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# The relationship between the hyperbolic Nizhnik–Novikov–Veselov equation and the stationary Davey–Stewartson II equation

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**Abstract**
A Lax system in three variables is presented, two equations of which form the Lax pair of the stationary Davey–Stewartson II equation. With certain nonlinear constraints, the full integrability condition of this Lax system contains the hyperbolic Nizhnik–Novikov–Veselov equation and its standard Lax pair. The Darboux transformation for the Davey–Stewartson II equation is used to solve the hyperbolic Nizhnik–Novikov–Veselov equation, and global multi-soliton solutions are obtained. It is proved that the $n \times n$ soliton solution derived by the Darboux transformation of order $n$ approaches zero uniformly and exponentially at spatial infinity and is asymptotic to $n^2$ solitons at temporal infinity.

## 1. Introduction

The Nizhnik–Novikov–Veselov (NNV) equation [19, 20, 22] is an important (2+1)-dimensional integrable equation which is a natural generalization of the KdV equation to (2+1) dimensions. It is useful in both mechanics and differential geometry [14, 15]. The NNV equation has been solved by various methods such as inverse scattering [3], the bilinear method [21], bilinear Bäcklund transformation [11], binary Darboux transformation [17] and so on [2, 9, 10, 12, 16, 18]. However, one cannot construct the usual Darboux transformation (without integration) because the principal part of the first equation of its Lax pair is a two-dimensional wave operator or Laplace operator.

Starting from the idea of nonlinearization [5], many high dimensional integrable systems were reduced to lower dimensional ones so that interesting solutions such as soliton solutions and quasi-periodic solutions can be obtained from lower dimensional systems. Especially, the KP equation [6, 13], the DSI equation and the (2 + 1)-dimensional $N$-wave equation [24] were related to some (1+1)-dimensional AKNS systems. Following this idea, in this paper, we present a Lax system of three variables, two equations of which form the Lax pair of the stationary Davey–Stewartson II (DSII) equation. With nonlinear constraints between the solution of the stationary DSII equation and the solution of its Lax pair, the full integrability condition of this Lax system contains the hyperbolic NNV equation and its standard Lax pair.

The DSII equation has a Darboux transformation without integration. With the relations given by the nonlinear constraints, the Darboux transformation for the DSII equation is used to solve the hyperbolic NNV equation. This Darboux transformation without integration is more suitable for symbolic calculation than the known binary Darboux transformation.

It is well known that the DSI equation has solutions approaching zero exponentially at spatial infinity [4], but the DSII equation does not, although it has solutions approaching zero rationally [1, 7]. However, we obtain a soliton solution $u$ of the hyperbolic NNV equation from that of the stationary DSII equation so that $u$ approaches zero exponentially at spatial infinity. These soliton solutions are different from the known one derived by the binary Darboux transformation or bilinear method, and the behavior of the solutions is more complicated.

In section 2, after reviewing the hyperbolic NNV equation and the stationary DSII equation together with their standard Lax pairs, a new Lax system is presented in which an extra equation is added to the standard Lax pair of the stationary DSII equation. With the nonlinear constraints, the integrability condition of this Lax system includes both the hyperbolic NNV equation and its standard Lax pair. The Darboux transformation for the new Lax system is given in section 3 and the general expression of multi-soliton solutions is presented in section 4. In section 5, the explicit expressions and behavior of single-soliton solutions are discussed. In section 6, it is proved that the solution derived by the Darboux transformation of order $n$ approaches zero uniformly and exponentially at spatial infinity. In section 7, it is proved that each solution derived by the Darboux transformation of order $n$ is asymptotic to $n^2$ solitons at temporal infinity. Hence, we call this solution as an $n \times n$ soliton solution. Finally, some linear algebraic lemmas are presented in appendix A.

## 2. Hyperbolic Nizhnik–Novikov–Veselov equation and Davey–Stewartson II equation

The hyperbolic NNV equation is

$$u_t = u_{\xi\xi\xi} + u_{\eta\eta\eta} + 3(uv)_\xi + 3(uw)_\eta, \qquad v_\eta = u_\xi, \qquad w_\xi = u_\eta, \tag{1}$$

which has a Lax pair

$$f_{\xi\eta} + uf = 0, \qquad f_t = f_{\xi\xi\xi} + f_{\eta\eta\eta} + 3vf_\xi + 3wf_\eta. \tag{2}$$

By taking the new coordinates $x = \xi - \eta$ and $y = \xi + \eta$, the hyperbolic NNV equation (1) becomes

$$\begin{aligned} &u_t = 2u_{yyy} + 6u_{xxy} + 3(u(v+w))_y + 3(u(v-w))_x, \\ &(\partial_y - \partial_x)v = (\partial_y + \partial_x)u, \qquad (\partial_y + \partial_x)w = (\partial_y - \partial_x)u, \end{aligned} \tag{3}$$

and the Lax pair (2) becomes

$$f_{yy} - f_{xx} + uf = 0, \qquad f_t = 2f_{yyy} + 6u_{xxy} + 3(v+w)f_y + 3(v-w)f_x. \tag{4}$$

On the other hand, the DSII equation is

$$-\mathrm{i}f_\tau = f_{xx} - f_{yy} - \mathrm{i}(g - \bar{g})f, \qquad (\partial_y - \mathrm{i}\partial_x)g = (\partial_x - \mathrm{i}\partial_y)(|f|^2), \tag{5}$$

which has a Lax pair

$$\begin{aligned} &\Psi_y = \mathrm{i}J\Psi_x + P\Psi, \\ &\Psi_\tau = 2\mathrm{i}J\Psi_{xx} + 2P\Psi_x + Q\Psi, \end{aligned} \tag{6}$$

where

$$J=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad P=\begin{pmatrix}&f\\-\bar f&\end{pmatrix},\qquad Q=\begin{pmatrix}g&f_x-\mathrm{i}f_y\\-\bar f_x-\mathrm{i}\bar f_y&\bar g\end{pmatrix}.\tag{7}$$

If $(f, g)$ is independent of $\tau$, (5) becomes the stationary DSII equation:

$$f_{xx}-f_{yy}-\mathrm{i}(g-\bar g)f=0,\qquad (\partial_y-\mathrm{i}\partial_x)g=(\partial_x-\mathrm{i}\partial_y)(|f|^2).\tag{8}$$

Taking $\Psi(x,y,\tau)=\Phi(x,y)\,\mathrm{e}^{2\mathrm{i}\lambda^2\tau}$ in (6), we get the Lax pair for (8) as

$$\Phi_y=\mathrm{i}J\Phi_x+P\Phi,\qquad 2\mathrm{i}\lambda^2\Phi=2\mathrm{i}J\Phi_{xx}+2P\Phi_x+Q\Phi.\tag{9}$$

