Letters in Mathematical Physics **16** (1988) 9–17.

On the Darboux Transformation for 1 + 2-Dimensional Equations

ZHOU ZIXIANG
Institute of Mathematics, Fudan University Shanghai, China

(Received: 3 February 1988)

Abstract. The Darboux transformations in 1 + 2-dimensional space are constructed, with a discussion of their composition and permutability. The auto-Bäcklund property of the Darboux transformation for the $N \times N$ system and KP hierarchy is also proved.

1. Introduction

The Darboux matrix method is a convenient form of Bäcklund transformations (BTs). In the 1 + 1-dimensional case, the Darboux matrices for AKNS system and its $N \times N$ generalizations have already been constructed. Besides, these Darboux transformations (DTs) are auto-BT (i.e. they change a solution of an equation to a solution of the same equation) [1–4, 7].

In the 1 + 2-dimensional case, there are still a lot of equations which have Lax pairs and BTs [5, 7, 8]. Tian [6] gave an expression of BT for the KP equation as (4.4). In [8], the BTs (in the form of integro-differential equations) for the 1 + 2-dimensional $N \times N$ Zakharov–Shabat–AKNS and Zakharov–Shabat–Gelfand–Dikij systems were constructed.

In the present Letter, we consider the DTs for general 1 + 2-dimensional systems. A differential operator which has a similar form as that given in [3] and plays the role of the Darboux matrix is constructed. As in the 1 + 1-dimensional case, this method can be used successively by using a purely algebraic algorithm which is universal for the whole hierarchies.

In Section 2, the integrability of a system of linear PDEs is discussed briefly and a general form of DT is given. The composition and permutability property of DT are discussed. Sections 3 and 4 are devoted to two special cases: the $N \times N$ system and KP hierarchy. The auto-Bäcklund property for these equations is also shown.

2. Integrable Equations and Darboux Transformation

We use the following notations: M_r is the set of all $r \times r$-matrix-valued smooth functions of (x, y, t).

$$D_r = \left\{ \sum_{k=0}^{n} A_k \frac{\partial}{\partial x^k} \middle| A_k \in M_r,\ n \text{ is a nonnegative integer} \right\}.$$

Thus, M_r is a subset of D_r. D always denotes $\partial/\partial x$.

In this part, we consider the following PDEs

$$\phi y = M\phi, \qquad \phi_t = N\phi. \tag{2.1}$$

Here $M, N \in D_r$. $\phi \in M_r$ is called a solution of (2.1) if ϕ satisfies (2.1), and $\det\phi \neq 0$.

We say that (2.1) is integrable if, for any (x_0, y_0, t_0) and any smooth function $\phi_0(x)$ defined near x_0, there exists a unique solution ϕ of (2.1) in some neighbourhood of (x_0, y_0, t_0) such that $\phi(x, y_0, t_0) = \phi_0(x)$.

Let

$$M = \sum_{j=0}^{m} M_{m-j}D^j, \qquad N = \sum_{j=0}^{n} N_{n-j}D^j. \tag{2.2}$$

If (2.1) is integrable,

$$M_t\phi - N_y\phi + [M, N]\phi = 0, \tag{2.3}$$

which is equivalent to

$$\sum_{k=0}^{m+n} Q_k D^k \phi = 0. \tag{2.4}$$

Here Q_k is the coefficient of D^k in $M_t - N_y + [M, N]$.

LEMMA 1. *If* (2.1) *is integrable, then* $Q_k = 0$, *for all* k.

Proof. If not, we get a nontrivial ODE (with respect to x) of ϕ given by (2.4). This is a contradiction to the solvability of (2.1) for any initial data.

Remark. The condition in the definition of integrability can be weakened. Instead of the existence of the solution in a whole neighbourhood W, we can require that the solution only exists in one of the subsets

$$W \cap \{(x, y, t) \mid (y - y_0)\xi \geqslant 0, (t - t_0)\eta \geqslant 0\}, \quad \xi, \eta = \pm 1.$$

If we define the integrability like that, all the conclusions in this Letter still hold.

