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Finite dimensional Hamiltonian system related to a Lax pair with symplectic and cyclic symmetries

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Abstract
For the 1 + 1 dimensional Lax pair with a symplectic symmetry and cyclic symmetries, it is shown that there is a natural finite-dimensional Hamiltonian system related to it by presenting a unified Lax matrix. The Liouville integrability of the derived finite-dimensional Hamiltonian systems is proved in a unified way. Any solution of these Hamiltonian systems gives a solution of the original PDE. As an application, the two-dimensional hyperbolic $C_n^{(1)}$ Toda equation is considered and the finite-dimensional integrable Hamiltonian system related to it is obtained from the general results.

Mathematics Subject Classification: 37K10, 70H06, 35Q51

1. Introduction

There are many integrable nonlinear PDEs in 1 + 1 dimensions which have important applications in physics, mechanics, geometry and so on [3, 24]. For quite a few of them, the related finite-dimensional Liouville integrable Hamiltonian systems have been obtained. By this nonlinearization method [4, 5], the nonlinear PDE is changed to a system of nonlinear ODEs which are Liouville integrable Hamiltonian systems. Any solution of this system of nonlinear ODEs gives a solution of the original nonlinear PDE. This considerably simplifies the original problem. It is an effective way to obtain interesting exact solutions, especially quasi-periodic solutions of the nonlinear PDEs [6, 7, 19, 21–23]. Soliton solutions can be obtained in this way by a limiting process [26]. Some integrable systems in higher dimensions have also been reduced to finite-dimensional Liouville integrable Hamiltonian systems [7, 8, 10, 31, 32].

Usually these finite-dimensional Hamiltonian systems have Lax matrices so that the Liouville integrability can be guaranteed [11, 12, 20, 27, 28]. Most results are obtained for specific nonlinear PDEs and specific hierarchies with less symmetries, and the integrability of the derived finite-dimensional Hamiltonian systems is proved case by case.

In this paper, we consider a quite general Lax pair with a symplectic symmetry and cyclic symmetries. The Lax matrix is presented so that the nonlinear constraint of the lowest order is generated naturally from it. The Hamiltonian function for the ODEs derived from the nonlinear constraint is expressed in terms of the Lax matrix. The Liouville integrability of this Hamiltonian system is proved by obtaining the r matrix and finding enough functionally independent conserved integrals. This system contains some known examples such as the MKdV equation and the nonlinear Schrödinger equation. It also contains any $n \times n$ AKNS system with $u(n)$ symmetry, where the symplectic structure is naturally derived from the complex structure, and the binary nonlinearization method [11] is recovered.

As an application, the general results are used for the two-dimensional $C_n^{(1)}$ hyperbolic Toda equation [14]. The two-dimensional affine Toda equations are all integrable [1, 6, 13, 14, 16–18, 25] and are important in the integrable massive deformation of conformal field theory [2, 9, 33]. Among these equations, the two-dimensional $C_n^{(1)}$ hyperbolic Toda equation has a natural symplectic structure. The finite-dimensional Hamiltonian systems related to it are constructed explicitly in this paper. These Hamiltonian systems are simpler than (with space of lower dimension) that presented in [29] where binary nonlinear constraint was constructed. The result for the x-part of the Lax pair is derived from the general result of this paper, while that for the t-part which has the λ^{-1} term is obtained independently.

The paper is organized as follows. In section 2, some notations and the Lax pair with a symplectic symmetry and cyclic symmetries are presented. In section 3, the Lax matrix and nonlinear constraint are obtained for this general system. The Hamiltonian function is also presented. The r matrix is obtained in section 4, which gives the involution of conserved integrals. The independence of the conserved integrals which are enough for Liouville integrability is proved in section 5. In section 6, the specific results, most of which are known, for the 2×2 real AKNS system, the MKdV equation, the nonlinear Schrödinger equation, the $u(n)$ AKNS system and the n wave equation are derived from the general conclusions. Finally, in section 7, the results for the two-dimensional $C_n^{(1)}$ Toda equation are derived.

In order to illustrate the main idea of this paper, here we review the well-known results for the nonlinearization of the x-part of the 2×2 real AKNS system [5].

The x-part of the 2×2 real AKNS system is

$$\Phi_x = U(\lambda)\Phi \equiv \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi + \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \Phi. \tag{1}$$

Take $\lambda_1, \ldots, \lambda_r$ to be r distinct real numbers. Let $\Phi_\sigma = \begin{pmatrix} \phi_{1\sigma} \\ \phi_{2\sigma} \end{pmatrix}$ be a solution of (1) with $\lambda = \lambda_\sigma$. Then a finite-dimensional Lax matrix can be constructed as

$$L(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sum_{\sigma=1}^{r} \frac{1}{\lambda - \lambda_\sigma} \begin{pmatrix} -\phi_{1\sigma}\phi_{2\sigma} & \phi_{1\sigma}^2 \\ -\phi_{2\sigma}^2 & \phi_{1\sigma}\phi_{2\sigma} \end{pmatrix}. \tag{2}$$

It satisfies the Lax equation

$$L(\lambda)_x = [U(\lambda), L(\lambda)] \tag{3}$$

if and only if (u, v) satisfies the constraints

$$u = \sum_{\sigma=1}^{r} \phi_{1\sigma}^2, \qquad v = -\sum_{\sigma=1}^{r} \phi_{2\sigma}^2. \tag{4}$$

Under the constraints (4), the Lax equation (1) becomes

$$\phi_{1\sigma,x} = \lambda_\sigma \phi_{1\sigma} + \sum_{i=1}^{r} \phi_{1i}^2 \phi_{2\sigma}, \qquad \phi_{2\sigma,x} = -\lambda_\sigma \phi_{2\sigma} - \sum_{i=1}^{r} \phi_{2i}^2 \phi_{1\sigma}, \tag{5}$$

which is a system of nonlinear ODEs for $\phi_{j\sigma}$ ($j = 1, 2;\ \sigma = 1, 2, \ldots r$).

This system of ODEs is a Hamiltonian system with the standard Poisson bracket

$$\{f, g\} = \sum_{\sigma=1}^{r} \left(\frac{\partial f}{\partial \phi_{1\sigma}} \frac{\partial g}{\partial \phi_{2\sigma}} - \frac{\partial f}{\partial \phi_{2\sigma}} \frac{\partial g}{\partial \phi_{1\sigma}} \right), \tag{6}$$

and the Hamiltonian function

$$H^x = \sum_{i=1}^{r} \lambda_i \phi_{1i} \phi_{2i} + \frac{1}{2} \sum_{i=1}^{r} \phi_{1i}^2 \sum_{j=1}^{r} \phi_{2j}^2. \tag{7}$$

The involutive conserved integrals can be constructed from $L(\lambda)$. Expand

$$\operatorname{tr} L(\lambda)^{2k} = 2 \sum_{j=0}^{\infty} E_j^{(2k)} \lambda^{-j}, \tag{8}$$

then $\{E_j^{(2)} \mid j = 1, \ldots, r\}$ are in involution and are functionally independent in a dense open subset of $\boldsymbol{R}^{2r}$. In particular,

$$\begin{aligned} E_1^{(2)} &= -2 \sum_{i=1}^{r} \phi_{1i} \phi_{2i}, \\ E_2^{(2)} &= -2 \sum_{i=1}^{r} \lambda_i \phi_{1i} \phi_{2i} - \sum_{i=1}^{r} \phi_{1i}^2 \sum_{j=1}^{r} \phi_{2j}^2 + \left(\sum_{i=1}^{r} \phi_{1i} \phi_{2i} \right)^2, \end{aligned} \tag{9}$$

and H^x can be expressed by them as

$$H^x = -\frac{1}{2} E_2^{(2)} + \frac{1}{8} (E_1^{(2)})^2. \tag{10}$$

This means that $E_j^{(2)}$ are conserved integrals of the Hamiltonian system (5) determined by H^x. Hence (5) is an integrable Hamiltonian system in Liouville sense.

The above results are equivalent to those in the pioneering work [5]. A lot of work has been done for various equations after that. In this paper, we will give the corresponding results for a quite general Lax pair with a symplectic symmetry and cyclic symmetries which covers a lot of concrete equations with extra symmetries. The proofs will be given in a unified way.

2. Notations and the Lax pair with symmetries

Let W be a $2n \times 2n$ invertible antisymmetric real matrix which gives a symplectic structure on $\boldsymbol{R}^{2n}$.

Let

$$G = \{A \in GL(2n, \boldsymbol{C}) \mid A^T W A = W\}, \tag{11}$$

which is isomorphic to $Sp(n, \boldsymbol{C})$, the complex symplectic algebra. The inner automorphism group of G is $G/\{\pm I\}$. Let $p : G \to G/\{\pm I\}$ be the natural projection. Let

$$\boldsymbol{g} = \{X \in gl(2n, \boldsymbol{C}) \mid X^T = -W X W^{-1}\} \tag{12}$$

be the Lie algebra of G.

Let G_0 be a finite subgroup of G such that each of its element A satisfies $\bar{A} A = \pm I$. Here $\bar{A}$ is the complex conjugation (without transpose) of A.

Lemma 1. *$p(G_0)$ is a finite Abelian subgroup of $G/\{\pm I\}$. Therefore, for any $A, B \in G_0$, either $BA = AB$ or $BA = -AB$ holds.*

Proof. For any $A, B \in G_0$, $AB^{-1} \in G_0$. Hence $A^{-1}BAB^{-1} = \pm\overline{AB^{-1}}AB^{-1} = \pm I$, which implies $BA = \pm AB$. The lemma is proved. □

Suppose $\Omega_1, \ldots, \Omega_N \in G_0$ so that $p(\Omega_a)$ $(a = 1, \ldots, N)$ are generators of $p(G_0)$ and suppose the order of $p(\Omega_a) \in p(G_0)$ is m_a. Then, Ω_a satisfy

$$\Omega_a^T W \Omega_a = W, \qquad \bar{\Omega}_a = \pm\Omega_a^{-1}, \qquad \Omega_a^{m_a} = \pm I. \tag{13}$$

Let $\Sigma = \{(\alpha_1, \ldots, \alpha_N) \,|\, \alpha_a \in \boldsymbol{Z}\, (a = 1, \ldots, N)\}$, $\Sigma_0 = \{\alpha = (\alpha_1, \ldots, \alpha_N) \in \Sigma \,|\, 0 \leqslant \alpha_a < m_a\, (a = 1, \ldots, N)\}$, then we can write $\Omega^\alpha = \Omega_1^{\alpha_1} \cdots \Omega_N^{\alpha_N}$, etc. for multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \Sigma$. Denote m_0 to be the exponent of $p(G_0)$, which is the minimal common multiple of $m_1, \ldots, m_N$.

Let $\omega : G_0 \to S^1 = \{z \in \boldsymbol{C} \,|\, |z| = 1\}$ be a group homomorphism such that $\omega(\pm I) = 1$. For any $a = 1, \ldots, N$, denote $\omega_a = \omega(\Omega_a)$, then $\omega_a^{m_a} = 1$.

For any fixed integer k, denote

$$\mathcal{D}_k = \{X \in \boldsymbol{g} \,|\, \bar{X} = X,\ \Omega X \Omega^{-1} = \omega(\Omega)^k X \text{ for any } \Omega \in G_0\}, \tag{14}$$

then $[\mathcal{D}_j, \mathcal{D}_k] \subset \mathcal{D}_{j+k}$. Moreover, if $X \in \mathcal{D}_k$, then $X^{2j-1} \in \mathcal{D}_{(2j-1)k}$ for any positive integer j. Let $\mathcal{D} = \sum_{k=0}^{\infty} \mathcal{D}_k$, which is a real Lie subalgebra of $\boldsymbol{g}$.

Denote $G_0 \otimes S^1 = \{cg \,|\, c \in S^1, g \in G_0\}$. For given integer h, denote

$$\begin{aligned}\Theta_h = \{\theta \in G_0 \otimes S^1 | \,\bar{\theta} = \theta, \widetilde{\omega}(\theta) = 1, \theta^T = W\theta W^{-1},\\ \text{and } \Omega\theta\Omega^{-1} = \omega(\Omega)^h\theta \text{ for any } \Omega \in G_0\}.\end{aligned} \tag{15}$$

Here $\widetilde{\omega} : G_0 \otimes S^1 \to S^1$ is defined as $\widetilde{\omega}(cg) = \omega(g)$ for any $g \in G_0$ and $c \in S^1$. It is well-defined since $\omega(\pm I) = 1$. Moreover, $\Theta_{h'} = \Theta_h$ if $h' \equiv h \bmod m_0$.

Θ_h may be empty. However, Θ_0 is always non-empty since $I \in \Theta_0$. For $h \neq 0$, Θ_h is also useful for some nonlinear PDEs. (See the example of the nonlinear Schrödinger equation in section 6.3.)

Lemma 2.

(i) $\theta^2 = \pm I$ for any $\theta \in \Theta_h$.
(ii) $\Theta_h \neq \emptyset$ only if $2h \equiv 0 \bmod m_0$.
(iii) $AB = BA$ and $AB \in \Theta_h$ hold for any $A \in \Theta_0$, $B \in \Theta_h$.
(iv) $\theta X = X\theta$ and $\theta X \in \mathcal{D}_{h+k}$ hold for any $\theta \in \Theta_h$ and $X \in \mathcal{D}_k$.

Proof. Suppose $\theta = cg$ where $g \in G_0$ and $c \in S^1$, then by (11) and (15), $W = g^T W g = (WgW^{-1})Wg = Wg^2$, which implies $g^2 = I$ and then $\theta^2 = c^2 I$. Moreover, $\overline{\theta^2} = \theta^2$ and $c \in S^1$ implies $c^2 = \pm 1$. Hence (i) is true.

Following (i), (ii) holds since $\omega(\Omega)^{2h}\theta^2 = \Omega\theta^2\Omega^{-1} = \theta^2$ for any $\Omega \in G_0$.

