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On the Darboux Matrices of Bäcklund Transformations for AKNS Systems★

GU CHAOHAO and ZHOU ZIXIANG
Institute of Mathematics, Fudan University, Shanghai, China

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Abstract. The auto-Bäcklund property of the Darboux matrices for AKNS systems is proved. A quite general form of the theorem of permutability is obtained in a universal way.

1. Introduction

The Darboux matrix method has been proved to be very useful in the construction of Bäcklund transformations for soliton equations [1–6]. As is known, the Bäcklund transformation is a method of constructing new solutions of partial differential equations from known solutions. In the generalized sense, a Bäcklund transformation changes a solution of an equation to a solution of another equation. (For example, the Bäcklund transformation between sine-Laplace and sinh-Laplace equations changes the solutions of these two equations with each other.) However, in the original sense and in many cases, the Bäcklund transformations do change a solution of an equation to another solution of the same equation. These Bäcklund transformations are called auto-Bäcklund transformations. In [7], an elegant method of constructing Darboux matrices for general AKNS systems was presented. However, the proof of the auto-Bäcklund property in [7] has some ambiguous points. In the present Letter, we give a complete proof of that property. In Section 2, we give a clear description of the system of equations which are the integrability conditions of the AKNS system. In Section 3, we rewrite the construction of Darboux matrices with complementary comments. In particular, the existence of such matrices is investigated. Section 4 is devoted to the proof of the auto-Bäcklund property by the method used in [6]. In Section 5, we present a more general permutability theorem which has universal characters. This treatment is valid for the hierarchies of KdV equations, MKdV equations, and nonlinear Schrödinger equations as particular cases.

2. The System of Partial Differential Equations

We consider the general AKNS system

$$\phi_x = M\phi, \qquad \phi_t = N\phi. \tag{2.1}$$

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Here

$$M = \begin{pmatrix} \lambda & p(x,t) \\ q(x,t) & -\lambda \end{pmatrix}, \qquad V = \begin{pmatrix} A(x,t,\lambda) & B(x,t,\lambda) \\ C(x,t,\lambda) & -A(x,t,\lambda) \end{pmatrix}.$$

λ is a complex parameter, p, q are two smooth functions, and A, B, C are polynomials of the parameter λ

$$\begin{aligned} A &= \sum_{k=0}^{m} a_k(x,t)\lambda^{m-k}, \\ B &= \sum_{k=0}^{m} b_k(x,t)\lambda^{m-k}, \\ C &= \sum_{k=0}^{m} c_k(x,t)\lambda^{m-k}. \end{aligned} \tag{2.2}$$

We use the subscripts x and t as the notations of derivatives with respect to x and t, and all the functions appearing in this Letter are assumed to be sufficiently smooth.

The integrable conditions of Equations (2.1) are

$$A_x = pC - qB, \qquad B_x = p_t + 2\lambda B - 2pA, \qquad C_x = q_t - 2\lambda C + 2qA. \tag{2.3}$$

From (2.2) and (2.3), we get

$$\begin{aligned} &a_{0,x} = 0, \qquad b_0 = c_0 = 0, \qquad b_{j+1} = pa_j - \tfrac{1}{2}b_{j,x}, \\ &c_{j+1} = qa_j - \tfrac{1}{2}c_{j,x}, \qquad a_{j+1,x} = pc_{j+1} - qb_{j+1}\ (0 \leqslant j \leqslant m-1) \end{aligned} \tag{2.4}$$

and

$$p_t = 2pa_m + b_{m,x}, \qquad q_t = -2qa_m + c_{m,x}. \tag{2.5}$$

We shall give a simple proof of the fact that a_j, b_j, c_j ($j = 1, 2, \ldots, m$) are polynomials of p, q and their x derivatives. Hence, (2.5) is a system of partial differential equations of p and q.

Let $P = \{\varphi \mid \varphi$ is a polynomial of $p, q, p_x, q_x, p_{xx}, q_{xx} \cdots$ whose coefficients are independent of $x\}$, $P_1 = \{\varphi_x \mid \varphi \in P\}$. By the notation $f \doteq g$ we mean that $f, g \in P$ and $f - g \in P_1$.