Now suppose (2.1) is integrable. Let H be a solution of (2.1). Let

$$S = D - H_x H^{-1} \tag{2.5}$$

(which means $SH = 0$). Then we have

LEMMA 2. *There exist* $M', N' \in D_r$ *such that for any solution* ϕ *of* (2.1), $\phi' = S\phi$ *satisfies*

$$\phi'_y = M'\phi', \qquad \phi'_t = N'\phi'. \tag{2.6}$$

Proof. Let

$$N' = \sum_{j=0}^{n} N'_{n-j}D^j. \tag{2.7}$$

N' are given inductively by

$$N'_0 = N_0$$

$$N'_{s+1} = N_{s+1} + N_{s,x} - H_x H^{-1} N_s + \sum_{i=0}^{s} C_{n-i}^{n-s} N'_i D^{s-i}(H_x H^{-1}) . \tag{2.8}$$

Then

$$N'S - SN - S_t \in M_r . \tag{2.9}$$

Moreover

$$(N'S - SN - S_t)H = -(SH)_t = 0 . \tag{2.10}$$

Hence

$$N'S = SN + S_t , \tag{2.11}$$

which is equivalent to

$$(S\phi)_t = N'S\phi .$$

This is the second equation of (2.6). The first one is the same. □

Next we consider the higher-order DT. Let $H_1 \ldots, H_r$ be solutions of (2.1). Let K be the block matrix $K = (K_{ij})$, $K_{ij} = D^{i-1}H_j$, $i, j = 1, 2, \ldots, r$.

THEOREM 1. *For a given* $H_1, \ldots, H_r$ *with* $\det K \neq 0$, (1) *There uniquely exists an operator* S *(depending on* $H_1, \ldots, H_r$*)*

$$S(H_1, \ldots, H_r) = \sum_{i=0}^{r} \sigma_{r-i} D^i \tag{2.12}$$

$$\sigma_i \in M_r , \quad \sigma_0 = 1$$

such that $SH_j = 0$, *for* $j = 1, 2, \ldots, r$.

(2) *If*

$$\det \begin{bmatrix} H_{i_1} & \cdots & H_{i_{r-1}} \\ DH_{i_1} & \cdots & DH_{i_{r-1}} \\ \cdots & & \cdots \\ D^{r-2}H_{i_1} & \cdots & D^{r-2}H_{i_{r-1}} \end{bmatrix} \neq 0 ,$$

then

$$S(H_{i_1}, \ldots, H_{i_r}) = S(S(H_{i_1}, \ldots, H_{i_{r-1}})H_{i_r})S(H_{i_1}, \ldots, H_{i_{r-1}}) . \tag{2.13}$$

Here $(i_1, \ldots, i_r)$ *is a permutation of* $(1, \ldots, r)$.

(3) *There exist* $M', N' \in D_r$ *such that for any solution* ϕ *of* (2.1)

$$(S\phi)_y = M' S\phi, \qquad (S\phi)_t = N' S\phi.$$

Proof. (1) The existence and uniqueness of S is equivalent to the solvability of

$$\sum_{i=0}^{r-1} \sigma_{r-i} D^i H_j = -D^r H_j, \tag{2.14}$$

which is $\det K \neq 0$.

(2) Without loss of generality, assume $(i, \dots, i_r) = (1, \dots, r)$.