Suppose $A \in \Theta_0$, $B \in \Theta_h$, then $BAB^{-1} = \widetilde{\omega}(B)^0 A = A$ implies $AB = BA$. Then it can be checked that $AB \in \Theta_h$ by definition (15). This proves (iii).

Suppose $\theta = cg \in \Theta_h$ where $g \in G_0$ and $c \in S^1$. Since $\omega(g) = 1$, we have $\theta X \theta^{-1} = gXg^{-1} = X$, i.e. $\theta X = X\theta$ for any $X \in \mathcal{D}_k$. Then $(\theta X)^T = X^T\theta^T = (-WXW^{-1})(W\theta W^{-1}) = -W(\theta X)W^{-1}$. Moreover, $\Omega\theta X\Omega^{-1} = \omega(\Omega)^{h+k}\theta X$ holds for any $\Omega \in G_0$. This proves (iv). The lemma is proved. □

For fixed integers p and h, let

$$\mathcal{F}_{p,h} = \left\{ f(\tau) = \sum_{j=1}^{s} \theta f_{s-j} \tau^{j-1} \,\middle|\, s \text{ is a positive integer, } \theta \in \Theta_h,\ f_{s-j} \in \boldsymbol{R}, \right. \left. \text{and } f_{s-j} \neq 0 \text{ holds only when } j \text{ is even and } h + j \equiv p + 1 \bmod m_0 \right\}. \tag{16}$$

Here the necessity of j being even when $f_{s-j} \neq 0$ guarantees that $K^{j-1} \in \mathcal{D}_{j-1}$ when $K \in \mathcal{D}_1$.

Note that $\mathcal{F}_{p',h'} = \mathcal{F}_{p,h}$ if $p' \equiv p \bmod m_0$ and $h' \equiv h \bmod m_0$.

Lemma 3.

(i) $p - h$ must be odd if m_0 is even and $\mathcal{F}_{p,h} \neq \{0\}$.
(ii) $[\mathcal{F}_{1,0}, \mathcal{F}_{p,h}] = 0$ holds for any integers p and h.
(iii) If $f \in \mathcal{F}_{p,h}$, then $f(K) \in \mathcal{D}_p$ when $K \in \mathcal{D}_1$.

Proof. Suppose $f(\tau) = \sum_{j=1}^{s} \theta f_{s-j} \tau^{j-1} \in \mathcal{F}_{p,h}$ and $f \neq 0$. (i) holds since $p - h \equiv j - 1 \bmod m_0$ and $j - 1$ is odd when $f_{s-j} \neq 0$. (ii) follows from (iii) of lemma 2. Now suppose $f_{s-j} \neq 0$, then j is even and $K^{j-1} \in \mathcal{D}_{j-1}$ since $K \in \mathcal{D}_1$. (iv) of lemma 2 implies $\theta f_{s-j} K^{j-1} \in \mathcal{D}_{h+j-1} = \mathcal{D}_p$ by the definition of $\mathcal{F}_{p,h}$. This proves (iii). The lemma is proved. □

Lemma 4. *Suppose $f \in \mathcal{F}_{1,0}$, $g \in \mathcal{F}_{p,h}$, then their composition $g \circ f \in \mathcal{F}_{p,h}$.*

Proof. Let

$$f(\tau) = \sum_{j=1}^{s} \theta_1 f_{s-j} \tau^{j-1}, \qquad g(\tau) = \sum_{k=1}^{t} \theta_2 g_{t-k} \tau^{k-1} \tag{17}$$

where $\theta_1 \in \Theta_0$, $\theta_2 \in \Theta_h$, $f_{s-j} \neq 0$ only if j is even and $j \equiv 2 \bmod m_0$, and $g_{t-k} \neq 0$ only if k is even and $k \equiv p + 1 - h \bmod m_0$. Then

$$\begin{aligned} g(f(\tau)) &= \sum_{k=1}^{t} \theta_2 g_{t-k} \left(\sum_{j=1}^{s} \theta_1 f_{s-j} \tau^{j-1} \right)^{k-1} \\ &= \sum_{k=1}^{t} \sum_{j_1=1}^{s} \cdots \sum_{j_{k-1}=1}^{s} \theta_2 \theta_1^{k-1} f_{s-j_1} \cdots f_{s-j_{k-1}} g_{t-k} \tau^{(j_1-1)+\cdots(j_{k-1}-1)}. \end{aligned} \tag{18}$$

A term in the above summation is non-zero only if $k, j_1, \ldots, j_{k-1}$ are all even, $j_1, \ldots, j_{k-1} \equiv 2 \bmod m_0$ and $k \equiv p + 1 - h \bmod m_0$. Then, $(j_1 - 1) + \cdots + (j_{k-1} - 1) + 1$ is even and $(j_1 - 1) + \cdots + (j_{k-1} - 1) + 1 \equiv p + 1 - h \bmod m_0$. Moreover, (iii) of lemma 2 implies that $\theta_2 \theta_1^{k-1} \in \Theta_h$. Hence $g(f(\tau)) \in \mathcal{F}_{p,h}$. □

The space $\mathcal{F}_{p,h}$ will be used in constructing nonlinear constraint in the next section.

In this paper, we will consider the linear system

$$\Phi_x = U(x, \lambda)\Phi \tag{19}$$

where

$$U(x, \lambda) = \sum_{j=0}^{p} U_j(x) \lambda^{p-j} \tag{20}$$

with $U_j \in \mathcal{D}_{p-j}$ $(j = 0, 1, \ldots, p)$. Equivalently, $U(x, \lambda)$ satisfies

$$\begin{aligned} &\overline{U(\lambda)} = U(\bar{\lambda}), && U(\lambda)^T = -WU(\lambda)W^{-1}, \\ &\Omega_a U(\lambda)\Omega_a^{-1} = U(\omega_a \lambda) && (a = 1, \ldots, N). \end{aligned} \tag{21}$$

Here the first equation in (21) means that the coefficients of λ in $U(\lambda)$ are real. The second equation and the third equation mean that $U(\lambda)$ satisfies a symplectic symmetry and cyclic symmetries, respectively.

The linear system (19) with symmetries (21) consists of many Lax pairs in 1+1 dimensions. We will consider this general system in the following sections 3, 4 and 5. The general results to this linear system can be used for some specific integrable systems which will be shown in sections 6 and 7.

3. Lax matrix and nonlinear constraint

Let $\lambda_1, \ldots, \lambda_r$ be non-zero real numbers such that λ_j^2 are distinct. For $\sigma = 1, \ldots, r$, let $\Phi_\sigma = (\phi_{1\sigma}, \ldots, \phi_{2n,\sigma})^T$ be a real column solution of the linear system

$$\Phi_{\sigma,x} = U(x, \lambda_\sigma)\Phi_\sigma. \tag{22}$$

We will construct a finite-dimensional Lax matrix first. For given $K \in \mathcal{D}_1$, let

$$L(\lambda) = K + \kappa \sum_{\alpha\in\Sigma_0} \sum_{\sigma=1}^{r} \frac{\Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T W}{\lambda - \omega^\alpha \lambda_\sigma} \tag{23}$$

where κ is a real constant. This construction has already been used in [27, 29] and is similar to those used in constructing Darboux transformations [15, 18, 30].

Lemma 5. *$L(\lambda)$ satisfies*

$$L(\bar{\lambda}) = \overline{L(\lambda)}, \tag{24}$$

$$(L(\lambda))^T = -WL(\lambda)W^{-1}, \tag{25}$$

$$\Omega_a L(\lambda)\Omega_a^{-1} = \omega_a L(\omega_a \lambda), \qquad a = 1, 2, \ldots, N. \tag{26}$$

Proof. Owing to (13), suppose $\bar{\Omega}_a = \varepsilon_a \Omega_a^{-1}$ with $\varepsilon_a = \pm 1$. (24) holds since $\overline{\Omega^\alpha} = \varepsilon^\alpha \Omega^{-\alpha}$, $\overline{\omega^\alpha} = \omega^{-\alpha}$ and $\varepsilon^{2\alpha} = 1$.

With $W^T = -W$, (25) follows from

$$\begin{aligned} (L(\lambda))^T &= K^T - \kappa \sum_{\alpha\in\Sigma_0} \sum_{\sigma=1}^{r} \frac{W\Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T}{\lambda - \omega^\alpha \lambda_\sigma} \\ &= -WKW^{-1} - \kappa W \sum_{\alpha\in\Sigma_0} \sum_{\sigma=1}^{r} \frac{\Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T W}{\lambda - \omega^\alpha \lambda_\sigma} W^{-1} \\ &= -WL(\lambda)W^{-1}. \end{aligned} \tag{27}$$

To prove (26), we have

$$\begin{aligned} L(\omega_a \lambda) &= K + \kappa \sum_{\alpha\in\Sigma_0} \sum_{\sigma=1}^{r} \frac{\Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T W}{\omega_a \lambda - \omega^\alpha \lambda_\sigma} \\ &= K + \omega_a^{-1} \kappa \Omega_a \sum_{\alpha\in\Sigma_0} \sum_{\sigma=1}^{r} \frac{\Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T \Omega_a^T W \Omega_a}{\lambda - \omega^\alpha \lambda_\sigma} \Omega_a^{-1} \\ &= \omega_a^{-1} \Omega_a L(\lambda) \Omega_a^{-1}. \end{aligned} \tag{28}$$

Here we have shifted α_a to α_a+1 at the second equality in (28). The last equality follows from $\Omega_a K \Omega_a^{-1} = \omega_a K$ and $\Omega_a^T W \Omega_a = W$. The lemma is proved. □

By lemma 5, $L(\lambda) \in \mathcal{D}$ for any $\lambda \in \boldsymbol{R}$. Moreover, if $L(\lambda)$ is expanded as $L(\lambda) = \sum_{j=0}^{\infty} \lambda^{-j} L_j$ with $L_0 = K$, then $L_j \in \mathcal{D}_{1-j}$.

Corollary 1. *Suppose $f \in \mathcal{F}_{p,h}$ $(p \geqslant 1)$, then $f(L(\lambda))$ satisfies*

$$f(L(\bar{\lambda})) = \overline{f(L(\lambda))}, \tag{29}$$

$$f(L(\lambda))^T = -W f(L(\lambda)) W^{-1}, \tag{30}$$

$$\Omega_a f(L(\lambda)) \Omega_a^{-1} = \omega_a^p f(L(\omega_a \lambda)), \qquad a = 1, 2, \ldots, N. \tag{31}$$

Proof. Suppose $f(\tau) = \sum_{j=1}^{s} \theta f_{s-j} \tau^{j-1}$ where $\theta \in \Theta_h$, $f_{s-j} \in \boldsymbol{R}$, then we can check that $f(L(\lambda)) = \sum_{j=1}^{s} \theta f_{s-j} (L(\lambda))^{j-1}$ satisfies (29)–(31) by lemma 5 and the definition of Θ_h and $\mathcal{F}_{p,h}$. This proves the corollary. □

For a Laurent series $N(\lambda) = \sum_{j=-\infty}^{n} N_j \lambda^j$, define

$$N(\lambda)_+ = \sum_{j=0}^{n} N_j \lambda^j, \qquad N(\lambda)_- = \sum_{j=-\infty}^{-1} N_j \lambda^j. \tag{32}$$

Write $M(\lambda) = f(L(\lambda))$ and expand it as

$$M(\lambda) = \sum_{j=0}^{\infty} M_j \lambda^{-j} \tag{33}$$

with $M_0 = f(K)$. Corollary 1 implies that $M_j \in \mathcal{D}_{p-j}$ if $f \in \mathcal{F}_{p,h}$.

Theorem 1. *Suppose $f \in \mathcal{F}_{p,h}$ $(p \geqslant 1)$, then $L(\lambda)$ satisfies*

$$L(\lambda)_x = [U(\lambda), L(\lambda)] \tag{34}$$

under the constraint $U(\lambda) = \widetilde{U}(\lambda)$ where

$$\widetilde{U}(\lambda) = \left(\lambda^p f(L(\lambda))\right)_+. \tag{35}$$

Moreover, $\widetilde{U}(\lambda) = \sum_{j=0}^{p} \widetilde{U}_j \lambda^{p-j}$ satisfies $\widetilde{U}_j \in \mathcal{D}_{p-j}$.

Proof. Using (13), (21) and (23),

$$(\Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T W)_x = [U(\omega^\alpha \lambda_\sigma), \Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T W]. \tag{36}$$

Hence

$$\begin{aligned} -L(\lambda)_x + [U(\lambda), L(\lambda)] &= -\sum_{j=0}^{p} [K, \lambda^{p-j} U_j] \\ &\quad + \kappa \sum_{\alpha \in \Sigma_0} \sum_{\sigma=1}^{r} \sum_{j=0}^{p} \frac{\lambda^{p-j} - (\omega^\alpha \lambda_\sigma)^{p-j}}{\lambda - \omega^\alpha \lambda_\sigma} [U_j, \Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T W] \end{aligned} \tag{37}$$

is a polynomial of λ. On the other hand, since $[f(L(\lambda)), L(\lambda)] = 0$, we have

$$\begin{aligned} \left(-L(\lambda)_x + [\widetilde{U}(\lambda), L(\lambda)]\right)_+ &= \left[\left(\lambda^p f(L(\lambda))\right)_+, L(\lambda)\right]_+ \\ &= -\left[\left(\lambda^p f(L(\lambda))\right)_-, L(\lambda)\right]_+ = 0. \end{aligned} \tag{38}$$

Hence, $L(\lambda)_x = [U(\lambda), L(\lambda)]$ holds identically. Moreover, corollary 1 implies that $\widetilde{U}_j \in \mathcal{D}_{p-j}$. The theorem is proved. □

Therefore, L satisfies the Lax equation (34) if $U(\lambda)$ satisfies the constraint $U(\lambda) = \left(\lambda^p f(L(\lambda))\right)_+$.