LEMMA 2.1. *For $0 \leqslant k \leqslant m$, $a_k, b_k, c_k \in P$.*

Proof. We shall prove this lemma by induction. It is clear for $k = 0$. Suppose that is true for $k \leqslant l - 1$, i.e., $a_k, b_k, c_k \in P$ for $k \leqslant l-1$ $(1 \leqslant l \leqslant m)$, then by (2.4), $b_l, c_l \in P$. If $1 \leqslant j \leqslant l-1$, from (2.4) and the hypothesis of induction, we have

$$\begin{aligned} &b_j c_{l+1-j} - c_j b_{l+1-j} \\ &\quad = b_j(qa_{l-j} - \tfrac{1}{2}c_{l-j,x}) - c_j(pa_{l-j} + \tfrac{1}{2}b_{l-j,x}) \\ &\quad = -a_{j,x}a_{l-j} - \tfrac{1}{2}(b_j c_{l-j,x} + c_j b_{l-j,x}) \\ &\quad \doteq a_j a_{l-j,x} + \tfrac{1}{2}(b_{j,x}c_{l-j} + c_{j,x}b_{l-j}) \end{aligned}$$

$$= (pa_j + \tfrac{1}{2}b_{j,x})c_{l-j} - (qa_j - \tfrac{1}{2}c_{j,x})b_{l-j}$$
$$= b_{j-1}c_{l-j} - c_{j-1}b_{l-j}. \tag{2.6}$$

Hence,

$$b_1c_l - c_1b_l \doteq b_lc_1 - c_lb_1 = -(b_1c_l - c_1b_l),$$

i.e., $b_1c_l - c_1b_l \doteq 0$. Thus, $pc_l - qb_l \doteq 0$, since $b_1 = a_0p$, $c_1 = a_0q$. Therefore, by (2.4), we have $a_l \in P$. □

Now we can define $\{a_k^0[p,q]\}$, $\{b_k^0[p,q]\}$, $\{c_k^0[p,q]\}$ as follows, $a_0^0 = 1$, and the others are defined inductively by integrating (2.4) with the conditions $a_k^0[0,0] = 0$ $(1 \leqslant k \leqslant m)$. These $\{a_k^0\}$, $\{b_k^0\}$, $\{c_k^0\}$ are uniquely defined polynomials of p, q and their x-derivatives. Moreover, for any p, q, $b_k^0[0,q] = 0$, $c_k^0[p,0] = 0$, $a_k^0[p,0] = a_k^0[0,q] = 0$ $(1 \leqslant k \leqslant m)$.

Thus, the general expressions for a_k, b_k, c_k are

$$\begin{aligned} a_k[p,q] &= \sum_{j=0}^{k} a_{k-j}[0,0]a_j^0(p,q], \\ b_k[p,q] &= \sum_{j=0}^{k} a_{k-j}[0,0]b_j^0(p,q], \\ c_k[p,q] &= \sum_{j=0}^{k} a_{k-j}[0,0]c_j^0[p,q]. \end{aligned} \tag{2.7}$$

They are uniquely determined by $a_k[0,0]$. Here, in general, $a_k[0,0]$ are functions of t.

Thus, we obtain the general expressions for Equations (2.5) and related coefficients a_k, b_k, c_k. In particular, (2.5) is a system of partial differential equations whose coefficients may depend on t.

EXAMPLE. For $m = 2$, the equations are

$$\begin{aligned} p_t &= s_0(t)(\tfrac{1}{2}p_{xx} - p^2q) + s_1(t)p_x + 2s_2(t)p, \\ q_t &= -s_0(t)(\tfrac{1}{2}q_{xx} - pq^2) + s_1(t)q_x - 2s_2(t)q. \end{aligned} \tag{2.8}$$

As is well known, if p, q are suitably related, from (2.5) we can get an equation of one unknown function. For instance, if $q = -\bar{p}$, $s_1 = s_2 = 0$, $s_0 = 2i$, from (2.8) we get the nonlinear Schrödinger equation

$$ip_t + P_{xx} + 2|P|^2P = 0.$$

3. The Construction of Darboux Matrices

The Darboux matrices for Bäcklund transformations of the AKNS system have been constructed in [7]. We rewrite the method of construction and prove that the method is available.

Suppose that (p, q) is a known solution to the partial differential equations (2.5) and

$$\phi(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{pmatrix} \tag{3.1}$$

is a solution to the linear system (2.1) with $\det\phi \not\equiv 0$. We call $\phi(\lambda)$ a representation of the solution (p, q).