Let K_{r-1} be the block matrix $(D^{i-1}H_j)_{1 \leqslant i, j \leqslant r-1}$.

$$S(H_1, \dots, H_{r-1}) = \sum_{i=0}^{r-1} \tau_{r-1-i} D^i \quad (\tau_0 = I),$$

then (2.14) is

$$(\tau_{r-1}, \dots, \tau_1) = -(D^{r-1}H_1, \dots, D^{r-1}H_{r-1})K_{r-1}^{-1}. \tag{2.15}$$

Hence

$$\begin{aligned} &\det(S(H_1, \dots, H_{r-1})H_r) \\ &\quad = \det(D^{r-1}H_r - (D^{r-1}H_1, \dots, D^{r-1}H_{r-1})K_{r-1}^{-1}(H_r, \dots, D^{r-2}H_r)^T) \\ &\quad = \det K / \det K_{r-1} \neq 0. \end{aligned} \tag{2.16}$$

Here the symbol T means to change the row to column without replacing $H_1, \dots, H_{r-1}$ by their transposes. Thus, $S(S(H_1, \dots, H_{r-1})H_r)$ is well-defined.

Clearly,

$$S(S(H_1, \dots, H_{r-1})H_r)S(H_1, \dots, H_{r-1})H_r = 0.$$

But

$$S(H_1, \dots, H_{r-1})H_j = 0, \quad \text{for } 1 \leqslant j \leqslant r-1.$$

Hence, (2.13) holds by the first part of the theorem.

(3) Since $\det K \neq 0$, we can find $(i_1, \dots, i_r)$ such that

$$\det \begin{bmatrix} H_{i_1} & \cdots & H_{i_k} \\ DH_{i_1} & \cdots & DH_{i_k} \\ \cdots & & \cdots \\ D^{k-1}H_{i_1} & \cdots & D^{k-1}H_{i_k} \end{bmatrix} = 0,$$

for $1 \leqslant k \leqslant r$. Hence, by (2), S is the composition of first-order DTs. The existence of M', N' follows from Lemma 2. □

By the construction of S, we know $S(H_1, \dots, H_r)$ is symmetric for H_i and H_j. Hence, we have the following permutability property.

COROLLARY. *Suppose H_1, H_2 are two solutions of* (2.1),

$$\det\begin{bmatrix} H_1 & H_2 \\ DH_1 & DH_2 \end{bmatrix} \neq 0 .$$

Then

$$S(S(H_1)H_2)S(H_1) = S(S(H_2)H_1)S(H_2) .$$

3. $N \times N$ System

Let

$$M = AD + U , \qquad N = \sum_{j=0}^{n} V_{n-j}D^j . \tag{3.1}$$

Then (2.1) becomes

$$\phi_y = A\phi_x + U\phi , \qquad \phi_t = \sum_{j=0}^{n} V_{n-j}D^j\phi . \tag{3.2}$$

Here $V_0 = B$, $A = \mathrm{diag}(a_1, \dots, a_N)$, $B = \mathrm{diag}(b_1, \dots, b_N)$. a_i, b_i are constants, $a_i \neq a_j$, $b_i \neq b_j$ $(i \neq j)$. U is an $N \times N$ off-diagonal matrix. $V_s = (\underset{s}{v}_{ij})$.

In this case

$$Q_{n-s} = U_t\,\delta_{sn} - V_{s,y} + [A, V_{s+1}] + AV_{s,x} + \\ + UV_s - \sum_{k=0}^{s} C_{n-k}^{n-s} V_k D^{s-k} U . \tag{3.3}$$

If (3.2) is integrable,

$$Q_{n-s} = 0 \quad (s = 0, 1, \dots, n) , \tag{3.4}$$

i.e.

$$\underset{s+1}{v}_{ij} = \frac{1}{a_i - a_j}\left\{\underset{s}{v}_{ij,y} - a_i\underset{s}{v}_{ij,x} - \sum_{k\neq i} u_{ik}\underset{s}{v}_{kj} + \sum_{l=0}^{s}\sum_{k\neq j} C_{n-l}^{n-s}\underset{l}{v}_{ik}D^{s-l}u_{kj}\right\} , \tag{3.5}$$

$$\underset{s}{v}_{ii,y} - a_i\underset{s}{v}_{ii,x} = \sum_{k\neq i} u_{ik}v_{ki} - \sum_{l=0}^{s}\sum_{k\neq i} C_{n-l}^{n-s}\underset{l}{v}_{ik}D^{s-l}u_{ki} , \tag{3.6}$$

$$u_{ij,t} = \underset{n}{v}_{ij,y} - a_i v_{ij,x} - \sum_{k\neq i} u_{ik}\underset{n}{v}_{kj} + \sum_{l=0}^{n}\sum_{k\neq j}\underset{l}{v}_{ik}D^{n-l}u_{kj} . \tag{3.7}$$