With the above constraint, (22) becomes a system of nonlinear ODEs

$$\Phi_{\sigma,x} = \left(\lambda^p f(L(\lambda_\sigma))\right)_+ \Phi_\sigma. \tag{39}$$

Theorem 2. *Suppose $f \in \mathcal{F}_{p,h}$ $(p \geqslant 1)$, then (39) is a Hamiltonian system with the Hamiltonian function*

$$H = \frac{1}{2\kappa m_1 \cdots m_N} \operatorname{tr}\left(\operatorname{Res} \lambda^p F(L(\lambda))\right) \tag{40}$$

where F is a matrix-valued polynomial satisfying $F'(\tau) = f(\tau)$ and $F(0) = 0$.

Proof. Expand the Lax matrix $L(\lambda)$ as

$$L(\lambda) = \sum_{j=0}^{\infty} \lambda^{-j} L_j \tag{41}$$

where

$$L_0 = K, \qquad L_j = \kappa \sum_{\alpha\in\Sigma_0} \sum_{\sigma=1}^{r} (\omega^\alpha \lambda_\sigma)^{j-1} \Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T W \quad (j \geqslant 1). \tag{42}$$

Using the expression $f(\tau) = \sum_{l=1}^{s} \theta f_{s-l} \tau^{l-1}$, we have $F(\tau) = \sum_{l=1}^{s} \frac{1}{l} \theta f_{s-l} \tau^l$.

Denote $\hat{W} = W^{-1}$, then

$$\begin{aligned}
2\kappa m_1 \cdots m_N \sum_{k=1}^{2n} \hat{W}_{jk} \frac{\partial H}{\partial \phi_{k\sigma}} &= \kappa \sum_{\alpha\in\Sigma_0} \sum_{a,b,k=1}^{2n} \sum_{\mu=1}^{+\infty} \operatorname{Res}\Big(\lambda^{p-\mu} \big(f(L(\lambda))\big)_{ab} (\omega^\alpha \lambda_\sigma)^{\mu-1} \\
&\quad \cdot \hat{W}_{jk}\big((\Omega^\alpha \Phi_\sigma)_b ((\Omega^\alpha)^T W)_{ka} + (\Omega^\alpha)_{bk} (\Phi_\sigma^T (\Omega^\alpha)^T W)_a\big)\Big) \\
&= \kappa \sum_{\alpha\in\Sigma_0} \sum_{a,b=1}^{2n} \sum_{\mu=1}^{+\infty} \operatorname{Res}\Big(\lambda^{p-\mu} \big(f(L(\lambda))\big)_{ab} (\omega^\alpha \lambda_\sigma)^{\mu-1} \\
&\quad \cdot \big((\Omega^\alpha \Phi_\sigma)_b (\Omega^{-\alpha})_{ja} + (\Omega^{-\alpha} W^{-1})_{jb} (\Phi_\sigma^T (\Omega^\alpha)^T W)_a\big)\Big) \\
&= 2\kappa \sum_{\alpha\in\Sigma_0} \sum_{\mu=1}^{+\infty} \operatorname{Res}\Big(\lambda^{p-\mu} (\omega^\alpha \lambda_\sigma)^{\mu-1} \big(\Omega^{-\alpha} f(L(\lambda)) \Omega^\alpha \Phi_\sigma\big)_j\Big).
\end{aligned} \tag{43}$$

Here we have used (13) and (30). Expand

$$f(L(\lambda)) = \sum_{\nu=0}^{\infty} M_\nu \lambda^{-\nu} \tag{44}$$

as in (33), then $M_j \in \mathcal{D}_{p-j}$, and

$$\begin{aligned}
2\kappa m_1 \cdots m_N \sum_{k=1}^{2n} \hat{W}_{jk} \frac{\partial H}{\partial \phi_{k\sigma}} &= 2\kappa \sum_{\alpha\in\Sigma_0} \sum_{\mu=1}^{+\infty} \sum_{\nu=0}^{+\infty} \operatorname{Res}\Big(\lambda^{p-\mu-\nu} (\omega^\alpha \lambda_\sigma)^{\mu-1} \big(\Omega^{-\alpha} M_\nu \Omega^\alpha \Phi_\sigma\big)_j\Big) \\
&= 2\kappa \sum_{\alpha\in\Sigma_0} \sum_{\nu=0}^{p} (\omega^\alpha \lambda_\sigma)^{p-\nu} \big(\Omega^{-\alpha} M_\nu \Omega^\alpha \Phi_\sigma\big)_j
\end{aligned}$$

$$= 2\kappa m_1 \cdots m_N \sum_{\nu=0}^{p} \lambda_\sigma^{p-\nu} \big(M_\nu \Phi_\sigma\big)_j$$

$$= 2\kappa m_1 \cdots m_N \Big(\big(\lambda^p f(L(\lambda))\big)_+ \Big|_{\lambda=\lambda_\sigma} \Phi_\sigma \Big)_j . \tag{45}$$

Therefore, the Hamiltonian equations given by the Hamiltonian function (40) are just (39). The theorem is proved. □

In many concrete integrable systems, say, the nonlinear Schrödinger equation in real form, the condition $U_{p-1} \in \mathcal{D}_{p-1} \cap (\ker \operatorname{ad} K)^\perp$ is needed where $\perp$ refers to the orthogonal compliment with respect to the Killing form $\langle \cdot, \cdot \rangle$ of $\boldsymbol{g}$. However, usually $M_1 \in \mathcal{D}_{p-1} \cap (\ker \operatorname{ad} K)^\perp$ is not guaranteed when $\mathcal{D}_{p-1} \cap \ker \operatorname{ad} K \neq \{0\}$. This problem can be solved with the help of the following theorem 3. Before that, we need an algebraic lemma.

Lemma 6. *Let*

$$\mathcal{M} = \begin{vmatrix} 1 & \mu_1 & \mu_1^2 & \cdots & \mu_1^{2n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \mu_n & \mu_n^2 & \cdots & \mu_n^{2n-1} \\ 0 & 1 & 2\mu_1 & \cdots & (2n-1)\mu_1^{2n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2\mu_n & \cdots & (2n-1)\mu_n^{2n-2} \end{vmatrix}, \tag{46}$$

then

$$\mathcal{M} = (-1)^{n(n-1)/2} \prod_{1 \leqslant j < k \leqslant n} (\mu_k - \mu_j)^4. \tag{47}$$

Proof. Denote $f(x) = (1, x, x^2, \ldots, x^{2n-1})^T$,

$$\mathcal{M}^{(j,k)}(x) = \det \left(\frac{\mathrm{d}^j f}{\mathrm{d}x^j}(x), f(\mu_2), \ldots, f(\mu_n), \frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x), \frac{\mathrm{d}f}{\mathrm{d}x}(\mu_2), \ldots, \frac{\mathrm{d}f}{\mathrm{d}x}(\mu_n) \right). \tag{48}$$

Then $\mathcal{M}^{(0,1)}(\mu_1) = \mathcal{M}$, $\mathcal{M}^{(0,k)}(\mu_2) = 0$, $\mathcal{M}^{(1,k)}(\mu_2) = 0$ and $\mathcal{M}^{(k,k)}(x) = 0$ for any $k \geqslant 1$. Moreover, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{M}^{(0,1)}(x) = \mathcal{M}^{(0,2)}(x), \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{M}^{(0,1)}(x) = \mathcal{M}^{(0,3)}(x) + \mathcal{M}^{(1,2)}(x), \\ &\frac{\mathrm{d}^3}{\mathrm{d}x^3}\mathcal{M}^{(0,1)}(x) = \mathcal{M}^{(0,4)}(x) + 2\mathcal{M}^{(1,3)}(x). \end{aligned} \tag{49}$$

This implies $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\mathcal{M}^{(0,1)}(\mu_2) = 0$ for $k = 0, 1, 2, 3$. Since $\mathcal{M}^{(0,1)}(x)$ is a polynomial of x, $\mathcal{M}$ must be of the form $(\mu_2 - \mu_1)^4 F_1(\mu_1, \ldots, \mu_n)$ where F_1 is a polynomial. Owing to the symmetry, $\mathcal{M} = \prod_{1 \leqslant j < k \leqslant n} (\mu_k - \mu_j)^4 F_2(\mu_1, \ldots, \mu_n)$ where F_2 is another polynomial. However, regarded as a polynomial of μ_1, $\mathcal{M}$ is of degree $4n-4$. Hence F_2 must be a constant. Comparing the coefficient of $\prod_{k=2}^n \mu_k^{2k-4}$, we obtain $F_2 = (-1)^{n(n-1)/2}$. The lemma is proved. □

Theorem 3. *Suppose $K \in \mathcal{D}_1$ is diagonalizable, $f \in \mathcal{F}_{p,h}$ $(p \geqslant 1)$. Expand $M(\lambda) = f(L(\lambda))$ as in (33) where $L(\lambda)$ is given by (23). Then there exists a polynomial ζ such that $\widetilde{M}(\lambda) \equiv \zeta(M(\lambda)) = M_0 + \lambda^{-1}\widetilde{M}_1 + o(\lambda^{-1})$ with $\widetilde{M}_1 \in \mathcal{D}_{p-1} \cap (\ker \operatorname{ad} K)^\perp$ and $M_1 - \widetilde{M}_1 \in \mathcal{D}_{p-1} \cap \ker \operatorname{ad} K$.*

Proof. Let $K = T\Lambda T^{-1}$ where Λ is a complex diagonal matrix and T is a complex invertible matrix. Let $\widetilde{m}_0 = m_0$ if m_0 is even and $\widetilde{m}_0 = 2m_0$ if m_0 is odd. Let $\mu_1, \ldots, \mu_l$ be all the distinct eigenvalues of $\Lambda^{\widetilde{m}_0}$. By lemma 6, there is a unique complex solution ζ_j $(j = 0, 1, \ldots, 2l-1)$ of the linear system

$$\sum_{k=0}^{2l-1} \zeta_k \mu_j^k = 0, \qquad \widetilde{m}_0 \mu_j \sum_{k=0}^{2l-1} k\zeta_k \mu_j^{k-1} = 1 \quad (j = 1, \ldots, l). \tag{50}$$

Then

$$\sum_{k=0}^{2l-1} \zeta_k K^{k\widetilde{m}_0} = 0, \qquad \widetilde{m}_0 \sum_{k=0}^{2l-1} k\zeta_k K^{k\widetilde{m}_0} = I. \tag{51}$$

Since K is real and ζ_j are unique, ζ_j must be real.

Let $\zeta(\tau) = \tau - \sum_{k=0}^{2l-1} \zeta_k \tau^{k\widetilde{m}_0+1}$, then $\zeta \in \mathcal{F}_{1,0}$ since $\widetilde{m}_0$ is always even, and $\zeta(K) = K$. For any $H \in \ker \operatorname{ad} K$,

$$\begin{aligned}
\langle H, \widetilde{M} - M\rangle &= -\sum_{k=0}^{2l-1} \zeta_k \langle H, M^{k\widetilde{m}_0+1}\rangle \\
&= -\sum_{k=0}^{2l-1} \zeta_k \left\langle H, K^{k\widetilde{m}_0+1} + \lambda^{-1} \sum_{j=0}^{k\widetilde{m}_0} K^j M_1 K^{k\widetilde{m}_0-j} \right\rangle + o(\lambda^{-1}) \\
&= -\sum_{k=0}^{2l-1} \langle H, \zeta_k K^{k\widetilde{m}_0+1} + \lambda^{-1}(k\widetilde{m}_0+1)\zeta_k K^{k\widetilde{m}_0} M_1\rangle + o(\lambda^{-1}) \\
&= -\lambda^{-1}\langle H, M_1\rangle + o(\lambda^{-1}).
\end{aligned} \tag{52}$$

Comparing the coefficients of λ^{-1}, we have $\langle H, \widetilde{M}_1\rangle = 0$. Since $\zeta \in \mathcal{F}_{1,0}$, lemma 4 and corollary 1 imply that $\widetilde{M}_1 \in \mathcal{D}_{p-1}$. Hence $\widetilde{M}_1 \in \mathcal{D}_{p-1} \cap (\ker \operatorname{ad} K)^{\perp}$.

On the other hand,

$$\begin{aligned}
[K, \widetilde{M} - M] &= -\sum_{k=0}^{2l-1} \zeta_k \left[K, K^{k\widetilde{m}_0+1} + \lambda^{-1} \sum_{j=0}^{k\widetilde{m}_0} K^j M_1 K^{k\widetilde{m}_0-j} \right] + o(\lambda^{-1}) \\
&= -\lambda^{-1} \sum_{k=0}^{2l-1} [\zeta_k K^{k\widetilde{m}_0+1}, M_1] + o(\lambda^{-1}) = o(\lambda^{-1}).
\end{aligned} \tag{53}$$

Comparing the coefficients of λ^{-1}, we obtain $M_1 - \widetilde{M}_1 \in \mathcal{D}_{p-1} \cap \ker \operatorname{ad} K$. The theorem is proved. □

Owing to lemma 4 and theorem 3, we can always want $U_1 \in \mathcal{D}_{p-1} \cap (\ker \operatorname{ad} K)^{\perp}$ if necessary, by replacing f with $f \circ \zeta$. However, when ζ is complicated, the Hamiltonian function need not be calculated from theorem 2. Instead, it is simpler to integrate it from the Hamiltonian equations directly. In this case, theorem 2 is still important because it shows that the Hamiltonian function is expressed by the Lax matrix, which is essential in the proof of Liouville integrability.

4. r matrix

In $\boldsymbol{R}^{2n\times r}$ with coordinates $\phi_{j\sigma}$ $(j = 1, \ldots, 2n; \sigma = 1, \ldots, r)$, define the symplectic form

$$\sum_{j,k=1}^{2n} \sum_{\sigma=1}^{r} W_{jk} \mathrm{d}\phi_{j\sigma} \wedge \mathrm{d}\phi_{k\sigma}. \tag{54}$$

Then, for any two smooth functions f and g, their Poisson bracket is

$$\{f, g\} = \sum_{j,k=1}^{2n} \sum_{\sigma=1}^{r} \hat{W}_{jk} \frac{\partial f}{\partial \phi_{j\sigma}} \frac{\partial g}{\partial \phi_{k\sigma}} \tag{55}$$

with $\hat{W} = W^{-1}$.