The first equation of (4) and the first equation of (8) are similar, and the second equation of (4) is of order 3. Hence, we introduce an extra equation to the Lax pair (9) so that the whole system becomes

$$\begin{aligned}&\Phi_y=M(\partial)\Phi\equiv\mathrm{i}J\Phi_x+P\Phi,\\&2\mathrm{i}\lambda^2\Phi=L(\partial)\Phi\equiv2\mathrm{i}J\Phi_{xx}+2P\Phi_x+Q\Phi,\\&\Phi_t=N(\partial)\Phi\equiv16\mathrm{i}J\Phi_{xxx}+16P\Phi_{xx}+R\Phi_x+S\Phi,\end{aligned}\tag{10}$$

where $J$, $P$ and $Q$ are given by (7):

$$\begin{aligned}&R=4\begin{pmatrix}3g+\mathrm{i}|f|^2&4f_x-2\mathrm{i}f_y\\-4\bar f_x-2\mathrm{i}\bar f_y&3\bar g-\mathrm{i}|f|^2\end{pmatrix},\\&S=2\begin{pmatrix}3g_x+2\mathrm{i}\bar f f_x+\bar f f_y-f\bar f_y&6f_{xx}-2\mathrm{i}f_{xy}-\mathrm{i}(g-\bar g)f+2|f|^2f\\-6\bar f_{xx}-2\mathrm{i}\bar f_{xy}+\mathrm{i}(g-\bar g)\bar f-2|f|^2\bar f&3\bar g_x-2\mathrm{i}f\bar f_x+f\bar f_y-\bar f f_y\end{pmatrix},\end{aligned}\tag{11}$$

and $L(\partial)$, $M(\partial)$ and $N(\partial)$ refer to differential operators with respect to $x$ whose coefficients are $2\times2$ matrices, $\partial=\partial_x$.

The integrability conditions of (10) include the following equations:

$$f_{yy}-f_{xx}+uf=0,\qquad f_t=2f_{yyy}+6f_{xxy}+3(v+w)f_y+3(v-w)f_x,\tag{12}$$

$$\begin{aligned}&(\partial_y-\mathrm{i}\partial_x)g=(\partial_x-\mathrm{i}\partial_y)(|f|^2),\\&\frac{\mathrm{i}}{2}g_t=-2g_{xxx}+2\bar f f_{xxy}+2f\bar f_{xxy}+4\mathrm{i}\bar f f_{xxx}+4\mathrm{i}f\bar f_{xxx}\\&\quad+2(\bar f_x-\mathrm{i}\bar f_y)f_{xy}+2(f_x-\mathrm{i}f_y)\bar f_{xy}+2(\mathrm{i}\bar f_x+2\bar f_y)f_{xx}+2(\mathrm{i}f_x+2f_y)\bar f_{xx}\\&\quad+(2|f|^2-\mathrm{i}(g-\bar g))(|f|^2)_y+(6\mathrm{i}|f|^2+(g-\bar g))(|f|^2)_x-2|f|^2\bar g_x+6\mathrm{i}gg_x,\end{aligned}\tag{13}$$

where

$$u=\mathrm{i}(g-\bar g),\qquad v=2|f|^2+(g+\bar g),\qquad w=2|f|^2-(g+\bar g).\tag{14}$$

Note that (12) is exactly the same as the original Lax pair (4) of the hyperbolic NNV equation. By direct calculation, we know that $(u, v, w)$ satisfies the hyperbolic NNV equation (3) provided that $f$ and $g$ satisfy (12)–(14). Therefore, explicit solutions of the hyperbolic NNV equation can be obtained from those of (12)–(14).

Clearly, the solutions of (12)–(14) are only part of those of the hyperbolic NNV equation. However, they include some interesting ones which will be shown in the rest of this paper.

## 3. Darboux transformation

The binary Darboux transformation for the hyperbolic NNV equation is well known [17]. Integrations are needed in constructing explicit solutions. However, for the DSII equation, the usual Darboux transformation without integration is known. This Darboux transformation is simpler than the binary Darboux transformation for the hyperbolic NNV equation, and can be easily used to the stationary DSII equation so that explicit solutions of the hyperbolic NNV equation can be constructed.

Note that the coefficients of $L(\partial)$, $M(\partial)$ and $N(\partial)$ satisfy

$$\mathrm{i}J,\ P,\ Q,\ R,\ S\in\Sigma, \tag{15}$$

where

$$\Sigma=\{A \text{ is a } 2\times 2 \text{ matrix} | KAK^{-1}=\bar{A}\}=\left\{\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix}\middle| a,b\in\mathbf{C}\right\}, \tag{16}$$

$K=\begin{pmatrix} & -1\\ 1 & \end{pmatrix}$. That is, $L(\partial)$, $M(\partial)$ and $N(\partial)$ satisfy

$$KL(\partial)K^{-1}=\bar{L}(\partial),\qquad KM(\partial)K^{-1}=\bar{M}(\partial),\qquad KN(\partial)K^{-1}=\bar{N}(\partial). \tag{17}$$

Hence, if $\Phi=\begin{pmatrix}\xi\\ \eta\end{pmatrix}$ is a solution of (10) with $\lambda=\lambda_0$, then $K\bar{\Phi}=\begin{pmatrix}-\bar{\eta}\\ \bar{\xi}\end{pmatrix}$ is a solution of (10) with $\lambda=\pm\mathrm{i}\bar{\lambda}_0$.

The Darboux transformation of an arbitrary order is constructed as follows [8, 23]. Suppose that

$$G(\partial)=\partial^n+G_1(x,y,t)\partial^{n-1}+\cdots+G_n(x,y,t) \tag{18}$$

is a Darboux operator for (10), i.e. there exist $L'(\partial)$, $M'(\partial)$ and $N'(\partial)$ which have the same form as $L(\partial)$, $M(\partial)$ and $N(\partial)$ with $f$ and $g$ replaced by certain $f'$ and $g'$, such that $\Phi'=G(\partial)\Phi$ satisfies

$$\lambda\Phi'=L'(\partial)\Phi',\qquad \Phi'_y=M'(\partial)\Phi',\qquad \Phi'_t=N'(\partial)\Phi'. \tag{19}$$

If so, then $G(\partial)$ satisfies

$$\begin{aligned} &L'(\partial)G(\partial)=G(\partial)L(\partial),\\ &M'(\partial)G(\partial)=G(\partial)M(\partial)+G_y(\partial),\\ &N'(\partial)G(\partial)=G(\partial)N(\partial)+G_t(\partial).\end{aligned} \tag{20}$$

Since $L(\partial)$, $M(\partial)$ and $N(\partial)$ satisfy relations (17), and $L'(\partial)$, $M'(\partial)$, $N'(\partial)$ satisfy the similar relations

$$KL'(\partial)K^{-1}=\bar{L}'(\partial),\qquad KM'(\partial)K^{-1}=\bar{M}'(\partial),\qquad KN'(\partial)K^{-1}=\bar{N}'(\partial), \tag{21}$$

we want that $G(\partial)$ satisfies $KG(\partial)K^{-1}=\bar{G}(\partial)$. Write

$$G_j=\begin{pmatrix} a_j & b_j\\ -\bar{b}_j & \bar{a}_j\end{pmatrix}. \tag{22}$$