We now consider the nonlinear PDEs (3.6)–(3.7), with unknowns

$$u_{ij}\,(i, j = 1, \dots, N, i \neq j) \quad \text{and} \quad \underset{s}{v}_{ii}\,(s = 1, \dots, n; i = 1, \dots, N) .$$

In these equations, $\underset{s}{v}_{ij}$ $(i \neq j)$ are given by (3.5).

The following theorem shows the auto-Bäcklund property for Equations (3.6)–(3.7).

THEOREM 2. *Let* u_{ij}, v_{ii} *be solutions of* (3.6)–(3.7). *H is a solution of* (3.2). *Then*

$$\tilde{U} = U + [A, H_x H^{-1}] \tag{3.8}$$

$$\underset{s+1}{\tilde{v}}_{ii} = \underset{s+1}{v}_{ii} + \underset{s}{v}_{ii,x} - \sum_j (H_x H^{-1})_{ij} \underset{s}{v}_{ji} + + \sum_{l=0}^{s} \sum_j C_{n-l}^{n-s} \underset{l}{\tilde{v}}_{ij} D^{s-l} (H_x H^{-1})_{ji} \tag{3.9}$$

are solutions of (3.6)–(3.7), *where* $\underset{s}{v}_{ij}$ $(i \neq j)$ *are given by*

$$\underset{s+1}{\tilde{v}}_{ij} = \frac{1}{a_i - a_j} \left\{ \underset{s}{\tilde{v}}_{ij,y} - a_i \underset{s}{\tilde{v}}_{ij,k} - \sum_{k \neq i} \tilde{u}_{ik} \underset{s}{\tilde{v}}_{kj} + + \sum_{l=0}^{s} \sum_{k \neq j} C_{n-l}^{n-s} \underset{s}{\tilde{v}}_{ik} D^{s-l} \tilde{u}_{kj} \right\}. \tag{3.10}$$

Moreover, for any solution ϕ *of* (3.2),

$$(S\phi)_y = A(S\phi)_k + \tilde{U} S\phi, \tag{3.11}$$

$$(S\phi)_t = \sum_{j=0}^{n} \tilde{V}_{n-j} D^j (S\phi), \tag{3.12}$$

where $S = D - H_x H^{-1}$.

Proof. (3.8) and (3.11) are given by (2.8) and (2.6). Let

$$\tilde{N} = \sum_{j=0}^{n} \tilde{V}_{n-j} D^j, \qquad N' = \sum_{j=0}^{n} V'_{n-j} D^j \tag{3.13}$$

in which $\tilde{V}_j$ is given by (3.9), (3.10), and $\tilde{V}_0 = B$. N' is given by Lemma 2.

Because of (2.6),

$$M'_t - N'_y + [M', N'] = 0. \tag{3.14}$$

Hence, (3.5) and (3.6) hold for $(\tilde{U}, V'_j)$.

We shall now prove $\tilde{V}_j = V'_j$. By definition, $\tilde{V}_0 = V'_0 = B$. Suppose $\tilde{V}_j = V'_j$ for $j \leqslant s$. Then, we have $\underset{s+1}{\tilde{v}}_{ii} = \underset{s+1}{v'}_{ii}$ by (3.9) and (2.8), $\underset{s+1}{\tilde{v}}_{ij} = \underset{s+1}{v'}_{ij}$ $(i \neq j)$ by (3.10) and (3.14). Hence, $\tilde{N} = N'$. Formula (3.14) implies that (3.8) and (3.9) satisfy (3.6) and (3.7). Equation (3.12) is just the second equation of (2.6). □

As for the reductive case, the DT given by Theorem 2 does not generally keep the reduction, so we need more restrictions on S (or on H).