Theorem 4. *For any* $\lambda, \mu \in \boldsymbol{C}$,

$$\{L_{ab}(\lambda), L_{cd}(\mu)\} = [r_1(\lambda,\mu), L(\lambda)\otimes I]_{abcd} + [r_2(\lambda,\mu), I\otimes L(\mu)]_{abcd} \tag{56}$$

holds where the Poisson bracket is given by (55) and

$$\begin{aligned}
(r_1(\lambda,\mu))_{abcd} &= \sum_{\gamma\in\Sigma_0} \frac{\kappa}{\mu-\omega^\gamma\lambda}\Big((\Omega^{-\gamma})_{ad}(\Omega^\gamma)_{cb} - (\Omega^{-\gamma}W^{-1})_{ac}(W\Omega^\gamma)_{db}\Big)\\
(r_2(\lambda,\mu))_{abcd} &= \sum_{\gamma\in\Sigma_0} \frac{\kappa\omega^\gamma}{\mu-\omega^\gamma\lambda}\Big((\Omega^{-\gamma})_{ad}(\Omega^\gamma)_{cb} - (\Omega^{-\gamma}W^{-1})_{ac}(W\Omega^\gamma)_{db}\Big)\\
&= -(r_1(\mu,\lambda))_{cdab}.
\end{aligned} \tag{57}$$

Here

$$[A, B]_{abcd} = \sum_{p,q=1}^{2n} (A_{apcq}B_{pbqd} - B_{apcq}A_{pbqd}) \tag{58}$$

for any two $(2n)^2 \times (2n)^2$ matrices A and B.

Proof. Written in components,

$$L_{ab}(\lambda) = K_{ab} + \sum_{\alpha\in\Sigma_0}\sum_{\sigma=1}^{r}\sum_{f,g,h=1}^{2n} \frac{\kappa(\Omega^\alpha)_{af}\phi_{f\sigma}\phi_{g\sigma}(\Omega^\alpha)_{hg}W_{hb}}{\lambda-\omega^\alpha\lambda_\sigma}, \tag{59}$$

$$\frac{\partial L_{ab}(\lambda)}{\partial\phi_{j\sigma}} = \sum_{\alpha\in\Sigma_0}\sum_{f,h=1}^{2n}\left(\frac{\kappa(\Omega^\alpha)_{aj}\phi_{f\sigma}(\Omega^\alpha)_{hf}W_{hb}}{\lambda-\omega^\alpha\lambda_\sigma} + \frac{\kappa(\Omega^\alpha)_{af}\phi_{f\sigma}(\Omega^\alpha)_{hj}W_{hb}}{\lambda-\omega^\alpha\lambda_\sigma}\right), \tag{60}$$

$$\frac{\partial L_{cd}(\mu)}{\partial\phi_{k\sigma}} = \sum_{\beta\in\Sigma_0}\sum_{q,r=1}^{2n}\left(\frac{\kappa(\Omega^\beta)_{ck}\phi_{q\sigma}(\Omega^\beta)_{rq}W_{rd}}{\mu-\omega^\beta\lambda_\sigma} + \frac{\kappa(\Omega^\beta)_{cq}\phi_{q\sigma}(\Omega^\beta)_{rk}W_{rd}}{\mu-\omega^\beta\lambda_\sigma}\right). \tag{61}$$

The Poisson bracket is

$$\begin{aligned}
\Delta_{abcd} &\equiv \{L_{ab}(\lambda), L_{cd}(\mu)\} = \sum_{\sigma=1}^{r}\sum_{j,k=1}^{2n}\hat{W}_{jk}\frac{\partial L_{ab}(\lambda)}{\partial\phi_{j\sigma}}\frac{\partial L_{cd}(\mu)}{\partial\phi_{k\sigma}}\\
&= \sum_{\sigma=1}^{r}\sum_{\alpha,\beta\in\Sigma_0}\sum_{j,k,f,h,q,r=1}^{2n}\kappa^2\left(\frac{\omega^\alpha}{\lambda-\omega^\alpha\lambda_\sigma} - \frac{\omega^\beta}{\mu-\omega^\beta\lambda_\sigma}\right)\frac{\phi_{f\sigma}\phi_{q\sigma}}{\omega^\alpha\mu-\omega^\beta\lambda}\hat{W}_{jk}\\
&\quad\cdot\Big((\Omega^\alpha)_{aj}(\Omega^\alpha)_{hf}W_{hb} + (\Omega^\alpha)_{af}(\Omega^\alpha)_{hj}W_{hb}\Big)\\
&\quad\cdot\Big((\Omega^\beta)_{ck}(\Omega^\beta)_{rq}W_{rd} + (\Omega^\beta)_{cq}(\Omega^\beta)_{rk}W_{rd}\Big)\\
&= \kappa^2\sum_{\sigma=1}^{r}\sum_{\alpha,\beta\in\Sigma_0}\left(\frac{1}{\mu-\omega^{\beta-\alpha}\lambda}\frac{1}{\lambda-\omega^\alpha\lambda_\sigma} + \frac{1}{\lambda-\omega^{\alpha-\beta}\mu}\frac{1}{\mu-\omega^\beta\lambda_\sigma}\right)D_{abcd\alpha\beta}
\end{aligned} \tag{62}$$

where

$$D_{abcd\alpha\beta} = (\Omega^{\alpha-\beta}W^{-1})_{ac}(W^T\Omega^\alpha\Phi_\sigma\Phi_\sigma^T(\Omega^\beta)^TW)_{bd} + (\Omega^{\alpha-\beta})_{ad}(W^T\Omega^\alpha\Phi_\sigma\Phi_\sigma^T(\Omega^\beta)^T)_{bc}$$
$$- (\Omega^{\beta-\alpha})_{cb}(\Omega^\alpha\Phi_\sigma\Phi_\sigma^T(\Omega^\beta)^TW)_{ad} - (W\Omega^{\alpha-\beta})_{bd}(\Omega^\alpha\Phi_\sigma\Phi_\sigma^T(\Omega^\beta)^T)_{ac}. \tag{63}$$

Here we have used (13).

Let $\gamma = \beta - \alpha$, $\Pi_\alpha^{(\sigma)} = \Omega^\alpha\Phi_\sigma\Phi_\sigma^T(\Omega^\alpha)^TW$, then $(\Pi_\alpha^{(\sigma)})^T = -W\Pi_\alpha^{(\sigma)}W^{-1}$, $\Pi_\beta^{(\sigma)} = \Omega^\gamma\Pi_\alpha^{(\sigma)}\Omega^{-\gamma}$.

Written in terms of $\Pi_\alpha^{(\sigma)}$,

$$D_{abcd\alpha\beta} = -(\Omega^{-\gamma}W^{-1})_{ac}(W\Omega^\gamma\Pi_\alpha^{(\sigma)})_{db} + (\Omega^{-\gamma})_{ad}(\Omega^\gamma\Pi_\alpha^{(\sigma)})_{cb}$$
$$- (\Omega^\gamma)_{cb}(\Pi_\alpha^{(\sigma)}\Omega^{-\gamma})_{ad} + (W\Omega^\gamma)_{db}(\Pi_\alpha^{(\sigma)}\Omega^{-\gamma}W^{-1})_{ac}. \tag{64}$$

On the other hand, written in terms of $\Pi_\beta^{(\sigma)}$,

$$D_{abcd\alpha\beta} = -(\Omega^{-\gamma}W^{-1})_{ac}(W\Omega^{-\gamma}\Pi_\beta^{(\sigma)})_{bd} + (\Omega^{-\gamma})_{ad}(\Pi_\beta^{(\sigma)}\Omega^\gamma)_{cb}$$
$$- (\Omega^\gamma)_{cb}(\Omega^{-\gamma}\Pi_\beta^{(\sigma)})_{ad} + (W\Omega^\gamma)_{db}(\Pi_\beta^{(\sigma)}\Omega^\gamma W^{-1})_{ca}. \tag{65}$$

Hence

$$\Delta_{abcd} = \sum_{\gamma\in\Sigma_0}\sum_{l=1}^{2n}\frac{\kappa}{\mu-\omega^\gamma\lambda}$$
$$\cdot\Big(- (\Omega^{-\gamma}W^{-1})_{ac}(W\Omega^\gamma(L(\lambda)-K))_{db} + (\Omega^{-\gamma})_{ad}(\Omega^\gamma(L(\lambda)-K))_{cb}$$
$$- (\Omega^\gamma)_{cb}((L(\lambda)-K)\Omega^{-\gamma})_{ad} + (W\Omega^\gamma)_{db}((L(\lambda)-K)\Omega^{-\gamma}W^{-1})_{ac}\Big)$$
$$+ \sum_{\gamma\in\Sigma_0}\sum_{l=1}^{2n}\frac{\kappa\omega^\gamma}{\mu-\omega^\gamma\lambda}$$
$$\cdot\Big(- (\Omega^{-\gamma}W^{-1})_{ac}(W\Omega^{-\gamma}(L(\mu)-K))_{bd} + (\Omega^{-\gamma})_{ad}((L(\mu)-K)\Omega^\gamma)_{cb}$$
$$- (\Omega^\gamma)_{cb}(\Omega^{-\gamma}(L(\mu)-K))_{ad} + (W\Omega^\gamma)_{db}((L(\mu)-K)\Omega^\gamma W^{-1})_{ca}\Big). \tag{66}$$

In Δ_{abcd}, the terms with K_{jk} are

$$\sum_{\gamma\in\Sigma_0}\frac{\kappa}{\mu-\omega^\gamma\lambda}\Big((\Omega^{-\gamma}W^{-1})_{ac}(W\Omega^\gamma K)_{db} - (\Omega^{-\gamma})_{ad}(\Omega^\gamma K)_{cb}$$
$$+ (\Omega^\gamma)_{cb}(K\Omega^{-\gamma})_{ad} - (W\Omega^\gamma)_{db}(K\Omega^{-\gamma}W^{-1})_{ac}\Big)$$
$$+ \sum_{\gamma\in\Sigma_0}\frac{\kappa\omega^\gamma}{\mu-\omega^\gamma\lambda}\Big((\Omega^{-\gamma}W^{-1})_{ac}(W\Omega^{-\gamma}K)_{bd} - (\Omega^{-\gamma})_{ad}(K\Omega^\gamma)_{cb}$$
$$+ (\Omega^\gamma)_{cb}(\Omega^{-\gamma}K)_{ad} - (W\Omega^\gamma)_{db}(K\Omega^\gamma W^{-1})_{ac}\Big) = 0, \tag{67}$$

in which we have used the relations in (13) and the fact $K \in \mathcal{D}_1$. Hence

$$\Delta_{abcd} = \sum_{\gamma\in\Sigma_0}\sum_{l=1}^{2n}\frac{\kappa}{\mu-\omega^\gamma\lambda}\Big(- (\Omega^{-\gamma}W^{-1})_{ac}(W\Omega^\gamma)_{dl}L_{lb}(\lambda) + (\Omega^{-\gamma})_{ad}(\Omega^\gamma)_{cl}L_{lb}(\lambda)$$
$$- L_{al}(\lambda)(\Omega^{-\gamma})_{ld}(\Omega^\gamma)_{cb} + L_{al}(\lambda)(\Omega^{-\gamma}W^{-1})_{lc}(W\Omega^\gamma)_{db}\Big)$$
$$+ \sum_{\gamma\in\Sigma_0}\sum_{l=1}^{2n}\frac{\kappa\omega^\gamma}{\mu-\omega^\gamma\lambda}\Big(- (\Omega^{-\gamma}W^{-1})_{ac}(W\Omega^\gamma)_{lb}L_{ld}(\mu) + (\Omega^{-\gamma})_{al}(\Omega^\gamma)_{cb}L_{ld}(\mu)$$
$$- L_{cl}(\mu)(\Omega^{-\gamma})_{ad}(\Omega^\gamma)_{lb} + L_{cl}(\mu)(\Omega^{-\gamma}W^{-1})_{al}(W\Omega^\gamma)_{db}\Big), \tag{68}$$

which is the result of the theorem. □

From theorem 4, it is easy to derive

Theorem 5. *Suppose* θ_1 *and* θ_2 *are constant* $2n \times 2n$ *matrices such that* $[\theta_j, L(\lambda)] = 0$ $(j = 1, 2)$*, then* $\{\mathrm{tr}(\theta_1 L(\lambda)^k), \mathrm{tr}(\theta_2 L(\mu)^l)\} = 0$ *holds for any positive integers k and l, and complex numbers* λ *and* μ.

Proof. According to theorem 4,

$$\begin{aligned}\frac{1}{kl}\{\mathrm{tr}(\theta_1 L(\lambda)^k), \mathrm{tr}(\theta_2\theta L(\mu)^l)\} &= \sum_{a,b,c,d=1}^{2n} \big(\theta_1 L(\lambda)^{k-1}\big)_{ba}\big(\theta_2 L(\mu)^{l-1}\big)_{dc}\big\{L(\lambda)_{ab}, L(\mu)_{cd}\big\}\\ &= \sum_{a,b,c,d,j=1}^{2n} \big(\theta_1 L(\lambda)^{k-1}\big)_{ba}\big(\theta_2 L(\mu)^{l-1}\big)_{dc}\\ &\quad\cdot\Big((r_1)_{ajcd}L(\lambda)_{jb} - L(\lambda)_{aj}(r_1)_{jbcd} + (r_2)_{abcj}L(\mu)_{jd} - L(\mu)_{cj}(r_2)_{abjd}\Big)\\ &= 0 \end{aligned} \tag{69}$$

by the relations

$$\sum_{b=1}^{2n} \big(\theta_1 L(\lambda)^{k-1}\big)_{ba} L(\lambda)_{jb} = \big(\theta_1 L(\lambda)^k\big)_{ja}, \tag{70}$$

etc. The theorem is proved. □

According to theorems 2 and 5, $\{H, \mathrm{tr}(\theta L(\lambda)^k)\} = 0$ holds for any positive integer k, complex number λ and matrix θ with $[\theta, L(\lambda)] = 0$.