Denote $L'(\partial)=2\mathrm{i}J\partial^2+2P'\partial+Q'$; then the first equation of (20) leads to

$$\begin{aligned}&(2\mathrm{i}J\partial^2+2P'\partial+Q')(\partial^n+G_1\partial^{n-1}+\cdots+G_n)\\ &\quad=(\partial^n+G_1\partial^{n-1}+\cdots+G_n)(2\mathrm{i}J\partial^2+2P\partial+Q),\end{aligned} \tag{23}$$

in which the coefficients of $\partial^{n+1}$ and $\partial^n$ give

$$\begin{aligned} &P'=P-\mathrm{i}[J,G_1],\\ &Q'=Q-2\mathrm{i}[J,G_2]-2[P,G_1]+2\mathrm{i}[J,G_1]G_1+2nP_x-4\mathrm{i}JG_{1,x}.\end{aligned} \tag{24}$$

Hence, after the action of Darboux transformation,

$$\begin{aligned} &f'=f-2\mathrm{i}b_1,\\ &g'=g-4\mathrm{i}a_{1,x}-2(\bar{f}b_1-f\bar{b}_1)-4\mathrm{i}|b_1|^2,\\ &u'=u+8|b_1|^2-4\mathrm{i}(\bar{f}b_1-f\bar{b}_1)+4(a_1+\bar{a}_1)_x,\\ &v'=v+8|b_1|^2-4\mathrm{i}(\bar{f}b_1-f\bar{b}_1)-4\mathrm{i}(a_1-\bar{a}_1)_x,\\ &w'=w+8|b_1|^2-4\mathrm{i}(\bar{f}b_1-f\bar{b}_1)+4\mathrm{i}(a_1-\bar{a}_1)_x.\end{aligned} \tag{25}$$

Now take $n$ distinct complex numbers $\lambda_1, \ldots, \lambda_n$ with $\lambda_j = \mu_j + \mathrm{i}\nu_j$ ($\mu_j$'s and $\nu_j$'s are real). Let $\Phi_j = \binom{\xi_j}{\eta_j}$ be a column solution of (10) with $\lambda = \lambda_j$; then $\Phi_{n+j} \equiv K\bar{\Phi}_j = \binom{-\bar{\eta}_j}{\bar{\xi}_j}$ is a solution of (10) with $\lambda = \pm\mathrm{i}\bar{\lambda}_j (j = 1, \ldots, n)$. The Darboux transformation is determined by the system of linear algebraic equations:

$$G(\partial)\Phi_j = 0 \qquad (j = 1, \ldots, 2n) \tag{26}$$

if it has a unique solution [23].

Denote

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \qquad \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}, \qquad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}; \tag{27}$$

then (26) becomes

$$T\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} \partial^n \xi \\ -\partial^n \bar{\eta} \end{pmatrix}, \tag{28}$$

where

$$T = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \tag{29}$$

$$A = (\partial^{n-1}\xi \quad \cdots \quad \xi), \qquad B = (\partial^{n-1}\eta \quad \cdots \quad \eta). \tag{30}$$

Equation (26) has a unique solution if and only if $\det T \neq 0$.

## 4. Expression of soliton solutions

For a zero seed solution $u = v = w = f = g = 0$, (10) becomes

$$\Phi_{xx} = \lambda^2 J\Phi, \qquad \Phi_y = \mathrm{i}J\Phi_x, \qquad \Phi_t = 16\mathrm{i}J\Phi_{xxx} \tag{31}$$

with $\Phi = (\xi, \eta)^T$. Hence, take

$$\xi_j = \kappa_j^{(1)}\left(\mathrm{e}^{\rho_j^{(1)}+\mathrm{i}\sigma_j^{(1)}} + \mathrm{e}^{-\rho_j^{(1)}-\mathrm{i}\sigma_j^{(1)}}\right), \qquad \eta_j = \kappa_j^{(2)}\left(\mathrm{e}^{\rho_j^{(2)}+\mathrm{i}\sigma_j^{(2)}} + \mathrm{e}^{-\rho_j^{(2)}-\mathrm{i}\sigma_j^{(2)}}\right), \tag{32}$$

where

$$\begin{aligned}
\rho_j^{(1)} &= \mathrm{Re}\left(\lambda_j x + \mathrm{i}\lambda_j y + 16\mathrm{i}\lambda_j^3 t\right) + \rho_{j0}^{(1)} = \mu_j x - \nu_j y + 16\left(\nu_j^3 - 3\mu_j^2\nu_j\right)t + \rho_{j0}^{(1)}, \\
\rho_j^{(2)} &= \mathrm{Re}\left(\mathrm{i}\lambda_j x + \lambda_j y - 16\lambda_j^3 t\right) + \rho_{j0}^{(2)} = -\nu_j x + \mu_j y - 16\left(\mu_j^3 - 3\mu_j\nu_j^2\right)t + \rho_{j0}^{(2)}, \\
\sigma_j^{(1)} &= \mathrm{Im}\left(\lambda_j x + \mathrm{i}\lambda_j y + 16\mathrm{i}\lambda_j^3 t\right) + \sigma_{j0}^{(1)} = \nu_j x + \mu_j y + 16\left(\mu_j^3 - 3\mu_j\nu_j^2\right)t + \sigma_{j0}^{(1)}, \\
\sigma_j^{(2)} &= \mathrm{Im}\left(\mathrm{i}\lambda_j x + \lambda_j y - 16\lambda_j^3 t\right) + \sigma_{j0}^{(2)} = \mu_j x + \nu_j y + 16\left(\nu_j^3 - 3\mu_j^2\nu_j\right)t + \sigma_{j0}^{(2)},
\end{aligned} \tag{33}$$

where $\kappa_j^{(1)}$ and $\kappa_j^{(2)}$ are non-zero constants, and $\rho_{j0}^{(1)}, \rho_{j0}^{(2)}, \sigma_{j0}^{(1)}$ and $\sigma_{j0}^{(2)}$ are real constants. By solving $a_j$'s and $b_j$'s from (28), the Darboux transformation (25) gives the multi-soliton solution

$$\begin{aligned}
&f = -2\mathrm{i}b_1, \qquad g = -4\mathrm{i}a_{1,x} - 4\mathrm{i}|b_1|^2, \qquad u = 8|b_1|^2 + 4(a_1 + \bar{a}_1)_x, \\
&v = 8|b_1|^2 - 4\mathrm{i}(a_1 - \bar{a}_1)_x, \qquad w = 8|b_1|^2 + 4\mathrm{i}(a_1 - \bar{a}_1)_x.
\end{aligned} \tag{34}$$

Hereafter, we omit the primes on $f, g, u, v, w$ for those obtained by the action of Darboux transformation.