Now consider Davey–Stewartson system, i.e.

$$A = B = \frac{i}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & p \\ \varepsilon \bar{p} & 0 \end{bmatrix}, \quad \varepsilon = \pm 1,$$

$$V_{n-j} = \begin{bmatrix} v_{n-j} & w_{n-j} \\ \varepsilon \bar{w}_{n-j} & \bar{v}_{n-j} \end{bmatrix}. \tag{3.15}$$

Suppose $[\alpha, \beta]^T$ is a solution of (3.2), then it is easy to see that $[i\varepsilon\bar{\beta}, i\bar{\alpha}]^T$ is also a solution.

Let

$$H = \begin{bmatrix} \alpha & i\varepsilon\bar{\beta} \\ \beta & i\bar{\alpha} \end{bmatrix},$$

then the DT is

$$\tilde{U} = U + [A, H_x H^{-1}] = U + \frac{i}{|\alpha|^2 - \varepsilon|\beta|^2} \begin{pmatrix} 0 & \varepsilon(\alpha\bar{\beta}_x - \bar{\beta}\alpha_x) \\ \beta\bar{\alpha}_x - \bar{\alpha}\beta_x & 0 \end{pmatrix}. \tag{3.16}$$

Clearly, $\tilde{U}$ satisfies the reduction.

EXAMPLE. More specifically, let $n = 2$,

$$V_1 = \begin{pmatrix} 0 & p \\ \varepsilon\bar{p} & 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} \sigma_1 & -ip_y + p_x/2 \\ i\varepsilon\bar{p}_y + \varepsilon\bar{p}_x/2 & \sigma_2 \end{pmatrix}, \tag{3.17}$$

where

$$\begin{aligned} \sigma_{1,y} - \frac{i}{2}\sigma_{1,x} &= i\varepsilon\left((|p|^2)_y + \frac{i}{2}(|p|^2)_x\right), \\ \sigma_{2,y} + \frac{i}{2}\sigma_{2,x} &= -i\varepsilon\left((|p|^2)_y - \frac{i}{2}(|p|^2)_x\right). \end{aligned} \tag{3.18}$$

If we let $q = 2\varepsilon|p|^2 + i(\sigma_1 - \sigma_2)$, then we have the Davey–Stewartson equation

$$\begin{aligned} ip_t &= p_{yy} - p_{xx}/4 - 2\varepsilon|p|^2 p + qp, \\ q_{yy} &+ q_{xx}/4 = \varepsilon(|p|^2)_{xx}. \end{aligned} \tag{3.19}$$

In this case, we know (3.16) is still an auto-BT, only to prove that the diagonal terms of $\tilde{V}_1$ are zeros. This is easy to check in (3.9).

4. KP Hierarchy

The system is

$$\begin{aligned} \phi_y &= \phi_{xx} + u\phi, \\ \phi_t &= \sum_{j=0}^{n} v_{n-j} D^j \phi \quad (v_0 = 1). \end{aligned} \tag{4.1}$$

If (4.1) is integrable, we have

$$2v_{s+1,x} = v_{s,y} - v_{s,xx} + \sum_{k=0}^{s-1} C_{n-k}^{n-s} v_k D^{s-k} u, \tag{4.2}$$

$$u_t = v_{n,y} - v_{n,xx} + \sum_{k=0}^{n-1} v_k D^{n-k} u . \tag{4.3}$$

We regard (4.2) and (4.3) as a group of differential equations of unknowns u and v_i ($i = 1, 2, \ldots, n$).