5. Independence of conserved integrals

According to (25), $\mathrm{tr}(\theta L(\lambda)^{2k-1}) = 0$ for any $K \in \mathcal{D}_1$, $\theta \in \Theta_h$, and positive integers k and h. It is only necessary to consider $\mathrm{tr}(\theta L(\lambda)^k)$ for even k to generate the conserved integrals.

For given $\theta \in \Theta_h$, expand

$$\mathrm{tr}(\theta L(\lambda)^{2k}) = \sum_{j=0}^{\infty} s_j^{(2k)}(\theta)\lambda^{-j}. \tag{71}$$

By (26), $s_j^{(2k)}(\theta) = \omega_a^{2k+h-j} s_j^{(2k)}(\theta)$ for all $a = 1, \ldots, N$. Hence $s_j^{(2k)}(\theta) = 0$ unless $j \equiv 2k + h \bmod m_0$.

To consider the non-zero $s_j^{(2k)}(\theta)$ let

$$E_p^{(k)}(\theta) = \frac{1}{2k} s^{(2k)}_{m_0(p-1)+2k+h}(\theta). \tag{72}$$

Theorem 6. *Suppose* $K \in \mathcal{D}_1$ *is diagonalizable. Suppose also that there exist* $\theta_k \in \Theta_{h_k}$ $(k = 1, \ldots, n)$ *such that* $[\theta_j, \theta_k] = 0$ *for all j, k, and* $\theta_j K^{2j-1}$ $(j = 1, \ldots, n)$ *are linearly independent. Then* $E_p^{(k)}(\theta_k)$ $(k = 1, \ldots, n;\ p = 1, \ldots, r)$ *are functionally independent in a dense open subset of* $\boldsymbol{R}^{2n\times r}$.

Proof. According to (iv) of lemma 2, $[\theta_j, K] = 0$ for all j since $\theta_j \in \Theta_{h_j}$ and $K \in \mathcal{D}_1$. Moreover, $\theta_j^2 = \pm I$ implies that θ_j are diagonalizable. Hence there exists a $2n \times 2n$ complex invertible matrix T such that $K^{(0)} = TKT^{-1}$ and $\theta_j^{(0)} = T\theta_j T^{-1}$ are all complex diagonal matrices. Let $\Xi = (\xi_{jk})_{1\leqslant j\leqslant n, 1\leqslant k\leqslant 2n}$ where ξ_{jk} is the (k, k) entry of $\theta_j^{(0)}(K^{(0)})^{2j-1}$. Since

$\theta_j K^{2j-1}$ $(j = 1, \dots, n)$ are linearly independent, $\mathrm{rank}(\Xi) = n$. Without loss of generality, suppose that the first n columns of Ξ are linearly independent.

Let

$$S = \{\Psi \in \boldsymbol{R}^{2n\times r} \mid \text{all the entries of } T\Psi \text{ are non-zero}\}, \tag{73}$$

then S is a dense subset of $\boldsymbol{R}^{2n\times r}$.

Now we compute the Jacobian matrix of $E_p^{(k)}(\theta_k)$ with respect to $\phi_{j\sigma}$. By the definition of $L(\lambda)$,

$$\begin{aligned}\frac{1}{2k}\,\mathrm{tr}(\theta_k L(\lambda)^{2k}) &= \frac{1}{2k}\,\mathrm{tr}\left(\theta_k\Big(K + \kappa\sum_{\alpha\in\Sigma_0}\sum_{\sigma=1}^{r}\frac{\Omega^\alpha\Phi_\sigma\Phi_\sigma^T(\Omega^\alpha)^T W}{\lambda-\omega^\alpha\lambda_\sigma}\Big)^{2k}\right)\\ &= \frac{1}{2k}\,\mathrm{tr}\Big(\theta_k K^{2k} + 2k\kappa\theta_k K^{2k-1}\sum_{\alpha\in\Sigma_0}\sum_{\sigma=1}^{r}\frac{\Omega^\alpha\Phi_\sigma\Phi_\sigma^T(\Omega^\alpha)^T W}{\lambda-\omega^\alpha\lambda_\sigma}\Big)+\cdots\end{aligned} \tag{74}$$

where '$\cdots$' represents the terms of $\phi_{j\sigma}$ whose degrees are higher than 2. Hence

$$\begin{aligned}E_p^{(k)}(\theta_k) &= \kappa\,\mathrm{tr}\Big(\sum_{\alpha\in\Sigma_0}\sum_{\sigma=1}^{r}(\omega^\alpha\lambda_\sigma)^{m_0(p-1)+2k+h_k-1}\theta_k K^{2k-1}\Omega^\alpha\Phi_\sigma\Phi_\sigma^T(\Omega^\alpha)^T W\Big)+\cdots\\ &= \kappa m_1\cdots m_N\sum_{\sigma=1}^{r}\lambda_\sigma^{m_0(p-1)+2k+h_k-1}\Phi_\sigma^T W\theta_k K^{2k-1}\Phi_\sigma+\cdots\end{aligned} \tag{75}$$

since $\omega^{m_0\alpha} = 1$, $K \in \mathcal{D}_1$, $\theta_k \in \Theta_{h_k}$ and (13) holds.

Denote $\Psi = (\phi_{j\sigma})_{1\leqslant j\leqslant 2n;1\leqslant\sigma\leqslant r}$. For $k = 1, \dots, n$, $j = 1, \dots, 2n$, $p = 1, \dots, r$, $\sigma = 1, \dots, r$,

$$\frac{\partial E_p^{(k)}(\theta_k)}{\partial\phi_{j\sigma}} = 2\kappa m_1\cdots m_N\lambda_\sigma^{m_0(p-1)+2k+h_k-1}(W\theta_k K^{2k-1}\Psi)_{j\sigma}+\cdots. \tag{76}$$

Here we have used the fact that $W\theta_k K^{2k-1}$ is symmetric. Then

$$\sum_{l=1}^{2n}(W^{-1})_{jl}\frac{\partial E_p^{(k)}(\theta_k)}{\partial\phi_{l\sigma}} = 2\kappa m_1\cdots m_N\lambda_\sigma^{m_0(p-1)+2k+h_k-1}(\theta_k K^{2k-1}\Psi)_{j\sigma}+\cdots. \tag{77}$$

Let

$$\mathcal{M}^{(s)} = (\lambda_\sigma^{m_0(p-1)+2k+h_k-1}(\theta_k K^{2k-1}\Psi)_{j\sigma})_{ns\times 2ns} \qquad (1\leqslant s\leqslant r) \tag{78}$$

where the row indices are $k = 1, \dots, n$ and $p = 1, \dots, s$, and the column indices are $j = 1, \dots, 2n$ and $\sigma = 1, \dots, s$. Write $\mathcal{M}^{(s)}$ as the block matrix $\mathcal{M}^{(s)} = (\mathcal{M}_{kj}^{(s)})_{1\leqslant k\leqslant n;1\leqslant j\leqslant 2n}$ where

$$\mathcal{M}_{kj}^{(s)} = \begin{pmatrix}\lambda_1^{2k+h_k-1}(\Psi^{(k)})_{j1} & \cdots & \lambda_s^{2k+h_k-1}(\Psi^{(k)})_{js}\\ \lambda_1^{m_0+2k+h_k-1}(\Psi^{(k)})_{j1} & \cdots & \lambda_s^{m_0+2k+h_k-1}(\Psi^{(k)})_{js}\\ \vdots & & \vdots\\ \lambda_1^{m_0(s-1)+2k+h_k-1}(\Psi^{(k)})_{j1} & \cdots & \lambda_s^{m_0(s-1)+2k+h_k-1}(\Psi^{(k)})_{js}\end{pmatrix} \tag{79}$$

are $s\times s$ matrices, and $\Psi^{(k)} = \theta_k K^{2k-1}\Psi$. Let

$$\mathcal{N}^{(s)} = \begin{pmatrix}\sigma^{(s)} & & \\ & \ddots & \\ & & \sigma^{(s)}\end{pmatrix}_{ns\times ns}, \qquad \sigma^{(s)} = \begin{pmatrix}1 & & & & \\ -\lambda_s^{m_0} & 1 & & & \\ & -\lambda_s^{m_0} & 1 & & \\ & & \ddots & \ddots & \\ & & & -\lambda_s^{m_0} & 1\end{pmatrix}_{s\times s}, \tag{80}$$

then

$$\sigma^{(s)}\mathcal{M}_{kj}^{(s)} = \begin{pmatrix} \lambda_1^{2k+h_k-1}(\Psi^{(k)})_{j1}\cdots\lambda_{s-1}^{2k+h_k-1}(\Psi^{(k)})_{j,s-1} & \lambda_s^{2k+h_k-1}(\Psi^{(k)})_{js} \\ \left((\lambda_b^{m_0}-\lambda_s^{m_0})\lambda_b^{(a-1)m_0+2k+h_k-1}(\Psi^{(k)})_{jb}\right)_{1\leqslant a,b\leqslant s-1} & 0_{(s-1)\times 1} \end{pmatrix} \tag{81}$$

and $\mathcal{M}^{(s)}$ is transformed to $\mathcal{N}^{(s)}\mathcal{M}^{(s)}$ under elementary transformations. Take another elementary transformation for $\mathcal{N}^{(s)}\mathcal{M}^{(s)}$ by changing the first, $(s+1)$th, $(2s+1)$th, …, $((n-1)s+1)$th rows to the bottom and changing the sth, $2s$th, $\cdots$, $2ns$th column to the right. Then $\mathcal{M}^{(s)}$ is changed to

$$\begin{pmatrix} \widetilde{M}^{(s-1)} & 0 \\ * & B_s \end{pmatrix} \tag{82}$$

where $\widetilde{\mathcal{M}}^{(s-1)} = (\widetilde{M}_{kj}^{(s-1)})_{1\leqslant k\leqslant n,1\leqslant j\leqslant 2n}$,

$$\begin{aligned} \widetilde{\mathcal{M}}_{kj}^{(s-1)} &= \mathcal{M}_{kj}^{(s-1)}\begin{pmatrix} \lambda_1^{m_0}-\lambda_s^{m_0} & & \\ & \ddots & \\ & & \lambda_{s-1}^{m_0}-\lambda_s^{m_0} \end{pmatrix}, \\ B_s &= \left(\lambda_s^{2k+h_k-1}(\Psi^{(k)})_{js}\right)_{1\leqslant k\leqslant n,1\leqslant j\leqslant 2n}. \end{aligned} \tag{83}$$

Then

$$\sum_{l=1}^{2n} T_{jl}(B_s)_{kl} = \lambda_s^{2k+h_k-1}(T\theta_k K^{2k-1}\Psi)_{js} = \lambda_s^{2k+h_k-1}(\theta_k^{(0)}(K^{(0)})^{2k-1}T\Psi)_{js}. \tag{84}$$

That is

$$(B_s T^T)_{jk} = \lambda_s^{2j+h_j-1}\xi_{jk}(T\Psi)_{ks}. \tag{85}$$

Hence

$$B_s T^T = (\lambda_s^{2j+h_j-1}\delta_{jk})_{n\times n}(\xi_{jk})_{n\times 2n}((T\Psi)_{js}\delta_{jk})_{2n\times 2n} \tag{86}$$

is of rank n provided that $\Psi \in S$.

Hence $\mathrm{rank}(B_s) = n$ if $\Psi \in S$. From (82), we have

$$\mathrm{rank}(\mathcal{M}^{(s)}) = \mathrm{rank}(\mathcal{M}^{(s-1)}) + n \tag{87}$$

if $\Psi \in S$, which implies $\mathrm{rank}(\mathcal{M}^{(r)}) = nr$ if $\Psi \in S$.

From (77), for given $\Psi_0 \in S$, there exists $\varepsilon_0 > 0$ such that

$$\mathrm{rank}\left(\sum_{l=1}^{2n}(W^{-1})_{jl}\frac{\partial E_p^{(k)}(\theta_k)}{\partial\phi_{l\sigma}}\right)_{\substack{1\leqslant k\leqslant n,1\leqslant p\leqslant r\\ 1\leqslant j\leqslant 2n,1\leqslant\sigma\leqslant r}}\Bigg|_{\Psi=\varepsilon\Psi_0} = nr \tag{88}$$

holds for all ε with $|\varepsilon| < \varepsilon_0$. Equivalently, $\mathrm{rank}\left(\frac{\partial E_p^{(k)}(\theta_k)}{\partial\phi_{j\sigma}}\right)_{\substack{1\leqslant k\leqslant n,1\leqslant p\leqslant r\\ 1\leqslant j\leqslant 2n,1\leqslant\sigma\leqslant r}}\Big|_{\Psi=\varepsilon\Psi_0} = nr$ holds for all ε with $|\varepsilon| < \varepsilon_0$. Because of the real analyticity, the above equality holds in a dense subset of $\boldsymbol{R}^{2n\times r}$. The theorem is proved. □

Summarizing the results in theorems 2, 5 and 6, we have the final theorem on the integrability.