Let $K_n = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$. Denote

$$\zeta = \begin{pmatrix} \xi \\ -\bar{\eta} \end{pmatrix}, \qquad R_{j\cdots k} = (\partial^j \zeta \quad \partial^{j-1}\zeta, \ldots, \partial^k \zeta) \tag{35}$$

for $j \geqslant k$; then

$$T = (R_{n-1\cdots 0} \quad K_n \bar{R}_{n-1\cdots 0}). \tag{36}$$

Let

$$\Pi = \begin{pmatrix} \partial^n \zeta & \partial^{n-1}\zeta & \partial^{n-2}\zeta & R_{n-3\cdots 0} & K_n \bar{R}_{n-1\cdots 0} & 0 & 0 \\ \partial^{n+1}\zeta & \partial^n \zeta & \partial^{n-1}\zeta & 0 & 0 & R_{n-2\cdots 0} & K_n \bar{R}_{n-1\cdots 0} \end{pmatrix}. \tag{37}$$

**Theorem 1.** *When* $\det T \neq 0$, *the multi-soliton solution u of the hyperbolic NNV equation given by (34) can be written as*

$$u = -8 \frac{\operatorname{Re} \det \Pi}{(\det T)^2}. \tag{38}$$

**Proof.** Solved from (28) by the Cramer rule,

$$a_1 = -(\det T)^{-1} |\partial^n \zeta \quad R_{n-2\cdots 0} \quad K_n \bar{R}_{n-1\cdots 0}|, \tag{39}$$

$$b_1 = -(\det T)^{-1} |R_{n-1\cdots 0} \quad \partial^n \zeta \quad K_n \bar{R}_{n-2\cdots 0}|. \tag{40}$$

$$\begin{aligned} a_1 + \bar{a}_1 &= -(\det T)^{-1} (|\partial^n \zeta \quad R_{n-2\cdots 0} \quad K_n \bar{R}_{n-1\cdots 0}| + \overline{|\partial^n \zeta \quad R_{n-2\cdots 0} \quad K_n \bar{R}_{n-1\cdots 0}|}) \\ &= -(\det T)^{-1} (|\partial^n \zeta \quad R_{n-2\cdots 0} \quad K_n \bar{R}_{n-1\cdots 0}| + |R_{n-1\cdots 0} \quad K_n \partial^n \bar{\zeta} \quad K_n \bar{R}_{n-2\cdots 0}|) \\ &= -(\det T)^{-1} (\det T)_x = -\operatorname{tr}(T^{-1} T_x), \end{aligned} \tag{41}$$

$$a_{1,x} + \bar{a}_{1,x} = -\operatorname{tr}(T^{-1} T_{xx}) + \operatorname{tr}((T^{-1} T_x)^2). \tag{42}$$

Denote $\widetilde{I}_k = \begin{pmatrix} I_{k\times k} \\ 0_{(n-k)\times k} \end{pmatrix}$. Let $h = \begin{pmatrix} \widetilde{a} \\ \widetilde{b} \end{pmatrix}$ be the solution of $Th = -\partial^{n+1}\zeta$ where $\widetilde{a} = (\widetilde{a}_1, \ldots, \widetilde{a}_n)^T$ and $\widetilde{b} = (\widetilde{b}_1, \ldots, \widetilde{b}_n)^T$; then

$$\begin{aligned} a_{1,x} + \bar{a}_{1,x} &= -\operatorname{tr} \begin{pmatrix} -\widetilde{a} & -a & \widetilde{I}_{n-2} & \bar{\widetilde{b}} & \bar{b} & 0 \\ -\widetilde{b} & -b & 0 & -\bar{\widetilde{a}} & -\bar{a} & \widetilde{I}_{n-2} \end{pmatrix} + \operatorname{tr} \begin{pmatrix} -a & \widetilde{I}_{n-1} & \bar{b} & 0 \\ -b & 0 & -\bar{a} & \widetilde{I}_{n-1} \end{pmatrix}^2 \\ &= a_1^2 + \bar{a}_1^2 + \widetilde{a}_1 + \bar{\widetilde{a}}_1 - a_2 - \bar{a}_2 - 2|b_1|^2. \end{aligned} \tag{43}$$

According to (34),

$$u = 8 \operatorname{Re}\big(a_1^2 + \widetilde{a}_1 - a_2\big). \tag{44}$$

Let $d = -(\det T)^2 \big(a_1^2 + \widetilde{a}_1 - a_2\big)$; then, by the Cramer rule,

$$\begin{aligned} d = &-|\partial^n \zeta \quad R_{n-2\cdots 0} \quad K_n \bar{R}_{n-1\cdots 0} \quad |^2 + \det T |\partial^{n+1}\zeta \quad R_{n-2\cdots 0} \quad K_n \bar{R}_{n-1\cdots 0} \quad | \\ &+ \det T |\partial^n \zeta \quad \partial^{n-1}\zeta \quad R_{n-3\cdots 0} \quad K_n \bar{R}_{n-1\cdots 0} \quad |. \end{aligned} \tag{45}$$

Using the Laplace expansion of $\det \Pi$, $d = \det \Pi$. Hence,

$$u = -8 \frac{\operatorname{Re} d}{(\det T)^2} = -8 \frac{\operatorname{Re} \det \Pi}{(\det T)^2}. \tag{46}$$

The theorem is proved. □

**Remark 1.** According to lemma 2 of appendix A, $\det T \geqslant 0$ holds everywhere. However, $\det T > 0$ may not hold everywhere when the parameters $\rho_{j0}^{(k)}$ and $\sigma_{j0}^{(k)}$ take some special values, as will be shown in the following section for a single soliton solution. On the other hand, $\det T > 0$ holds everywhere in a generic case, which will be shown here.

The Darboux operator $G(\partial)$ of order $n$ can be constructed by composing $n$ Darboux operators of order 1 as follows. For given $\lambda_1, \ldots, \lambda_n$ and $\Phi_1, \ldots, \Phi_n$ as above, let $H_j = (\Phi_j, \Phi_{n+j})(j = 1, \ldots, n)$. If $\det H_1 \neq 0$, then $\Delta_1(\partial) = \partial - H_{1,x}H_1^{-1}$ is a Darboux operator of order 1. It transforms $(u, v, w, f, g)$ to $(u^{(1)}, v^{(1)}, w^{(1)}, f^{(1)}, g^{(1)})$ and transforms $H_j$ to $H_j^{(1)} = \Delta_1(\partial)H_j = H_{j,x} - H_{1,x}H_1^{-1}H_j (j = 2, 3, \ldots, n)$. Again, if $\det H_2^{(1)} \neq 0$, then $\Delta_2(\partial) = \partial - H_{2,x}^{(1)}\left(H_2^{(1)}\right)^{-1}$ is a Darboux operator of order 1 for the Lax pair with $(u^{(1)}, v^{(1)}, w^{(1)}, f^{(1)}, g^{(1)})$. It transforms $(u^{(1)}, v^{(1)}, w^{(1)}, f^{(1)}, g^{(1)})$ to $(u^{(2)}, v^{(2)}, w^{(2)}, f^{(2)}, g^{(2)})$ and transforms $H_j^{(1)}$ to $H_j^{(2)} = \Delta_2(\partial)H_j^{(1)} = H_{j,x}^{(1)} - H_{2,x}^{(1)}\left(H_2^{(1)}\right)^{-1}H_j^{(1)} (j = 3, 4, \ldots, n)$. Continuing this process, we get $H_j^{(k)} (k = 1, \ldots, n-1;\ j = k+1, \ldots, n)$ and $\Delta_j(\partial)(j = 1, \ldots, n)$. According to Zhou [23],