A Darboux matrix is a 2×2 matrix of the form

$$s(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \lambda^l + \sum_{j=0}^{l-1} \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} \lambda^j \tag{3.2}$$

which satisfies

(i) $$\det s(\lambda) = \prod_{j=1}^{2l} (\lambda - \lambda_j) \tag{3.3}$$

where λ_j are $2l$ distinct complex numbers such that $\det\phi(\lambda_i) \neq 0$ $(i = 1, 2, \ldots, 2l)$,

(ii) the two columns of the matrix

$$\phi_1(\lambda) = s(\lambda)\phi(\lambda) = \begin{pmatrix} \tilde{a}_{11}(\lambda) & \tilde{a}_{12}(\lambda) \\ \tilde{a}_{21}(\lambda) & \tilde{a}_{22}(\lambda) \end{pmatrix} \tag{3.4}$$

have a linear relation with constant coefficients $\mu_i : \nu_i$ at $\lambda = \lambda_i$:

$$\mu_i \tilde{a}_{\alpha 1} + \nu_i \tilde{a}_{\alpha 2} = 0 \quad (\alpha = 1, 2;\ i = 1, 2, \ldots, 2l). \tag{3.5}$$

For simplicity we assume $\mu_i \neq 0$ and denote ν_i/μ_i by b_i. Equation (3.5) becomes

$$\begin{aligned} &\sum_{j=0}^{l-1} (\rho_i \alpha_j + \tau_i \beta_j)\lambda_i^j = -\lambda_i^l \rho_i, \\ &\sum_{j=0}^{l-1} (\rho_i \gamma_j + \tau_i \delta_j)\lambda_i^j = -\lambda_i^l \tau_i. \end{aligned} \tag{3.6}$$

Here

$$\rho_i = a_{11}(\lambda_i) + b_i a_{12}(\lambda_i), \quad \tau_i = a_{21}(\lambda_i) + b_i a_{22}(\lambda_i). \tag{3.7}$$

Let

$$\sigma_i = \frac{\tau_i}{\rho_i} = \frac{a_{21}(\lambda_i) + b_i a_{22}(\lambda_i)}{a_{11}(\lambda_i) + b_i a_{12}(\lambda_i)}. \tag{3.8}$$

Since $\det\phi(\lambda_i) \neq 0$, τ_i, ρ_i cannot identically equal zero. Moreover, σ_i satisfies the Riccati equation

$$\sigma_{i,x} = q - 2\lambda_i \sigma_i - p\sigma_i^2. \tag{3.9}$$

Equation (3.6) is a system of $4l$ linear equations with $4l$ unknowns $\alpha_j, \beta_j, \gamma_j, \delta_j$. If the system admits a unique solution in some region Ω in the (t, x)-plane, we say that it is regular in Ω.

In particular, for the case $l = 1$, we have

$$S = \frac{1}{\sigma_2 - \sigma_1}\begin{pmatrix} (\sigma_2 - \sigma_1)\lambda + \lambda_2\sigma_1 - \lambda_1\sigma_2 & \lambda_1 - \lambda_2 \\ (\lambda_2 - \lambda_1)\sigma_1\sigma_2 & (\sigma_2 - \sigma_1)\lambda + \lambda_1\sigma_1 - \lambda_2\sigma_2 \end{pmatrix}. \tag{3.10}$$

Since $\lambda_1 \neq \lambda_2$, we have $\sigma_1 \neq \sigma_2$ whenever $\sigma_1 \neq 0$. Hence, for generic points in the (t, x) plane S is defined regularly.

THEOREM 3.1. *For given constants $\lambda_1, \lambda_2, \ldots, \lambda_{2l}, b_1, b_2, \ldots, b_{2l}$ such that*

$$\det\phi(\lambda_i) \neq 0\,, \quad \lambda_i \neq \lambda_j\,(i, j = 1, 2, \ldots, 2l, i \neq j)\,,$$

the Darboux matrix exists for generic points in the (t, x) plane.

Proof. By using the constants $(\lambda_1, \lambda_2, b_1, b_2)$, we construct the Darboux matrix (3.8) and denote it by $S^{(1)}(\lambda) = S(\lambda, \lambda_1, \lambda_2, b_1, b_2, \phi)$ and let

$$\phi^{(1)}(\lambda) = S^{(1)}(\lambda)\phi(\lambda)\,. \tag{3.11}$$

Then, using the constants $(\lambda_3, \lambda_4, b_3, b_4)$ and the matrix $\phi^{(1)}(\lambda)$, we construct the Darboux matrix (3.8) and denote it by $S^{(2)}(\lambda) = S(\lambda, \lambda_3, \lambda_4, b_3, b_4, \phi^{(1)})$.