Theorem 7. *Suppose $K \in \mathcal{D}_1$ is diagonalizable, $f \in \mathcal{F}_{p,h}$ $(p \geqslant 1)$. Suppose also that there exist $\theta_k \in \Theta_{h_k}$ $(k = 1, \ldots, n)$ such that $[\theta_j, \theta_k] = 0$ for all j, k, and $\theta_j K^{2j-1}$ $(j = 1, \ldots, n)$ are linearly independent. Then the system (39) is an integrable Hamiltonian system in the Liouville sense with the Hamiltonian function given by (40).*

With stronger conditions on K, we have

Corollary 2. *Suppose $K \in \mathcal{D}_1$ is diagonalizable and K^2 has at least n distinct eigenvalues. Suppose also that $f \in \mathcal{F}_{p,h}$ $(p \geqslant 1)$. Then the system (39) is an integrable Hamiltonian system in the Liouville sense with the Hamiltonian function given by (40).*

Proof. Take $\theta_1 = \cdots = \theta_n = I$ in theorem 7. Since K^2 has at least n distinct non-zero eigenvalues, $K, K^3, \cdots, K^{2n-1}$ are linearly independent. The result follows from theorem 7. □

6. Some examples

In this section, we will recover some known results for certain important integrable equations from the general results in this paper. Hereafter, we always write

$$\pi_{j,k}^{(l)} = \sum_{\sigma=1}^{r} \lambda_\sigma^l \phi_{j\sigma}\phi_{k\sigma}. \tag{89}$$

6.1. 2 × 2 *real AKNS system*

The 2 × 2 real AKNS system has already been discussed in the introduction. The corresponding $n = 1$, $N = 0$, $W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\kappa = -1$. Moreover, $\mathcal{D}_0 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a, b, c \in \boldsymbol{R} \right\}$, $\mathcal{D}_0 \cap \ker \mathrm{ad} K = \left\{ \begin{pmatrix} a & \\ & -a \end{pmatrix} \middle| a \in \boldsymbol{R} \right\}$, $\Theta_0 = \{\pm I_{2\times 2}\}$. There is no cyclic symmetry here ($N = 0$).

Noting that $\mathcal{D}_0 \cap \ker \mathrm{ad} K \neq \{0\}$, we take $f^x(\tau) = -\frac{1}{2}(\tau^3 - 3\tau) \in \mathcal{F}_{3,0} = \mathcal{F}_{1,0}$ as in theorem 3. The nonlinear constraint $u = \pi_{1,1}^{(0)}$, $v = -\pi_{2,2}^{(0)}$ is obtained from theorem 1. Under this constraint, the Lax pair (1) becomes

$$\Phi_{\sigma,x} = (\lambda f^x(L(\lambda)))_+|_{\lambda=\lambda_\sigma} \Phi_\sigma, \tag{90}$$

which is the same as (5).

According to theorem 2 and corollary 2, the system (90) is an integrable Hamiltonian system with Hamiltonian function

$$H^x = \frac{1}{16} \operatorname{tr} \operatorname{Res} \lambda (L(\lambda)^4 - 6L(\lambda)^2) = \pi_{1,2}^{(1)} + \frac{1}{2}\pi_{1,1}^{(0)}\pi_{2,2}^{(0)}, \tag{91}$$

which is the same as (7).

6.2. MKdV equation

The MKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{92}$$

has the Lax pair

$$\begin{aligned}
\Phi_x &= \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi + \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \Phi, \\
\Phi_t &= -4\lambda^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi - 4\lambda^2 \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \Phi \\
&\quad - 2\lambda \begin{pmatrix} u^2 & u_x \\ u_x & -u^2 \end{pmatrix} \Phi - \begin{pmatrix} 0 & u_{xx} + 2u^3 \\ -u_{xx} - 2u^3 & 0 \end{pmatrix} \Phi.
\end{aligned} \tag{93}$$

Now $n = 1$, $N = 1$, $m_1 = 2$, $\omega_1 = -1$, $W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\Omega_1 = W$, $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\mathcal{D}_0 = \left\{ \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} \middle| c \in \boldsymbol{R} \right\}$, $\mathcal{D}_1 = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \middle| a, b \in \boldsymbol{R} \right\}$, $\mathcal{D}_0 \cap \ker \operatorname{ad} K = \{0\}$, $\Theta_0 = \{\pm I_{2\times 2}\}$, $\Theta_1 = \emptyset$.

Take $\kappa = -1$. Let Φ_σ $(\sigma = 1, \ldots, r)$ be column solutions of (93) with $\lambda = \lambda_\sigma$. By (23), the Lax matrix is

$$L(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sum_{k=0}^{\infty} \lambda^{-2k-1} \begin{pmatrix} 0 & \pi_{1,1}^{(2k)} + \pi_{2,2}^{(2k)} \\ -\pi_{1,1}^{(2k)} - \pi_{2,2}^{(2k)} & 0 \end{pmatrix}$$
$$+ \sum_{k=0}^{\infty} \lambda^{-2k-2} \begin{pmatrix} -2\pi_{1,2}^{(2k+1)} & \pi_{1,1}^{(2k+1)} - \pi_{2,2}^{(2k+1)} \\ \pi_{1,1}^{(2k+1)} - \pi_{2,2}^{(2k+1)} & 2\pi_{1,2}^{(2k+1)} \end{pmatrix}. \tag{94}$$

Take

$$f^x(\tau) = \tau \in \mathcal{F}_{1,0}, \qquad f^t(\tau) = 2(\tau^3 - 3\tau) \in \mathcal{F}_{3,0} = \mathcal{F}_{1,0}, \tag{95}$$

then $f^x(K) = K$, $f^t(K) = -4K$. According to theorem 1, the nonlinear constraint is

$$u = \pi_{1,1}^{(0)} + \pi_{2,2}^{(0)}. \tag{96}$$

Then

$$u_x = 2(\pi_{1,1}^{(1)} - \pi_{2,2}^{(1)}), \qquad u_{xx} = 4(\pi_{1,1}^{(2)} + \pi_{2,2}^{(2)}) + 8u\pi_{1,2}^{(1)}. \tag{97}$$

Under the constraint (96), the Lax pair (93) becomes

$$\Phi_{\sigma,x} = (\lambda f^x(L(\lambda)))_+|_{\lambda=\lambda_\sigma} \Phi_\sigma, \qquad \Phi_{\sigma,t} = (\lambda^3 f^t(L(\lambda)))_+|_{\lambda=\lambda_\sigma} \Phi_\sigma. \tag{98}$$

Remark 1. $f^x(\tau) = -\frac{1}{2}(\tau^3 - 3\tau) \in \mathcal{F}_{1,0} = \mathcal{F}_{3,0}$ which is proportional to $f^t(\tau)$ will give the same equation as $f(\tau) = \tau$.

According to theorem 2, the systems in (98) are Hamiltonian systems with Hamiltonian functions

$$H^x = -\frac{1}{8} \operatorname{tr} \operatorname{Res} \lambda L(\lambda)^2 = \frac{1}{32} \operatorname{tr} \operatorname{Res} \lambda (L(\lambda)^4 - 6L(\lambda)^2) = \pi_{1,2}^{(1)} + \frac{1}{4}(\pi_{1,1}^{(0)} + \pi_{2,2}^{(0)})^2,$$
$$H^t = -\frac{1}{8} \operatorname{tr} \operatorname{Res} \lambda^3 (L(\lambda)^4 - 6L(\lambda)^2) = -4\pi_{1,2}^{(3)} - 2(\pi_{1,1}^{(0)} + \pi_{2,2}^{(0)})(\pi_{1,1}^{(2)} + \pi_{2,2}^{(2)})$$
$$+ (\pi_{1,1}^{(1)} - \pi_{2,2}^{(1)})^2 - 2(\pi_{1,1}^{(0)} + \pi_{2,2}^{(0)})^2 \pi_{1,2}^{(1)} - \frac{1}{4}(\pi_{1,1}^{(0)} + \pi_{2,2}^{(0)})^4. \tag{99}$$

These are involutive Hamiltonian systems which are integrable. The solutions of the corresponding Hamiltonian equations satisfy the MKdV equation. Therefore, we have recovered some known results [27] from our general results.

6.3. Nonlinear Schrödinger equation

The nonlinear Schrödinger equation, written in real form, is

$$\begin{aligned} u_t &= v_{xx} + 2(u^2 + v^2)v, \\ -v_t &= u_{xx} + 2(u^2 + v^2)u. \end{aligned} \tag{100}$$

Denote $I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and $J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ which play the role of 1 and $\mathrm{i} = \sqrt{-1}$, respectively. The Lax pair in real form is

$$\begin{aligned} \Phi_x &= \lambda \begin{pmatrix} I & \\ & -I \end{pmatrix} \Phi + \begin{pmatrix} & uI + vJ \\ -uI + vJ & \end{pmatrix} \Phi, \\ \Phi_t &= -2\lambda^2 \begin{pmatrix} J & \\ & -J \end{pmatrix} \Phi - 2\lambda \begin{pmatrix} & -vI + uJ \\ -vI - uJ & \end{pmatrix} \Phi \\ &\quad - \begin{pmatrix} (u^2+v^2)J & -v_x I + u_x J \\ v_x I + u_x J & -(u^2+v^2)J \end{pmatrix} \Phi. \end{aligned} \tag{101}$$

Now $n = 2$, $N = 2$, $m_1 = 2$, $m_2 = 2$, $\omega_1 = -1$, $\omega_2 = 1$,

$$\Omega_1 = W = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} \mathrm{i}J & \\ & \mathrm{i}J \end{pmatrix}, \qquad K = \begin{pmatrix} I & \\ & -I \end{pmatrix}. \tag{102}$$

Then

$$\begin{aligned} \mathcal{D}_0 &= \left\{ \begin{pmatrix} aJ & bI + cJ \\ -bI + cJ & -aJ \end{pmatrix} \middle| a, b, c \in \boldsymbol{R} \right\}, \\ \mathcal{D}_1 &= \left\{ \begin{pmatrix} aI & bI + cJ \\ bI - cJ & -aI \end{pmatrix} \middle| a, b, c \in \boldsymbol{R} \right\}, \\ \mathcal{D}_0 \cap \ker \mathrm{ad} K &= \left\{ \begin{pmatrix} aJ & \\ & -aJ \end{pmatrix} \middle| a \in \boldsymbol{R} \right\}, \\ \Theta_0 &= \{\pm I_{4\times 4}\}, \quad \Theta_1 = \left\{ \pm \begin{pmatrix} J & \\ & J \end{pmatrix} \right\}. \end{aligned} \tag{103}$$

Take $\kappa = -1$. Let Φ_σ ($\sigma = 1, \ldots, r$) be column solutions of (101) with $\lambda = \lambda_\sigma$. The Lax matrix is

$$\begin{aligned} L(\lambda) &= \begin{pmatrix} I & \\ & -I \end{pmatrix} + \sum_{k=0}^{\infty} \lambda^{-2k-1} \begin{pmatrix} -q_1^{(2k)} J & q_2^{(2k)} I + q_3^{(2k)} J \\ -q_2^{(2k)} I + q_3^{(2k)} J & q_1^{(2k)} J \end{pmatrix} \\ &\quad + \sum_{k=0}^{\infty} \lambda^{-2k-2} \begin{pmatrix} -q_1^{(2k+1)} I & q_2^{(2k+1)} I + q_3^{(2k+1)} J \\ q_2^{(2k+1)} I - q_3^{(2k+1)} J & q_1^{(2k+1)} I \end{pmatrix} \end{aligned} \tag{104}$$

where

$$\begin{aligned} &q_1^{(2k)} = 2(\pi_{1,4}^{(2k)} + \pi_{2,3}^{(2k)}), \qquad q_2^{(2k)} = \pi_{1,1}^{(2k)} - \pi_{2,2}^{(2k)} + \pi_{3,3}^{(2k)} - \pi_{4,4}^{(2k)}, \\ &q_3^{(2k)} = 2(\pi_{1,2}^{(2k)} - \pi_{3,4}^{(2k)}), \qquad q_1^{(2k+1)} = 2(\pi_{1,3}^{(2k+1)} - \pi_{2,4}^{(2k+1)}), \\ &q_2^{(2k+1)} = \pi_{1,1}^{(2k+1)} - \pi_{2,2}^{(2k+1)} - \pi_{3,3}^{(2k+1)} + \pi_{4,4}^{(2k+1)}, \qquad q_3^{(2k+1)} = 2(\pi_{1,2}^{(2k+1)} + \pi_{3,4}^{(2k+1)}). \end{aligned} \tag{105}$$

Denote $\theta = \begin{pmatrix} J & \\ & J \end{pmatrix} \in \Theta_1$, and let

$$f^x(\tau) = -\frac{1}{2}(\tau^3 - 3\tau) \in \mathcal{F}_{1,0}, \qquad f^t(\tau) = -\frac{1}{4}\theta(3\tau^5 - 10\tau^3 + 15\tau) \in \mathcal{F}_{2,1}, \tag{106}$$

then $f^x(K) = K$, $f^t(K) = -2\theta K$.

Remark 2. If we take $f^x(\tau) = \tau$, then $f^x(K) = K$, but L_1 has a non-zero projection in $\mathcal{D}_0 \cap \ker \mathrm{ad} K \neq \{0\}$. To solve this problem, we use theorem 3 to obtain $\zeta(\tau) = -\frac{1}{2}(\tau^3 - 3\tau)$, which gives $f^x(x)$ in (106).

Remark 3. $f^x(\tau) = \frac{1}{8}(3\tau^5 - 10\tau^3 + 15\tau) = -\frac{1}{2}\theta f^t(\tau)$ plays the same role as $f^x(\tau)$ in (106) does.