$$\begin{aligned} G(\partial) &= \Delta_n(\partial)\Delta_{n-1}(\partial)\cdots\Delta_1(\partial), \\ \det T &= \det\left(H_n^{(n-1)}\right)\det\left(H_{n-1}^{(n-2)}\right)\cdots\det\left(H_2^{(1)}\right)\det(H_1). \end{aligned} \tag{47}$$

Hence, $\det T \neq 0$ if all $\det H_j^{(j-1)} \neq 0$.

Suppose that $H_j^{(j-1)} = \begin{pmatrix} \xi_j^{(j-1)} & -\bar{\eta}_j^{(j-1)} \\ \eta_j^{(j-1)} & \bar{\xi}_j^{(j-1)} \end{pmatrix}$; then $\det H_j^{(j-1)} = \left|\xi_j^{(j-1)}\right|^2 + \left|\eta_j^{(j-1)}\right|^2 = 0$ if and only if $\xi_j^{(j-1)} = 0$ and $\eta_j^{(j-1)} = 0$ hold simultaneously. For fixed $j$, this gives a system of four real equations:

$$\operatorname{Re}\xi_j^{(j-1)} = 0, \qquad \operatorname{Im}\xi_j^{(j-1)} = 0, \qquad \operatorname{Re}\eta_j^{(j-1)} = 0, \qquad \operatorname{Im}\eta_j^{(j-1)} = 0 \tag{48}$$

for three real variables $x, y$ and $t$. It has no solution unless the parameters $\rho_{j0}^{(k)}$ and $\sigma_{j0}^{(k)} (j = 1, \ldots, n; k = 1, 2)$ take special values. This shows that $\det T > 0$ holds everywhere for generic $\rho_{j0}^{(k)}$ and $\sigma_{j0}^{(k)}$. Therefore, the multi-soliton solution $u$ is global for generic $\rho_{j0}^{(k)}$ and $\sigma_{j0}^{(k)}$.

## 5. Single soliton solution

By taking $n = 1$, the single soliton solution can be obtained as

$$u = \frac{16A}{B^2}, \tag{49}$$

where

$$\begin{aligned} B &= \left(\kappa_1^{(1)}\right)^2\cosh\left(2\rho_1^{(1)}\right) + \left(\kappa_1^{(2)}\right)^2\cosh\left(2\rho_1^{(2)}\right) + \left(\kappa_1^{(1)}\right)^2\cos\left(2\sigma_1^{(1)}\right) + \left(\kappa_1^{(2)}\right)^2\cos\left(2\sigma_1^{(2)}\right), \\ A &= -\left(\mu_1^2 - \nu_1^2\right)\left(\kappa_1^{(1)}\right)^4\cosh\left(2\rho_1^{(1)}\right)\cos\left(2\sigma_1^{(1)}\right) - 2\mu_1\nu_1\left(\kappa_1^{(1)}\right)^4\sinh\left(2\rho_1^{(1)}\right)\sin\left(2\sigma_1^{(1)}\right) \\ &\quad - \left(\mu_1^2 + \nu_1^2\right)\left(\kappa_1^{(1)}\right)^2\left(\kappa_1^{(2)}\right)^2\sinh\left(2\rho_1^{(1)}\right)\sin\left(2\sigma_1^{(2)}\right) + \left(\mu_1^2 - \nu_1^2\right)\left(\kappa_1^{(2)}\right)^4\cosh\left(2\rho_1^{(2)}\right) \\ &\quad \times \cos\left(2\sigma_1^{(2)}\right) + 2\mu_1\nu_1\left(\kappa_1^{(2)}\right)^4\sinh\left(2\rho_1^{(2)}\right)\sin\left(2\sigma_1^{(2)}\right) + \left(\mu_1^2 + \nu_1^2\right)\left(\kappa_1^{(1)}\right)^2\left(\kappa_1^{(2)}\right)^2 \\ &\quad \times \sinh\left(2\rho_1^{(2)}\right)\sin\left(2\sigma_1^{(1)}\right) + \left(\mu_1^2 - \nu_1^2\right)\left(\left(\kappa_1^{(2)}\right)^4 - \left(\kappa_1^{(1)}\right)^4\right). \end{aligned} \tag{50}$$

The solution is singular if $B = 0$, i.e. $|\xi_1|^2 + |\eta_1|^2 = 0$. This is equivalent to $\rho_1^{(1)} = \rho_1^{(2)} = 0, 2\sigma_1^{(1)} = j\pi + \pi/2, 2\sigma_1^{(2)} = k\pi + \pi/2$ for certain integers $j$ and $k$. In contrast, the solution is global if and only if $|\xi_1|^2 + |\eta_1|^2 \neq 0$ everywhere, i.e. the parameters satisfy

$$\mu_1\left(\rho_{10}^{(1)} - \sigma_{10}^{(2)} + k\pi + \pi/2\right) + \nu_1\left(\rho_{10}^{(2)} + \sigma_{10}^{(1)} - j\pi - \pi/2\right) \neq 0. \tag{51}$$

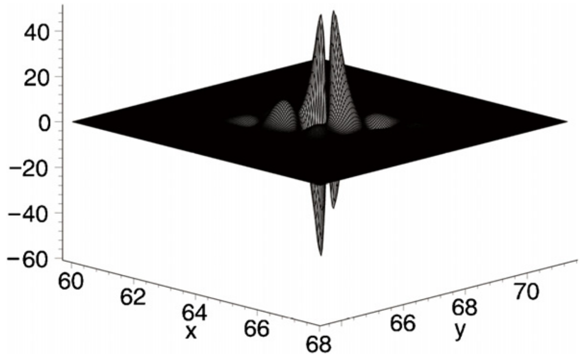


**Figure 1.** Single soliton solution $u$: $\lambda_1 = 2+0.5\mathrm{i}$, $\kappa_1^{(1)} = 1$, $\kappa_1^{(2)} = 1.2$, $\rho_{10}^{(1)} = 0$, $\rho_{10}^{(2)} = 1$, $\sigma_{10}^{(1)} = \sigma_{10}^{(2)} = 0$, $t = 1$.

