^\infty(\Omega, \mathcal{M}_N).$$

However, for the fundamental solution matrix Ψ of $\Psi_x = S\Psi$, (11) gives $D(\partial)\Psi = 0$, This means $D(\partial)=0$ as a matrix. QED.

Proof of Theorem 1. Suppose H is an NxN nondegenerate matrix solution of (2), $S = H_x H^{-1}$. Then (2) leads to (11) immediately.

Conversely, suppose $G(\partial) = \partial - S(x,y,t)$ is a DO of (2), we need to find a solution H of (2) such that $S=H_x H^{-1}$, or equivalently, we need to solve

$$\begin{cases} H_x = SH \\ H_y = U(\partial)H \\ H_t = V(\partial)H. \end{cases} \qquad (14)$$

Again, this is equivalent to

$$\begin{cases} H_x = SH \\ H_y = U(S)H \\ H_t = V(S)H \end{cases} \tag{15}$$

by (10). Therefore, we only need to verify the integrability condition of (15).

Let Ψ be a fundamental solution matrix of $\Psi_x = S\Psi$. From (6),

$$(\Psi_y - U(\partial)\Psi)_x = (S\Psi)_y - \partial U(\partial)\Psi = S(\Psi_y - U(\partial)\Psi),$$

thus,

$$(V_y(\partial) + V(\partial)U(\partial))\Psi = (V(\partial)\Psi)_y - V(\partial)(\Psi_y - U(\partial)\Psi)$$
$$= (V(S)\Psi)_y - V(S)(\Psi_y - U(S)\Psi) = V(S)_y\Psi + V(S)U(S)\Psi.$$

We have a similar equation by changing U and V. These lead to

$$U(S)_t - V(S)_y + [U(S),V(S)] = 0 \tag{16}$$

by the integrability condition (4), since $\det\Psi \neq 0$.

The lemma implies that other two integrability conditions $H_{xy} = H_{yx}$, $H_{xt} = H_{tx}$ hold. Therefore, (15) has an NxN nondegenerate matrix solution. QED.

This theorem implies that any nondegenerate DO of first order can be determined only by an NxN matrix solution of the Lax pair. Thus, we obtain infinite number of solutions of (4) in the usually way.[5]

3. Application to 1+1 dimensional problems

We consider the equation (or the system of equations)

$$F(x,t,u,u_x,u_t,u_{xx},\dots) = 0 \tag{17}$$

defined in Ω (a simply connected domain in $\mathbb{R}^2$) which possesses the Lax pair

$$\begin{cases} \lambda\Phi = U(\partial)\Phi \\ \Phi_t = V(\partial)\Phi. \end{cases} \tag{18}$$

Here

$$\begin{cases} U(\partial) = U(x,t,u,u_x,\dots,\partial) = \sum_{j=0}^{m} U_{m-j}(x,t,u,u_x,\dots)\partial^j \\ V(\partial) = V(x,t,u,u_x,\dots,\partial) = \sum_{j=0}^{n} V_{n-j}(x,t,u,u_x,\dots)\partial^j, \end{cases} \tag{19}$$

U_o is nondegenerate.

Also, we assume that (18) is integrable, in the sense that for any $\lambda \in \mathbb{C}$, $(x_o,t_o) \in \Omega$ and $\Phi_o,\dots,\Phi_{m-1} \in \mathcal{M}_N$, there exists a local solution Φ of (18) such that $\partial^j\Phi(x_o,t_o) = \Phi_j$ $(j=0,1,\dots,m-1)$. The integrability condition (necessary) of (18) is

$$U_t(\partial) + [U(\partial),V(\partial)] = 0. \tag{20}$$

The simple examples of (20) are KdV equation and Boussinesq equation, the DO for the latter is given in [9] .

For an unreduced equation (20), the entries of U_j, V_j are independent unknowns, then $G(x,t,\partial) \in \mathcal{D}_N(\Omega)$ is a DO of (18) if and only if there exist $\tilde{U}(\partial), \tilde{V}(\partial) \in \mathcal{D}_N(\Omega)$ such that for any solution Φ of (18), $\tilde{\Phi} = G(\partial)\Phi$ satisfies

$$\begin{cases} \lambda\tilde{\Phi} = \tilde{U}(\partial)\tilde{\Phi} \\ \tilde{\Phi}_t = \tilde{V}(\partial)\tilde{\Phi}. \end{cases} \tag{21}$$

The nondegenerate DO of first order $G(x,t,\partial) = \partial - S(x,t)$ is given as follows, using the conclusions in 1+2 dimensions.

Theorem 2. $\partial - S(x,t)$ is a DO of (18) if and only if $S = H_x H^{-1}$ where H is an NxN nondegenerate matrix solution of

$$\begin{cases} H\Lambda = U(\partial)H \\ H_t = V(\partial)H, \end{cases} \tag{22}$$

and Λ is a constant matrix.

Proof. Suppose $\partial - S(x,t)$ is a DO of (18). Let

$$\Delta(\partial) = (\partial - S)U(\partial) - \tilde{U}(\partial)(\partial - S).$$

For any solution Φ of (18), $\Delta(\partial)\Phi = 0$ by (21). This implies

$$\lambda(\tilde{U}_o U_o^{-1} - 1)\Phi_x + \lambda(\tilde{U}_1 - \tilde{U}_o U_o^{-1}(U_1 + U_{o,x}) + SU_o - \tilde{U}_o S)U_o^{-1}\Phi + M(\partial)\Phi = 0$$

by (18), where $M(\partial)$ is independent of λ, and the order is less than m. From the integrability of (18),

$$\tilde{U}_o = U_o,\ \tilde{U}_1 = U_1 + U_{o,x} + [U_o,S].$$

Now it is easy to check that the order of $\Delta(\partial)$ is less than m. Hence $\Delta(\partial) = 0$ since it annihilates any solution Φ of (18). Thus

$$\begin{cases} 0 = (\partial - S)U(\partial) - \tilde{U}(\partial)(\partial - S) \\ S_t = (\partial - S)V(\partial) - \tilde{V}(\partial)(\partial - S). \end{cases} \tag{23}$$

(The second one is obtained from (21) directly.)

Now consider the equation

$$\begin{cases} \Psi_y = U(\partial)\Psi \\ \Psi_t = V(\partial)\Psi \end{cases} \tag{24}$$

for $(x,t) \in \Omega$, $y \in \mathbb{R}$. From (23), $\partial - S$ is a DO of (24). According to

Theorem 1, there exists a solution $K(x,y,t)$ of (24) such that $S=K_xK^{-1}$. Let $\Lambda = K^{-1}K_y$, then we can check that Λ is indeed a constant matrix by (11) and (16). Hence

$$K(x,y,t) = H(x,t)\exp(\Lambda y)$$

where H satisfies (22) and $S = H_xH^{-1}$.

Conversely, if H is a solution of (22), $S = H_xH^{-1}$, then it is easy to see that S satisfies (23), or equivalently $\partial - S$ is a DO of (18). QED.

If N=1, H must be a solution of the Lax pair (18). This gives the DO for Gelfand-Dikij system.

For the Lax pair

$$\begin{cases} \Phi_y = U(y,t,\lambda)\Phi \\ \Phi_t = V(y,t,\lambda)\Phi \end{cases} \tag{25}$$

where U,V are two polynomials of λ, we can get the similar conclusions as Theorem 2, which are the partial results in [13].

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