Consider the matrix

$$S^{(2,1)}(\lambda) = S^{(2)}(\lambda)S^{(1)}(\lambda)\,. \tag{3.12}$$

It is easy to see that $S^{(2,1)}(\lambda)$ is of form (3.2) with $l = 2$. Moreover, the conditions (3.3) and (3.4) are satisfied with constants $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, b_1, b_2, b_3, b_4)$. Consequently, $S^{(2,1)}(\lambda)$ is nothing more than a Darboux matrix corresponding to the coefficients $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, b_1, b_2, b_3, b_4)$. Thus, we have proved the existence of the Darboux matrix with $l = 2$. By using the same procedure we can obtain the Darboux matrix with any given l for generic points in the (t, x)-plane. □

For the regular case, the conclusion of the above theorem is obvious and the Darboux matrix exists uniquely. From the proof we see that the case of $l = 1$ is fundamental. For the nonregular case we may choose the Darboux matrix which is constructed in the above proof.

4. The Proof of the Auto-Bäcklund Property

Let $S(\lambda)$ be the Darboux matrix (3.2) and $\phi_1(\lambda) = S(\lambda)\phi(\lambda)$. Equation (3.9) is valid for $i = 1, 2, \ldots, 2l$. By using these identities, it can be proved by direct calculation [7] that

$$\phi_{1,x} = \begin{pmatrix} \lambda & p_1 \\ q_1 & -\lambda \end{pmatrix}\phi_1 = M_1\phi_1 \tag{4.1}$$

with

$$p_1 = p - 2\beta_{l-1}\,, \qquad q_1 = q + 2\gamma_{l-1}\,. \tag{4.2}$$

Write

$$\sigma_{1t} = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} \sigma_1 = N_1 \phi_1 . \tag{4.3}$$

Then

$$\begin{aligned} N_1 &= SNS^{-1} + S_t S^{-1} \\ &= SNS^{-1} + \sum_{i=1}^{2l} [C(\lambda_i) - 2A(\lambda_i)\sigma_i - B(\lambda_i)\sigma_i^2] S\sigma_i S^{-1} . \end{aligned} \tag{4.4}$$

Here $S\sigma_i$ is the derivative of S with respect to σ_i and σ_{it} has been eliminated by the known formula

$$\sigma_{it} = C(\lambda_i) - 2A(\lambda_i)\sigma_i - B(\lambda_i)\sigma_i^2 . \tag{4.5}$$

N_1 is a polynomial of λ of degree m. This can be obtained by verifying $(SNS^* + S_t S^*)_{\lambda = \lambda_i} = 0$ together with some calculations, where $S^* = (\det S)S^{-1}$.

A_1, B_1 and C_1 can be written as

$$A_1 = \sum_{k=0}^{m} a_{k1} \lambda^{m-k} , \quad B_1 = \sum_{k=0}^{m} b_{k1} \lambda^{m-k} , \quad C_1 = \sum_{k=0}^{m} c_{k1} \lambda^{m-k} . \tag{4.6}$$

Equations (2.4) and (2.5) hold if we replace a_j, b_j, c_j by a_{j1}, b_{j1}, c_{j1} and p, q by p_1, q_1. Moreover, a_{j1}, b_{j1}, c_{j1} are polynomials of p_1, q_1 and its x-derivatives. The structure of these polynomials is described in Section 2.

We are going to prove

$$a_{j1}[p_1, q_1] = a_j[p_1, q_1] , \quad b_{j1}[p_1, q_1] = b_j[p_1, q_1] , \quad c_{j1}[p_1, q_1] = c_j[p_1, q_1] . \tag{4.7}$$

If (4.7) is true, then (p_1, q_1) satisfies Equation (2.5), and (4.2) is an auto-Bäcklund transformation.

Owing to (4.2) and (4.4), we see that (4.7) consists of some equalities between p and q, their x-derivatives and σ_i, provided that p, q, σ_i are related by (3.9) and that p and q are solutions to (2.5). However, it is more convenient to prove (4.7) for arbitrary functions p, q and σ_i which are related by (3.9) at any instant t, say $t = 0$.

Now let p and q be two arbitrary functions of x and

$$\begin{aligned} \hat{p} &= p + t(2pa_m[p, q] + b_{m,x}[p, q]) , \\ \hat{q} &= q - t(-2qa_m[p, q] + c_{m,x}[p, q]) . \end{aligned} \tag{4.8}$$

Moreover, let $\phi(x)$ be a solution to the equation $\phi_x = M[p, q]\phi$ with $\det \phi \not\equiv 0$. Define

$$\hat{\phi}(x, t) = \phi(x) - tN[p, q]\phi(x) .$$

From the definition of $\hat{p}$, $\hat{q}$ and N, it is easily seen that

$$\hat{M}_t|_{t=0} - N_x[p, q] + [M, N] = 0 . \tag{4.9}$$

Here $\hat{M} = M[\hat{p}, \hat{q}]$.