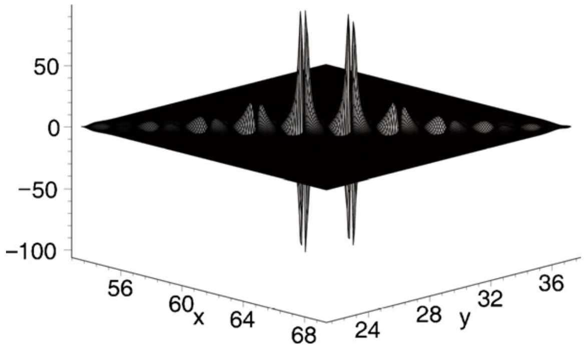


**Figure 2.** Single soliton solution $u$: $\lambda_1 = 1.1 + 0.9\mathrm{i}$, $\kappa_1^{(1)} = 1$, $\kappa_1^{(2)} = 1.2$, $\rho_{10}^{(1)} = 0$, $\rho_{10}^{(2)} = 1$, $\sigma_{10}^{(1)} = \sigma_{10}^{(2)} = 0$, $t = 1$.

We always suppose that (51) is satisfied, which is equivalent to $\det T \neq 0$.

When $\mu_1^2 \neq \nu_1^2$, the solution $u$ approaches zero exponentially at spatial infinity, and the peaks appear when neither $\rho_1^{(1)}$ nor $\rho_1^{(2)}$ is large. Hence, the center of the soliton locates near $\rho_1^{(1)} = 0$ and $\rho_1^{(2)} = 0$, i.e.

$$x = 64\mu_1\nu_1 t - \frac{\mu_1\rho_{10}^{(1)} + \nu_1\rho_{10}^{(2)}}{\mu_1^2 - \nu_1^2}, \qquad y = 16\left(\mu_1^2 + \nu_1^2\right)t - \frac{\nu_1\rho_{10}^{(1)} + \mu_1\rho_{10}^{(2)}}{\mu_1^2 - \nu_1^2}. \tag{52}$$

The solutions are shown in figures 1 and 2 for different parameters. The figure of the solution contains several peaks rather than a single peak, and the shape depends on the angle $\arctan \frac{\mu_1^2-\nu_1^2}{2\mu_1\nu_1}$ between the straight lines $\rho_1^{(1)} = 0$ and $\rho_1^{(2)} = 0$. Nevertheless, we still call it a single soliton solution because it is generated from the zero solution by the Darboux transformation of order 1, and the peaks in the solution never separate.

Note that although $u$ is localized, $v$ and $w$ are not.

If $\nu_1 = \mu_1 \neq 0$, the solution is invariant when $(x, y)$ is changed to $\left(x + \frac{m\pi}{2\mu_1}, y + \frac{m\pi}{2\mu_1}\right)$ for any integer $m$. Hence, the solution is periodic. Moreover, $\rho_1^{(1)} + \rho_1^{(2)} = \rho_{10}^{(1)} + \rho_{10}^{(2)}$. The

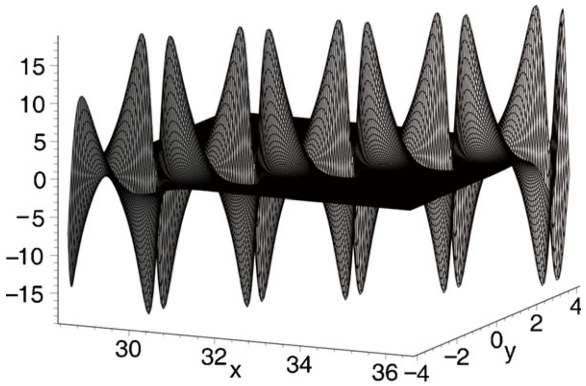


**Figure 3.** Periodic solution $u$: $\lambda_1 = 1+\mathrm{i}$, $\kappa_1^{(1)} = 1$, $\kappa_1^{(2)} = 1.2$, $\rho_{10}^{(1)} = 0$, $\rho_{10}^{(2)} = 1$, $\sigma_{10}^{(1)} = \sigma_{10}^{(2)} = 0$, $t = 1$.

peaks appear when neither $\rho_1^{(1)}$ nor $\rho_1^{(2)}$ is large. Hence, the peaks lie near the straight line $x - y - 32\mu_1^2 t + \frac{\rho_{10}^{(1)} - \rho_{10}^{(2)}}{2\mu_1} = 0$. The solution is shown in figure 3.

Similarly, the solution is also periodic if $\nu_1 = -\mu_1 \neq 0$.

## 6. Localization of multi-soliton solutions

In this section, we will prove that the multi-soliton solutions approach zero uniformly and exponentially at spatial infinity. In order to get global solutions, we always suppose that $\det T \neq 0$ everywhere, which is true for generic parameters $\rho_{j0}^{(k)}$ and $\sigma_{j0}^{(k)} (j = 1, \ldots, n; k = 1, 2)$.

Note that the solution of (28) is invariant if both $\xi_j$ and $\eta_j$ (for fixed $j$) are multiplied by a common function. Let

$$\omega_j = \begin{cases} \xi_j & \text{if} \quad |\xi_j| \geqslant |\eta_j| \\ \eta_j & \text{if} \quad |\xi_j| < |\eta_j| \end{cases}, \tag{53}$$

$$\mathring{T} = \mathrm{diag}(\omega_1, \ldots, \omega_n, \qquad \bar{\omega}_1, \ldots, \bar{\omega}_n). \tag{54}$$

Let $\widetilde{T} = \mathring{T}^{-1} T$; then the norm of each entry of $\widetilde{T}$ cannot exceed 1. Although $\widetilde{T}$ is not continuous, $|\det \widetilde{T}|$ is continuous.

Let $x = r\cos\theta$ and $y = r\sin\theta$. Since $\rho_j^{(1)}$'s and $\rho_j^{(2)}$'s in (33) depend on $x$ and $y$ linearly, we can write, for $k = 1, 2$,

$$\rho_j^{(k)}(r\cos\theta, r\sin\theta, t) = \varepsilon_j^{(k)}(\theta)\alpha_j^{(k)}(\theta) r + \beta_j^{(k)}, \tag{55}$$

where $\varepsilon_j^{(k)}(\theta) = \pm 1 (j = 1, \ldots, n)$ so that $\alpha_j^{(k)}(\theta) \geqslant 0$. Here, the variable $t$ is omitted in $\alpha_j^{(k)}(\theta)$, $\beta_j^{(k)}$ and $\varepsilon_j^{(k)}(\theta)$.

Clearly, $\alpha_j^{(k)}$'s are continuous functions. Also note that $\varepsilon_j^{(k)}(\theta)$ is not well defined when $\alpha_j^{(k)}(\theta) = 0$.