According to theorem 1, the nonlinear constraint is

$$u = \pi_{1,1}^{(0)} - \pi_{2,2}^{(0)} + \pi_{3,3}^{(0)} - \pi_{4,4}^{(0)}, \qquad v = 2(\pi_{1,2}^{(0)} - \pi_{3,4}^{(0)}). \tag{107}$$

Then

$$\begin{aligned} u_x &= 2(\pi_{1,1}^{(1)} - \pi_{2,2}^{(1)} - \pi_{3,3}^{(1)} + \pi_{4,4}^{(0)}) - 4v(\pi_{1,4}^{(0)} + \pi_{2,3}^{(0)}), \\ v_x &= 4(\pi_{1,2}^{(1)} + \pi_{3,4}^{(1)}) + 4u(\pi_{1,4}^{(0)} + \pi_{2,3}^{(0)}). \end{aligned} \tag{108}$$

Under the constraint (107), the Lax pair becomes

$$\Phi_{\sigma,x} = (\lambda f^x(L(\lambda)))_+|_{\lambda=\lambda_\sigma}\Phi_\sigma, \qquad \Phi_{\sigma,t} = (\lambda^2 f^t(L(\lambda)))_+|_{\lambda=\lambda_\sigma}\Phi_\sigma. \tag{109}$$

According to theorem 2, these are Hamiltonian systems with Hamiltonian functions

$$\begin{aligned} H^x &= \frac{1}{64}\,\mathrm{tr\,Res}\left(\lambda\left(L(\lambda)^4 - 6L(\lambda)^2\right)\right) \\ &= -\frac{1}{128}\,\mathrm{tr\,Res}\left(\lambda\left(L(\lambda)^6 - 5L(\lambda)^4 + 15L(\lambda)^2\right)\right) \\ &= \pi_{1,3}^{(1)} - \pi_{2,4}^{(1)} + \left(\pi_{1,2}^{(0)} - \pi_{3,4}^{(0)}\right)^2 + \frac{1}{4}\left(\pi_{1,1}^{(0)} - \pi_{2,2}^{(0)} + \pi_{3,3}^{(0)} - \pi_{4,4}^{(0)}\right)^2, \\ H^t &= \frac{1}{64}\,\mathrm{tr\,Res}\left(\lambda^2\theta\left(L(\lambda)^6 - 5L(\lambda)^4 + 15L(\lambda)^2\right)\right) \\ &= 2(\pi_{1,4}^{(2)} + \pi_{2,3}^{(2)}) + 2(\pi_{1,2}^{(1)} + \pi_{3,4}^{(1)})(\pi_{1,1}^{(0)} - \pi_{2,2}^{(0)} + \pi_{3,3}^{(0)} - \pi_{4,4}^{(0)}) \\ &\quad - 2(\pi_{1,2}^{(0)} - \pi_{3,4}^{(0)})(\pi_{1,1}^{(1)} - \pi_{2,2}^{(1)} - \pi_{3,3}^{(1)} + \pi_{4,4}^{(1)}) \\ &\quad + (\pi_{1,4}^{(0)} + \pi_{2,3}^{(0)})\left((\pi_{1,1}^{(0)} - \pi_{2,2}^{(0)} + \pi_{3,3}^{(0)} - \pi_{4,4}^{(0)})^2 + 4(\pi_{1,2}^{(0)} - \pi_{3,4}^{(0)})^2\right). \end{aligned} \tag{110}$$

According to theorem 7 with $\theta_1 = I$ and $\theta_2 = \theta$, these Hamiltonian systems are integrable in the Liouville sense. The solutions of the corresponding Hamiltonian equations satisfy the nonlinear Schrödinger equation. This recovers the results in [27].

6.4. $u(n)$ AKNS system

Denote I and J as in the above subsection. The x part of the $u(n)$ AKNS system is the linear system

$$\Phi_x = (\lambda K + P)\Phi. \tag{111}$$

Here $K = (a_j J\delta_{jk})_{1\leqslant j,k\leqslant n}$, a_j $(j = 1, \ldots, n)$ are real numbers such that $a_1, \ldots, a_n$ are distinct. $P = (u_{jk}I + v_{jk}J)_{1\leqslant j,k\leqslant n}$ with $u_{jj} = v_{jj} = 0$, $u_{kj} = -u_{jk}$, $v_{kj} = v_{jk}$ $(j, k = 1, \ldots, n)$.

Here we have written the $u(n)$ AKNS system in real form, which is equivalent to usual complex form.

Now $m_1 = 2$, $N = 1$, $\omega_1 = 1$, $\Omega_1 = W = (-J\delta_{jk})_{1\leqslant j,k\leqslant n}$. Then

$$\begin{aligned} \mathcal{D}_0(= \mathcal{D}_1) &= \Big\{(a_{jk}I + b_{jk}J)_{1\leqslant j,k\leqslant n} \,|\, a_{jk}, b_{jk} \in \boldsymbol{R}, \\ &\qquad a_{kj} = -a_{jk}, b_{kj} = b_{jk}\ (j, k = 1, \ldots, n)\Big\}, \\ \mathcal{D}_0 \cap \ker\, \mathrm{ad}K &= \Big\{(c_j J\delta_{jk})_{1\leqslant j,k\leqslant n} \,|\, c_j \in \boldsymbol{R}\ (j = 1, \ldots, n)\Big\}, \\ \Theta_0 &= \{\pm I_{2n\times 2n}\}. \end{aligned} \tag{112}$$

Let $f^x = \zeta$ where ζ is given by theorem 3, then $f^x(K) = K$.

Take $\kappa = -1$. Let Φ_σ $(\sigma = 1, \ldots, r)$ be column solutions of (111) with $\lambda = \lambda_\sigma$. By (23), the Lax matrix is $L(\lambda) = (L_{jk}(\lambda))_{1\leqslant j,k\leqslant n}$ with

$$L_{jk}(\lambda) = \begin{pmatrix} 0 & -a_j \\ a_j & 0 \end{pmatrix} \delta_{jk} + \sum_{\sigma=1}^{N} \frac{1}{\lambda - \lambda_\sigma} \begin{pmatrix} \phi_{2j-1,\sigma}\phi_{2k,\sigma} - \phi_{2j,\sigma}\phi_{2k-1,\sigma} & -\phi_{2j-1,\sigma}\phi_{2k-1,\sigma} - \phi_{2j,\sigma}\phi_{2k,\sigma} \\ \phi_{2j-1,\sigma}\phi_{2k-1,\sigma} + \phi_{2j,\sigma}\phi_{2k,\sigma} & \phi_{2j-1,\sigma}\phi_{2k,\sigma} - \phi_{2j,\sigma}\phi_{2k-1,\sigma} \end{pmatrix}. \tag{113}$$

By theorems 1 and 3, the nonlinear constraint is

$$u_{jk} = \pi^{(0)}_{2j-1,2k} - \pi^{(0)}_{2j,2k-1}, \qquad v_{jk} = \pi^{(0)}_{2j-1,2k-1} + \pi^{(0)}_{2j,2k} \quad (j \neq k). \tag{114}$$

Under this constraint, the Lax pair becomes a system of ODEs

$$\Phi_{\sigma,x} = (\lambda\zeta(L(\lambda)))_+|_{\lambda=\lambda_\sigma}\Phi_\sigma. \tag{115}$$

It is too complicated to derive the Hamiltonian functions from theorem 2 directly. However, it can be easily integrated from (115) to obtain the Hamiltonian functions since the action of ζ is simply to remove the ker ad$K$ component from $\widetilde{P}$. The result is that (115) is a Hamiltonian system with the Hamiltonian function

$$H^x = \frac{1}{2}\sum_{j=1}^{n} a_j(\pi^{(1)}_{2j-1,2j-1} + \pi^{(1)}_{2j,2j}) + \frac{1}{4}\sum_{\substack{j,k=1\\ j\neq k}}^{n}(\pi^{(0)}_{2j,2k-1} - \pi^{(0)}_{2j-1,2k})^2 + \frac{1}{4}\sum_{\substack{j,k=1\\ j\neq k}}^{n}(\pi^{(0)}_{2j-1,2k-1} + \pi^{(0)}_{2j,2k})^2. \tag{116}$$

Remark 4. This process is just the binary nonlinearization [11] for the $u(n)$ AKNS system [12]. In fact, for any Lax pair with unitary symmetry, the complex structure induces a natural symplectic structure. Therefore, for any finite-dimensional Hamiltonian systems derived by the nonlinearization method from the $u(n)$ AKNS system, their conserved integrals, r matrices and the Liouville integrability are derived naturally from the results of this paper, although the Lax matrix and the Hamiltonian functions may be derived more simply by direct computation.

Remark 5. The nonlinear Schrödinger equation is also included in the $u(2)$ AKNS system. Hence the nonlinear constraint given here is also applicable to the nonlinear Schrödinger equation [28]. However, it is different from that in section 6.2 because the symplectic structure here is derived directly from the complex structure, while that in section 6.2 is the standard one in $sl(2, \mathbf{C})$ which is isomorphic to $sp(1, \mathbf{C})$.

6.5. n-wave equation

The n-wave equation is the integrability condition of the Lax pair

$$\Phi_x = (\lambda K + P)\Phi, \qquad \Phi_t = (\lambda K' + Q)\Phi. \tag{117}$$

Here $K = (a_j J\delta_{jk})_{1\leqslant j,k\leqslant n}$, $K' = (b_j J\delta_{jk})_{1\leqslant j,k\leqslant n}$, a_j, b_j $(j = 1, \ldots, n)$ are real numbers such that $a_1, \ldots, a_n$ are distinct. $P = (u_{jk}I + v_{jk}J)_{1\leqslant j,k\leqslant n}$ with $u_{jj} = v_{jj} = 0$, $u_{kj} = -u_{jk}$, $v_{kj} = v_{jk}$ $(j, k = 1, \ldots, n)$. Moreover, $Q = \left(\frac{b_j - b_k}{a_j - a_k}(u_{jk}I + v_{jk}J)\right)_{1\leqslant j,k\leqslant n}$. Then $[K, Q] = [K', P]$. Clearly the n wave equation is a special equation in the $u(n)$ AKNS system. Hence we only need to consider the t-part of the Lax pair.

For the n wave equation, N, m_1, ω_1, $\Omega_1 = W$, $\mathcal{D}_0$, Θ, the Lax matrix $L(\lambda)$ and the nonlinear constraint (114) are the same as those in the last subsection for the $u(n)$ AKNS system.

Since $\det((\sqrt{-1}a_j)^{k-1})_{1\leqslant j,k\leqslant n} \neq 0$, the linear system

$$\sum_{k=0}^{n-1}(a_j J)^{n-k-1}(\alpha_k I + \beta_k J) = b_j J \qquad (j = 1, \ldots, n) \tag{118}$$

has a unique real solution α_j, β_j $(j = 1, \ldots, n)$. Let

$$\hat{f}_j = ((\alpha_j I + \beta_j J)\delta_{ab})_{1\leqslant a,b\leqslant n}, \qquad \hat{f}(\tau) = \sum_{j=1}^{n} \hat{f}_{n-j}\tau^{j-1}, \tag{119}$$

then $\hat{f}(K) = K'$. Let $f^t = \hat{f} \circ \zeta$ where ζ is given by theorem 3, then $f^t(K) = K'$.

Under the constraint (114), the Lax pair (117) becomes two systems of ODEs

$$\Phi_{\sigma,x} = (\lambda f^x(L(\lambda)))_+|_{\lambda=\lambda_\sigma}\Phi_\sigma, \qquad \Phi_{\sigma,t} = (\lambda f^t(L(\lambda)))_+|_{\lambda=\lambda_\sigma}\Phi_\sigma. \tag{120}$$

Expand $\widetilde{L}(\lambda) = \zeta(L(\lambda))$ as

$$\begin{aligned}\widetilde{L} &= K + \lambda^{-1}\widetilde{P} + o(\lambda^{-1}),\\ \widetilde{L}^k &= K^k + \lambda^{-1}\sum_{j=0}^{k-1} K^j\widetilde{P}K^{k-j-1} + o(\lambda^{-1}).\end{aligned} \tag{121}$$

With the identity

$$\frac{b_\mu - b_\nu}{a_\mu - a_\nu} I = \sum_{k=1}^{n}\sum_{j=0}^{k-2}(\alpha_{n-k}I + \beta_{n-k}J)(a_\mu J)^j(a_\nu J)^{k-j-2}, \tag{122}$$

we have

$$\begin{aligned} f^t(L)_{\mu\nu} &= \hat{f}(\widetilde{L})_{\mu\nu}\\ &= K'_{\mu\nu} + \lambda^{-1}\sum_{k=1}^{n}\sum_{j=0}^{k-2}(\alpha_{n-k}I + \beta_{n-k}J)(a_\mu J)^j\widetilde{P}_{\mu\nu}(a_\nu J)^{k-j-2} + o(\lambda^{-1})\\ &= K'_{\mu\nu} + \lambda^{-1}\frac{b_\mu - b_\nu}{a_\mu - a_\nu}\widetilde{P}_{\mu\nu} + o(\lambda^{-1}).\end{aligned} \tag{123}$$

This gives the constraint on Q: $\widetilde{Q}_{\mu\nu} = \frac{b_\mu - b_\nu}{a_\mu - a_\nu}\widetilde{P}_{\mu\nu}$.

By integration, (120) becomes Hamiltonian systems with Hamiltonian functions

$$\begin{aligned} H^x &= \frac{1}{2}\sum_{j=1}^{n} a_j(\pi^{(1)}_{2j-1,2j-1} + \pi^{(1)}_{2j,2j})\\ &\quad + \frac{1}{4}\sum_{\substack{j,k=1\\ j\neq k}}^{n}(\pi^{(0)}_{2j,2k-1} - \pi^{(0)}_{2j-1,2k})^2 + \frac{1}{4}\sum_{\substack{j,k=1\\ j\neq k}}^{n}(\pi_{2j-1,2k-1}(0) + \pi^{(0)}_{2j,2k})^2,\\ H^t &= \frac{1}{2}\sum_{j=1}^{n} b_j(\pi^{(1)}_{2j-1,2j-1} + \pi^{(1)}_{2j,2j})\\ &\quad + \frac{1}{4}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\frac{b_j - b_k}{a_j - a_k}(\pi^{(0)}_{2j,2k-1} - \pi^{(0)}_{2j-1,2k})^2 + \frac{1}{4}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\frac{b_j - b_k}{a_j - a_k}(\pi^{(0)}_{2j-1,2k-1} + \pi^{(0)}_{2j,2k})^2.\end{aligned} \tag{124}$$

These are involutive Hamiltonian systems which are integrable in the Liouville sense. Each solution of these Hamiltonian systems gives a solution of the n wave equation [12].