THEOREM 3. *Let* u, v_i ($i = 1, \ldots, n$) *be solutions of* (4.2)–(4.3), H *be a solution of* (4.1). *Then*

$$\tilde{u} = u + 2(H_x/H)_x , \tag{4.4}$$

$$\tilde{v}_{s+1} = v_{s+1} + v_{s,x} - v_s H_x/H + \sum_{k=0}^{s} C_{n-k}^{n-s} \tilde{v}_k D^{s-k}(H_x/H) \tag{4.5}$$

are solutions of (4.2)–(4.3). *Moreover, for any solution* ϕ *of* (4.1),

$$(S\phi)_y = (S\phi)_{xx} + \tilde{u} S\phi , \tag{4.6}$$

$$(S\phi)_t = \sum_{j=0}^{n} \tilde{v}_{n-j} D^j (S\phi) . \tag{4.7}$$

The *proof*, which is similar to the proof of Theorem 2, is omitted.

If $n = 3$, we can study a differential equation (KP equation) rather than the system of equations (4.2)–(4.3). By (4.2)

$$\begin{aligned} v_1 &= c_1(y, t) , \\ v_2 &= \tfrac{3}{2}u + \tfrac{1}{2}c_{1,y}x + c_2(y, t) , \end{aligned} \tag{4.8}$$

where c_1, c_2 are two arbitrary smooth functions of y, t. Equation (4.3) can be written as

$$\begin{aligned} u_{tx} = {} & \tfrac{1}{4}(u_{xx} + 6uu_x)_x + \tfrac{3}{4}u_{yy} + \tfrac{1}{4}c_{1,yyy}x + c_{1,y}(\tfrac{3}{2}u_x + \tfrac{1}{2}xu) + \\ & + \tfrac{1}{2}c_{2,yy} + c_2 u_{xx} . \end{aligned} \tag{4.9}$$

Fir (4.9), the auto-Bäcklund property also holds, i.e.

COROLLARY. *Let* u *be a solution of* (4.9). H *is a solution of* (4.1). *Then* $\tilde{u} = u + 2(H_x/H)_x$ *is also a solution of* (4.9). *Moreover, for any solution* ϕ *of* (4.1), *Equations* (4.6) *and* (4.7) *hold.*

Proof. By Theorem 3, we only need to check that $(\tilde{u}, v_1, \tilde{v}_2)$ satisfies (4.8) as (u, v_1, v_2) with the same $c_1(y, t)$, $c_2(y, t)$. This is true by simple calculation. □

Acknowledgements

I am greatly indebted to Prof. Gu Chaohao for helpful advice. This work was supported by the Chinese Fund of Natural Sciences and the Chinese Fund of Doctor Programmes.

References

1. Neugerbauer, G. and Meinel, R., General N-soliton solution of the AKNS class on arbitrary background, *Phys. Lett.* **100A**, 467 (1984).
2. Gu, C. H. and Zhou, Z. X., On the Darboux matrices of Bäcklund transformations for the AKNS systems, *Lett. Math. Phys.* **13**, 179 (1987).
3. Gu, C. H., On the Bäcklund transformations for $N \times N$ AKNS systems, to appear in *Soliton Theory and its Application*, Zhejiang Science and Technology Press.
4. Lin, J. M., Report on seminar 'Soliton' in Fudan University.
5. Novikov, S., Manakov, S. V., Pitaevskii, L. P., and Zakharov, V. E., *Theory of Solitons*, Consultants Bureau (1984).
6. Tian, C., Generalized KP equation and Miura transformation (preprint) (1986).
7. Li, Y. S., Gu, X. S., and Zou, M. R., Three kinds of Darboux transformations for the evolution equations which connect with the AKNS eigenvalue problem (1986), (to appear in *Acta Math. Sinica*).
8. Boiti, M., Konopelchenko, B. G., and Pempinelli, F., Bäcklund transformations via gauge transformations in 2 + 1 dimensions, *Inverse Problem* **1**, 33 (1985).