**Theorem 2.** *Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct non-zero complex numbers such that $\bar{\lambda}_j \neq \pm\mathrm{i}\lambda_l$ for all $j, l = 1, \ldots, n$. $u$ is the multi-soliton solution given by (38). Then for fixed t, there are positive constants $r_0$, $\chi$ and $C$ such that*

$$|u(r\cos\theta, r\sin\theta, t)| \leqslant C\,\mathrm{e}^{-\chi r} \tag{56}$$

*for* $r > r_0$ *and all* $e^{i\theta} \in S^1$. *Hence,* $u(r\cos\theta, r\sin\theta, t) \to 0$ *uniformly and exponentially as* $r \to +\infty$.

**Proof.** The proof is divided into four steps.

*Step 1*. Obtain the asymptotic behavior of $\xi_j$'s and $\eta_j$'s.

Let $\lambda_j = \mu_j + i\nu_j$ where $\mu_j$'s and $\nu_j$'s are real; then $\mu_j \neq \pm\nu_j$ for all $j = 1, \ldots, n$.

Let $Z^{(\varepsilon)} = \{e^{i\theta} \in S^1 | \tan\theta = \varepsilon\}$ for $\varepsilon = \pm 1$, $Z = Z^{(+1)} \cup Z^{(-1)}$. If $e^{i\theta} \in Z^{(\varepsilon)}$, then by (33), $\alpha_j^{(1)}(\theta) = \alpha_j^{(2)}(\theta) = \frac{1}{\sqrt{2}}|\mu_j - \varepsilon\nu_j| > 0$, $\varepsilon_j^{(2)}(\theta) = \varepsilon\varepsilon_j^{(1)}(\theta)$ and $\varepsilon_j^{(1)}(\theta)(\mu_j - \varepsilon\nu_j)\cos\theta > 0$ for all $j = 1, \ldots, n$. If $e^{i\theta} \in S^1 \backslash Z$, then $\alpha_j^{(1)}(\theta) = |\mu_j\cos\theta - \nu_j\sin\theta|, \alpha_j^{(2)}(\theta) = |-\nu_j\cos\theta + \mu_j\sin\theta|$ with $\alpha_j^{(1)}(\theta) \neq \alpha_j^{(2)}(\theta)$.

For $\delta \in (0, \pi/4)$, define

$$\begin{aligned} &\Omega_\delta^{(\varepsilon)} = \{e^{i\theta} | \text{there exists } e^{i\theta_0} \in Z^{(\varepsilon)} \text{ such that } |\theta - \theta_0| < \delta\} \quad (\varepsilon = \pm 1), \\ &\Omega_\delta^{(0)} = \{e^{i\theta} || \theta - \theta_0| > \delta/2 \text{ for all } e^{i\theta_0} \in Z\}. \end{aligned} \tag{57}$$

Then $\Omega_\delta^{(+1)} \cup \Omega_\delta^{(-1)} \cup \Omega_\delta^{(0)} = S^1$, and there exists $\delta \in (0, \pi/4)$ and $\omega > 0$ such that

$$\begin{aligned} &\alpha_j^{(1)}(\theta) > \omega, \qquad \alpha_j^{(2)}(\theta) > \omega, \qquad \varepsilon_j^{(2)}(\theta) = \varepsilon\varepsilon_j^{(1)}(\theta) \qquad \text{if} \quad e^{i\theta} \in \Omega_\delta^{(\varepsilon)}, \\ &|\alpha_j^{(1)}(\theta) - \alpha_j^{(2)}(\theta)| > \omega \qquad \text{if} \quad e^{i\theta} \in \Omega_\delta^{(0)}. \end{aligned} \tag{58}$$

Recall that

$$\begin{aligned} \xi_j &= \kappa_j^{(1)}\left(e^{\varepsilon_j^{(1)}(\theta)\lambda_j x + i\varepsilon_j^{(1)}(\theta)\lambda_j y + 16i\varepsilon_j^{(1)}(\theta)\lambda_j^3 t} + e^{-\varepsilon_j^{(1)}(\theta)\lambda_j x - i\varepsilon_j^{(1)}(\theta)\lambda_j y - 16i\varepsilon_j^{(1)}(\theta)\lambda_j^3 t}\right) \\ &= \kappa_j^{(1)} e^{\alpha_j^{(1)}(\theta)r + \varepsilon_j^{(1)}(\theta)\beta_j^{(1)} + i\varepsilon_j^{(1)}(\theta)\sigma_j^{(1)}(\theta,r)}\left(1 + e^{-2\alpha_j^{(1)}(\theta)r - 2\varepsilon_j^{(1)}(\theta)\beta_j^{(1)} - 2i\varepsilon_j^{(1)}(\theta)\sigma_j^{(1)}(\theta,r)}\right), \\ \eta_j &= \kappa_j^{(2)}\left(e^{i\varepsilon_j^{(2)}(\theta)\lambda_j x + \varepsilon_j^{(2)}(\theta)\lambda_j y - 16\varepsilon_j^{(2)}(\theta)\lambda_j^3 t} + e^{-i\varepsilon_j^{(2)}(\theta)\lambda_j x - \varepsilon_j^{(2)}(\theta)\lambda_j y + 16\varepsilon_j^{(2)}(\theta)\lambda_j^3 t}\right) \\ &= \kappa_j^{(2)} e^{\alpha_j^{(2)}(\theta)r + \varepsilon_j^{(2)}(\theta)\beta_j^{(2)} + i\varepsilon_j^{(2)}(\theta)\sigma_j^{(2)}(\theta,r)}\left(1 + e^{-2\alpha_j^{(2)}(\theta)r - 2\varepsilon_j^{(2)}(\theta)\beta_j^{(2)} - 2i\varepsilon_j^{(2)}(\theta)\sigma_j^{(2)}(\theta,r)}\right). \end{aligned} \tag{59}$$

Let

$$Y_j(\theta, r) = \left(\kappa_j^{(1)}\right)^{-1}\kappa_j^{(2)}\, e^{(\alpha_j^{(2)}(\theta) - \alpha_j^{(1)}(\theta))r + \varepsilon_j^{(2)}(\theta)\beta_j^{(2)} - \varepsilon_j^{(1)}(\theta)\beta_j^{(1)} + i\varepsilon_j^{(2)}(\theta)\sigma_j^{(2)}(\theta,r) - i\varepsilon_j^{(1)}(\theta)\sigma_j^{(1)}(\theta,r)}. \tag{60}$$

When $r \to +\infty$, the following limits hold uniformly.