Moreover, we have

$$\hat{\phi}_x = \hat{M}[\hat{p}, \hat{q}]\hat{\phi} = M[p, q]\hat{\phi}, \quad \hat{\phi}_t = \hat{N}[\hat{p}, \hat{q}]\hat{\phi} = N[p, q]\hat{\phi}, \quad \hat{\phi}_{xt} = (\hat{M}[\hat{p}, \hat{q}]\hat{\phi})_t \tag{4.10}$$

at $t = 0$.

Define $\hat{\sigma}_i$, $\hat{S}$ by (3.8) and (3.2) on the base of $\hat{\phi}$ and let

$$\hat{\phi}_1 = \hat{S}\hat{\phi}, \qquad \hat{M}_1 = \hat{S}\hat{M}\hat{S}^{-1} + \hat{S}_x\hat{S}^{-1}, \qquad \hat{N}_1 = \hat{S}\hat{N}S^{-1} + \hat{S}_tS^{-1}. \tag{4.11}$$

We have

$$\hat{\phi}_{1x} = \hat{M}_1\hat{\phi}_1, \qquad \hat{\phi}_{1t} = \hat{N}_1\hat{\phi}_1, \tag{4.12}$$

$$\hat{M}_{1t} - \hat{N}_{1t} + [\hat{M}_1, \hat{N}_1] = 0 \tag{4.13}$$

at $t = 0$. Here

$$\hat{M}_1 = \begin{pmatrix} \lambda & \hat{p}_1 \\ \hat{q}_1 & -\lambda \end{pmatrix} \tag{4.14}$$

with

$$\hat{p}_1 = \hat{p} - 2\hat{\beta}_{l-1}, \qquad \hat{q}_1 = \hat{q} + 2\hat{\gamma}_{l-1}. \tag{4.15}$$

Equation (4.15) coincides with (4.2) at $t = 0$.

Since $\hat{\sigma}_i|_{t=0} = \sigma_i$, $\hat{\sigma}_{it}|_{t=0} = C(\lambda_i) - 2A(\lambda_i)\sigma_i - B(\lambda_i)\sigma_i^2$, by the definition of $\hat{N}_1$, we know $\hat{N}_1|_{t=0} = N_1$.

Considering the algorithm for $\hat{N}_1|_{t=0} = N_1$, we see that the coefficients of λ^{m-k} are just $a_{j1}[p_1, q_1]$, $b_{j1}[p_1, q_1]$, $c_{j1}[p_1, q_1]$, appearing in (4.7), at $t = 0$.

Now we prove (4.7) by induction.

For $j = 0$, $b_0[p_1, q_1] = 0$, $c_0[p_1, q_1] = 0$. Compare the coefficient of

$$\lambda^{m-l} \text{ in } SN + S_t = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} S.$$

we get

$$a_{01} = a_0[p_1, q_1] \quad (= a_0[0, 0]).$$

Suppose (4.7) is true for $j = k - 1$. By (4.9) and (2.4),

$$b_{k1} = b_{k1}[p_1, q_1], \qquad c_{k1} = c_k[p_1, q_1], \qquad a_{k1x} = a_{kx}[p_1, q_1].$$

Let $\mathscr{F}_k = a_{j1} - a_k[p_1, q_1]$, then for given $\{p, q, \sigma_i\}$, $\mathscr{F}_k$ is a constant.

What we must prove is that this constant is zero.

We need the following lemma.

LEMMA. *Let $\mathscr{F}[p, q, \sigma]$ be a polynomial of p, q, their derivatives and σ_i. If $\mathscr{F}[p, q, \sigma] = $ const. for any smooth functions p, q and σ_i which are defined in an interval* *and related by* (3.9), *then*, $\mathscr{F}[p, q, \sigma]$ *is a constant which is independent of* p, q *and* σ_i.