7. Two-dimensional hyperbolic $C_n^{(1)}$ Toda equation

The two-dimensional hyperbolic $C_n^{(1)}$ Toda equation is

$$\begin{aligned} u_{1,xt} &= \mathrm{e}^{2u_1} - \mathrm{e}^{u_2-u_1}, \qquad u_{n,xt} = \mathrm{e}^{u_n-u_{n-1}} - \mathrm{e}^{-2u_n}, \\ u_{j,xt} &= \mathrm{e}^{u_j-u_{j-1}} - \mathrm{e}^{u_{j+1}-u_j} \quad (2 \leqslant j \leqslant n-1). \end{aligned} \tag{125}$$

It has a Lax pair

$$\Phi_x = (\lambda K + P(x,t))\Phi, \qquad \Phi_t = \lambda^{-1}Q(x,t)\Phi \tag{126}$$

where $K = (\delta_{j+1,k})_{1\leqslant j,k\leqslant 2n}$, $P = (p_j\delta_{jk})_{1\leqslant j,k\leqslant 2n}$ with $p_j = u_{j,x}$ for $j = 1,\ldots,n$ and $p_j = -u_{2n+1-j,x}$ for $j = n+1,\ldots,2n$, $Q = (q_k\delta_{j,k+1})_{1\leqslant j,k\leqslant 2n}$ with $q_k = \mathrm{e}^{u_{k+1}-u_k}$ for $k = 1,\ldots,n-1$, $q_n = \mathrm{e}^{-2u_n}$, $q_k = \mathrm{e}^{u_{2n+1-k}-u_{2n-k}}$ for $k = n+1,\ldots,2n-1$, $q_{2n} = \mathrm{e}^{2u_1}$. Note that $p_j + p_{2n+1-j} = 0$, $q_j = q_{2n-j}$ and $q_1q_2\cdots q_{2n} = 1$. Here we use the convention $q_{2n+j} = q_j$ etc.

Written in components, (126) is

$$\phi_{j,x} = \lambda\phi_{j+1} + p_j\phi_j, \qquad \phi_{j,t} = \lambda^{-1}q_{j-1}\phi_{j-1}. \tag{127}$$

(125) is equivalent to

$$Q_x = [P,Q], \qquad P_t + [K,Q] = 0, \tag{128}$$

or

$$q_{k,x} = (p_{k+1}-p_k)q_k, \qquad p_{k,t} = q_{k-1} - q_k, \tag{129}$$

which are equivalent to the integrability condition of (126).

Now $N = 1$, $m_1 = 2n$, $\omega_1 = \omega = \rho^2$ where $\rho = \exp\left(\frac{\pi\mathrm{i}}{2n}\right)$, $W = ((-1)^j\delta_{j,2n+1-k})_{1\leqslant j,k\leqslant 2n}$, $\Omega_1 = (\rho^{-2j+1}\delta_{jk})_{1\leqslant j,k\leqslant 2n}$. Then

$$\begin{aligned} &\mathcal{D}_k = \{(a_{ij})_{2n\times 2n} \,|\, a_{ij} \neq 0 \text{ only when } j-i \equiv k \bmod 2n, \\ &\qquad \text{and satisfy } (-1)^k a_{i,i+k} + a_{1-k-i,1-i} = 0\,(1\leqslant i\leqslant 2n)\}, \\ &\mathcal{D}_0 \cap \ker \mathrm{ad}K = \{0\}, \\ &\Theta_0 = \{\pm I_{2n\times 2n} \,|\, c \in \boldsymbol{R}\}, \quad \Theta_k = \{0\} \quad (k = 1,2,\ldots,2n-1). \end{aligned} \tag{130}$$

We have $P \in \mathcal{D}_0$, $Q \in \mathcal{D}_{-1}$.

Take $\kappa = \frac{1}{2n}$, Let Φ_σ $(\sigma = 1,\ldots,r)$ be column solutions of (126) with $\lambda = \lambda_\sigma$. By (23), the Lax matrix is

$$L(\lambda) = K + \frac{1}{2n}\sum_{\alpha=1}^{2n}\sum_{\sigma=1}^{r}\frac{\Omega^\alpha\Phi_\sigma\Phi_\sigma^T(\Omega^\alpha)^TW}{\lambda-\omega^\alpha\lambda_\sigma}, \tag{131}$$

whose entries are

$$L_{jk}(\lambda) = K_{jk} + \sum_{\sigma=1}^{r}\frac{(-1)^{k-1}\lambda_\sigma^{\{j-k\}}\lambda^{2n-1-\{j-k\}}}{\lambda^{2n}-\lambda_\sigma^{2n}}\phi_{j\sigma}\phi_{2n+1-k,\sigma}, \tag{132}$$

where $\{k\}$ is the remainder of k divided by $2n$. Here we have used the identity

$$\sum_{\alpha=0}^{2n-1}\frac{\omega^{-p\alpha}}{\lambda-\omega^\alpha\lambda_\sigma} = \frac{2n\lambda_\sigma^{\{p\}}\lambda^{2n-1-\{p\}}}{\lambda^{2n}-\lambda_\sigma^{2n}}, \tag{133}$$

(see lemma 2 in [29]).

Theorem 8. *Under the constraint*

$$e^{u_j} = \Gamma^{j-\frac{1}{2}}\gamma_0^{-\frac{1}{2}}\prod_{k=1}^{j-1}\gamma_k^{-1} \tag{134}$$

where

$$\gamma_j = 1-(-1)^j\pi^{(-1)}_{j,-j}, \qquad \Gamma = \prod_{k=1}^{2n}\gamma_k^{\frac{1}{2n}}, \tag{135}$$

the Lax pair (126) of the two-dimensional hyperbolic $C_n^{(1)}$ Toda equation is changed to a system of ODEs

$$\phi_{j\sigma,x} = \lambda_\sigma\phi_{j+1,\sigma} + (-1)^{j-1}\pi^{(0)}_{j,1-j}\phi_{j,\sigma}, \qquad \phi_{j\sigma,t} = \lambda_\sigma^{-1}\prod_{k=1}^{2n}\gamma_k^{\frac{1}{2n}}\gamma_{j-1}^{-1}\phi_{j-1,\sigma}, \tag{136}$$

or equivalently,

$$\Phi_{\sigma,x} = (\lambda L(\lambda))_+|_{\lambda=\lambda_\sigma}\Phi_\sigma, \qquad \Phi_{\sigma,t} = \lambda_\sigma^{-1}\left(\frac{1}{2n}\operatorname{tr}(L(0)^{2n})\right)^{\frac{1}{2n}-1}L(0)^{2n-1}\Phi_\sigma. \tag{137}$$

These ODEs are Liouville integrable Hamiltonian systems with the Hamiltonian functions

$$\begin{aligned} H^x &= \frac{1}{2}\operatorname{tr}\operatorname{Res}\left(\lambda L^2(\lambda)\right) = \sum_{j=1}^{2n}(-1)^j\pi^{(1)}_{j,2n+2-j} + \frac{1}{2}\sum_{j=1}^{2n}(\pi^{(0)}_{j,2n+1-j})^2, \\ H^t &= -n\left(\frac{1}{2n}\operatorname{tr}(L(0)^{2n})\right)^{\frac{1}{2n}} = -n\prod_{j=1}^{2n}\left(1-(-1)^{j-1}\pi^{(-1)}_{j,-j}\right)^{\frac{1}{2n}}. \end{aligned} \tag{138}$$

Proof. Take $f^x(\tau) = \tau$. According to theorem 1, the nonlinear constraint is

$$p_j = (-1)^{j-1}\pi^{(0)}_{j,1-j}. \tag{139}$$

We should mention that (134) is compatible with (139) under the relation $p_j = u_{j,x}$. In fact, by the definition of γ_j and the constraint (139),

$$-\frac{\gamma_{j,x}}{\gamma_j} = p_{j+1} - p_j. \tag{140}$$

Hence $\Gamma_x = 0$. (This can also be obtained from (151).) Then from (134),

$$u_{j,x} = -\frac{1}{2}\frac{\gamma_{0,x}}{\gamma_0} - \sum_{k=1}^{j-1}\frac{\gamma_{k,x}}{\gamma_k} = p_j \tag{141}$$

with the relation $p_1 + p_0 = 0$.

Under the constraint (139), the first equation of the Lax pair (126) becomes

$$\Phi_{\sigma,x} = (\lambda L(\lambda))_+|_{\lambda=\lambda_\sigma}\Phi_\sigma = (\lambda_\sigma K + \widetilde{P})\Phi_\sigma \tag{142}$$

where $\widetilde{P} = ((-1)^{j-1}\pi^{(0)}_{j,1-j})_{1\leqslant j,k\leqslant 2n}$.

According to theorem 2, this is a Hamiltonian system with Hamiltonian function

$$H^x = \frac{1}{2}\operatorname{tr}\operatorname{Res}\left(\lambda L^2(\lambda)\right) = \sum_{j=1}^{2n}(-1)^j\pi^{(1)}_{j,2n+2-j} + \frac{1}{2}\sum_{j=1}^{2n}(\pi^{(0)}_{j,2n+1-j})^2. \tag{143}$$

The coefficient of the second equation of the Lax pair (126) is not a polynomial of λ. Hence we cannot use the above general method and should construct its nonlinear constraint and Hamiltonian function directly. Similar to (36), we have

$$(\Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T W)_t = \frac{1}{\omega^\alpha \lambda_\sigma}[Q, \Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T W]. \tag{144}$$

Hence

$$L_t - \frac{1}{\lambda}[Q, L] = -\frac{1}{\lambda}[Q, K - \hat{K}] \tag{145}$$

where

$$\begin{aligned} \hat{K} = (\hat{K}_{jk}) &= \frac{1}{2n}\sum_{\sigma=1}^{r}\sum_{\alpha=0}^{2n-1}(\omega^\alpha \lambda_\sigma)^{-1}\Omega^\alpha \Phi_\sigma \Phi_\sigma^T (\Omega^\alpha)^T W, \\ \hat{K}_{jk} &= (-1)^j \sum_{\sigma=1}^{r} \lambda_\sigma^{-1} (\Phi_\sigma \Phi_\sigma^T)_{j,-j}\delta_{j,k-1} = (1-\gamma_j)\delta_{j,k-1}. \end{aligned} \tag{146}$$

$[Q, K - \hat{K}] = 0$ holds if and only if $\gamma_j q_j = \gamma_{j+1} q_{j+1}$. This is equivalent to

$$q_j = \gamma_j^{-1}\widetilde{\Gamma} \tag{147}$$

for a certain function $\widetilde{\Gamma}$. However, since $q_1 q_2 \cdots q_{2n} = 1$, we have

$$\widetilde{\Gamma} = \left(\prod_{k=1}^{2n}\gamma_k\right)^{\frac{1}{2n}} = \Gamma, \tag{148}$$

and the nonlinear constraint becomes

$$q_j = \Gamma \gamma_j^{-1}, \tag{149}$$

which is equivalent to (134). Meanwhile, the second equation of the Lax pair (126) can be written as the second equation of (136).

From (132), we have

$$(L(0))_{jk} = \delta_{j+1,k} - \sum_{\sigma=1}^{r}(-1)^j \lambda_\sigma^{-1}\phi_{j\sigma}\phi_{-j,\sigma}\delta_{j+1,k} = \gamma_j \delta_{j+1,k}. \tag{150}$$

Hence

$$\mathrm{tr}(L(0))^{2n} = 2n\prod_{k=1}^{2n}\gamma_k \tag{151}$$

and

$$(L(0)^{2n-1})_{jk} = \prod_{\substack{l=1 \\ l\neq j-1}}^{2n}\gamma_l \delta_{j,k+1}. \tag{152}$$

With the constraint (134), the second equation of (136) can be written as the second equation of (137).

With H^t in (138), we have

$$
\begin{aligned}
\sum_{k=1}^{2n} \hat{W}_{jk} \frac{\partial H^t}{\partial \phi_{k\sigma}} &= \frac{1}{2} \sum_{k,a,b=1}^{2n} (-1)^j \delta_{j+k,1} \left(\frac{1}{2n} \operatorname{tr} L(0)^{2n} \right)^{\frac{1}{2n}-1} \left(L(0)^{2n-1} \right)_{ba} \frac{\partial L(0)_{ab}}{\partial \phi_{k\sigma}} \\
&= \frac{1}{2} \sum_{k,a,b=1}^{2n} \sum_{\sigma=1}^{r} (-1)^{j-1} \delta_{j+k,1} \left(\frac{1}{2n} \operatorname{tr} L(0)^{2n} \right)^{\frac{1}{2n}-1} \left(L(0)^{2n-1} \right)_{ba} \lambda_\sigma^{-1} \\
&\quad \cdot (-1)^a (\delta_{ak} \phi_{-a,\sigma} \delta_{a+1,b} + \phi_{a\sigma} \delta_{-a,k} \delta_{a+1,b}) \\
&= \frac{1}{2} \sum_{k=1}^{2n} \sum_{\sigma=1}^{r} (-1)^{j+k-1} \lambda_\sigma^{-1} \left(\frac{1}{2n} \operatorname{tr} L(0)^{2n} \right)^{\frac{1}{2n}-1} \\
&\quad \cdot \left((L(0)^{2n-1})_{k+1,k} + (L(0)^{2n-1})_{1-k,-k} \right) \phi_{-k,\sigma} \delta_{j+k,1} \\
&= \lambda_\sigma^{-1} \frac{\left(\frac{1}{2n} \operatorname{tr} L(0)^{2n} \right)^{\frac{1}{2n}}}{1 - (-1)^{j-1} \pi^{(-1)}_{j-1,1-j}} \phi_{j-1,\sigma}.
\end{aligned}
\tag{153}
$$

Hence H^t is the Hamiltonian function of the second equation of (137).

According to theorems 5 and 6, the Hamiltonian systems given by both H^x and H^t are Liouville integrable. The theorem is proved. □

Therefore, any solution of the integrable Hamiltonian systems with Hamiltonian functions (138) gives a solution of the two-dimensional hyperbolic $C_n^{(1)}$ Toda equation. The corresponding symplectic structure is the natural one of $C_n^{(1)}$. The Hamiltonian systems (138) are simpler than (with space of lower dimension) those presented in [29] where the symplectic structure is derived from the complex structure.

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