Proof. Let $p^{(r)}$, $q^{(s)}$ be the derivatives of p and q of the highest order contained in $\mathscr{F}$. We write

$$\mathscr{F}[p, q, \sigma] = F(p, p', \ldots, p^{(r)}, q, q', \ldots, q^{(s)}, \sigma_i).$$

From the hypothesis, we have

$$F_{p^{(r)}}p^{(r+1)} + F_{p^{(r-1)}}p^{(r)} + \ldots + F_p p' + F_{q^{(s)}}q^{(s-1)} + F_{q^{(s-1)}}q^{(s-1)} + \ldots$$
$$+ F_q q' + \sum_i F\sigma_i(q - 2\lambda_i\sigma_i - p\sigma_i^2) = 0 .$$

Hence, $F_{p^{(r)}} = 0$, i.e., F is independent of $p^{(r)}$. It is a contradiction. Hence, $F = F(\sigma_i)$. Since σ_i can take arbitrary values in different points, F must be a constant. The lemma is proved. □

From this lemma, we see that $\mathscr{F}_k$ is a constant which is independent of p, q and σ_i. Now we shall give a special set of p, q, σ_i to show that this constant is zero.

Let

$$p = q = 0, \qquad \sigma_i = e^{-2\lambda_i x} \quad (1 \leqslant i \leqslant l), \qquad \sigma_i = 0 \quad (l + 1 \leqslant 1 \leqslant 2l).$$

Then σ_i $(1 \leqslant i \leqslant 2l)$ satisfy the condition (3.9). By the definition of α_j, γ_{l-1}, we know α_j, $\sigma_i = 0$ and $\gamma_{l-1} = 0$. Thus, by (4.4) and (4.2), $A_1 = A[0, 0]$ and $q = 0$. Hence, $\mathscr{F}_k = 0$ by Section 2. Therefore, we have proved

THEOREM 4.1. *The transformation*

$$\{(p, q), \phi\} \to \{(p_1, q_1), \phi_1\}$$

defined by (4.2) *and* $\phi_1 = S\phi$ *is an auto-Bäcklund transformation.*

REMARK. If ϕ is known, the Bäcklund transformation can be carried out purely algebraically. Moreover, its algorithm is universal for the whole hierarchies of equations. In particular, for $l = 1$, (4.2) becomes

$$p_1 = p + 2\,\frac{\lambda_2 - \lambda_1}{\sigma_2 - \sigma_1}, \qquad q_1 = q + 2\,\frac{\sigma_1\sigma_2(\lambda_2 - \lambda_1)}{\sigma_2 - \sigma_1}.$$

5. Theorem of Permutability

The theorem of permutability can be proved very easily.

Let $S^{(1)}(\lambda, \lambda_1, \lambda_2, b_1, b_2)$ be the Darboux matrix based on $\phi_1(\lambda)$ and $S^{(2)}(\lambda, \lambda_3, \lambda_4, b_3, b_4)$ be the Darboux matrix based on $S^{(1)}\phi_1(\lambda)$. We have

THEOREM 5.1 (Theorem of permutability)

$$S^{(2)}(\lambda, \lambda_3, \lambda_4, b_3, b_4)S^{(1)}(\lambda, \lambda_1, \lambda_2, b_1, b_2)$$
$$= S^{(2)}(\lambda, \lambda_1, \lambda_2, b_1, b_2)S^{(1)}(\lambda, \lambda_3, \lambda_4, b_3, b_4)$$

$$= S^{(2)}(\lambda, \lambda_1, \lambda_4, b_1, b_4)S^{(1)}(\lambda, \lambda_2, \lambda_3, b_2, b_3)$$
$$= S^{(2)}(\lambda, \lambda_2, \lambda_3, b_2, b_3)S^{(1)}(\lambda, \lambda_1, \lambda_4, b_1, b_4)$$
$$= S^{(2)}(\lambda, \lambda_2, \lambda_4, b_2, b_4)S^{(1)}(\lambda, \lambda_1, \lambda_3, b_1, b_3)$$
$$= S^{(2)}(\lambda, \lambda_1, \lambda_3, b_1, b_3)S^{(1)}(\lambda, \lambda_2, \lambda_4, b_2, b_4).$$

Proof. For the regular case, all these $S^{(2)}S^{(1)}$'s are equal to the Darboux matrix which is uniquely determined by the constants $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, b_1, b_2, b_3, b_4)$. Hence, they are equal to each other. For the nonregular case we can get the equalities by a limiting process. □

REMARK. This theorem can be generalized to the more general case when the Darboux matrices are of degree l_1, l_2, l_3 and l_4 with $l_1 + l_2 = l_3 + l_4$ and both sides have the same set of parameters.

It is noticed that the theorem of permutability for generalized KdV hierarchies, MKdV hierarchies, and nonlinear Schrödinger hierarchies are special cases of this theorem.

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