For $e^{i\theta} \in \Omega_\delta^{(\varepsilon)}$ with $\alpha_j^{(1)}(\theta) \geqslant \alpha_j^{(2)}(\theta)$,

$$\xi_j^{-1}\partial^k\xi_j \to \left(\varepsilon_j^{(1)}(\theta)\lambda_j\right)^k, \qquad \xi_j^{-1}\partial^k\eta_j - \left(i\varepsilon_j^{(2)}(\theta)\lambda_j\right)^k Y_j(\theta, r) \to 0. \tag{61}$$

For $e^{i\theta} \in \Omega_\delta^{(\varepsilon)}$ with $\alpha_j^{(1)}(\theta) \leqslant \alpha_j^{(2)}(\theta)$,

$$\eta_j^{-1}\partial^k\xi_j - \left(\varepsilon_j^{(1)}(\theta)\lambda_j\right)^k Y_j(\theta, r)^{-1} \to 0, \qquad \eta_j^{-1}\partial^k\eta_j \to \left(i\varepsilon_j^{(2)}(\theta)\lambda_j\right)^k. \tag{62}$$

For $e^{i\theta} \in \Omega_\delta^{(0)}$ with $\alpha_j^{(1)}(\theta) > \alpha_j^{(2)}(\theta)$,

$$\xi_j^{-1}\partial^k\xi_j \to \left(\varepsilon_j^{(1)}(\theta)\lambda_j\right)^k, \qquad \xi_j^{-1}\partial^k\eta_j \to 0. \tag{63}$$

For $e^{i\theta} \in \Omega_\delta^{(0)}$ with $\alpha_j^{(1)}(\theta) < \alpha_j^{(2)}(\theta)$,

$$\eta_j^{-1}\partial^k\xi_j \to 0, \qquad \eta_j^{-1}\partial^k\eta_j \to \left(i\varepsilon_j^{(2)}(\theta)\lambda_j\right)^k. \tag{64}$$

*Step 2*. There exist $r_0 > 0$ and $c_0 > 0$ such that $\det\widetilde{T} > c_0$ when $r > r_0$.

When $e^{i\theta} \in Z^{(\varepsilon)} (\varepsilon = \pm 1)$, $\alpha_j^{(1)}(\theta) = \alpha_j^{(2)}(\theta)$ for $j = 1, \ldots, n$. Equation (59) implies

$$\frac{|\eta_j(\theta, r)|}{|\xi_j(\theta, r)|} \to \gamma_j(\theta) \equiv \frac{\left|\kappa_j^{(2)}\right| e^{\varepsilon_j^{(2)}(\theta)\beta_j^{(2)}}}{\left|\kappa_j^{(1)}\right| e^{\varepsilon_j^{(1)}(\theta)\beta_j^{(1)}}} \tag{65}$$

as $r \to +\infty$. By (61),

$$\det \widetilde{T}(\theta, r) = \left(\prod_{|\gamma_j(\theta)|>1} |\gamma_j(\theta)|\right)^{-2} \begin{vmatrix} A(\theta, r) & B(\theta, r) \\ -\bar{B}(\theta, r) & \bar{A}(\theta, r) \end{vmatrix} + o(1), \tag{66}$$

where $A(\theta, r)$ and $B(\theta, r)$ are $n \times n$ matrices, whose entries are

$$\begin{aligned} A_{jk}(\theta, r) &= \left(\varepsilon_j^{(1)}(\theta)\lambda_j\right)^k, \\ B_{jk}(\theta, r) &= \left(\mathrm{i}\varepsilon_j^{(2)}(\theta)\lambda_j\right)^k Y_j(\theta, r) = \left(\mathrm{i}\varepsilon\varepsilon_j^{(1)}(\theta)\lambda_j\right)^k Y_j(\theta, r), \end{aligned} \tag{67}$$

and $o(1)$ refers to the terms which tend to zero as $r \to +\infty$. Let

$$\begin{aligned} \Lambda &= \left(\varepsilon_1^{(1)}(\theta)\lambda_1, \ldots, \varepsilon_n^{(1)}(\theta)\lambda_n\right), \\ \Gamma &= \mathrm{diag}(Y_1(\theta, r), \ldots, Y_n(\theta, r)). \end{aligned} \tag{68}$$

By lemma 3 of appendix A, there exist $r_1 > 0$ and $c_1 > 0$ such that $\det \widetilde{T} > c_1$ for $\mathrm{e}^{\mathrm{i}\theta} \in Z^{(\varepsilon)}$ and $r > r_1$.

When $\mathrm{e}^{\mathrm{i}\theta} \in \Omega_\delta^{(\varepsilon)} \backslash Z(\varepsilon = \pm 1)$ with $0 < \theta - \theta_0 < \delta$ ($\theta_0 \in Z^{(\varepsilon)}$), $\varepsilon_j^{(2)}(\theta) = \varepsilon\varepsilon_j^{(1)}(\theta)$. Suppose that $\alpha_j^{(1)}(\theta) > \alpha_j^{(2)}(\theta)$ for $j = 1, \ldots, m$ and $\alpha_j^{(1)}(\theta) < \alpha_j^{(2)}(\theta)$ for $j = m+1, \ldots, n$. By (61) and (62),

$$\det \widetilde{T}(\theta, r) = \begin{vmatrix} A(\theta) & B(\theta, r) \\ C(\theta, r) & D(\theta) \\ -\bar{B}(\theta, r) & \bar{A}(\theta) \\ -\bar{D}(\theta) & \bar{C}(\theta, r) \end{vmatrix} + o(1) = \begin{vmatrix} A(\theta) & B(\theta, r) \\ \bar{D}(\theta) & -\bar{C}(\theta, r) \\ -\bar{B}(\theta, r) & \bar{A}(\theta) \\ C(\theta, r) & D(\theta) \end{vmatrix} + o(1), \tag{69}$$

where $A(\theta)$ and $B(\theta, r)$ are $m \times n$ matrices, and $C(\theta, r)$ and $D(\theta)$ are $(n-m) \times n$ matrices, whose entries are given by

$$\begin{aligned} &A_{jk}(\theta) = \left(\varepsilon_j^{(1)}(\theta)\lambda_j\right)^k, \qquad B_{jk}(\theta, r) = \left(\mathrm{i}\varepsilon\varepsilon_j^{(1)}(\theta)\lambda_j\right)^k Y_j(\theta, r) \quad (j = 1, \ldots, m), \\ &\bar{D}_{jk}(\theta) = \left(-\mathrm{i}\varepsilon_j^{(2)}(\theta)\bar{\lambda}_j\right)^k, \qquad -\bar{C}_{jk}(\theta, r) = \left(\varepsilon\varepsilon_j^{(2)}(\theta)\bar{\lambda}_j\right)^k \left(-\bar{Y}_j(\theta, r)^{-1}\right) \\ &\qquad (j = m+1, \ldots, n), \end{aligned} \tag{70}$$

and $o(1)$ refers to the terms which tend to zero uniformly as $r \to +\infty$.

Let

$$\begin{aligned} \Lambda &= \left(\varepsilon_1^{(1)}(\theta)\lambda_1, \ldots, \varepsilon_m^{(1)}(\theta)\lambda_m, -\mathrm{i}\varepsilon_{m+1}^{(2)}(\theta)\bar{\lambda}_{m+1}, \ldots, -\mathrm{i}\varepsilon_n^{(2)}(\theta)\bar{\lambda}_n\right), \\ \Gamma &= \mathrm{diag}(Y_1(\theta, r), \ldots, Y_m(\theta, r), -\bar{Y}_{m+1}(\theta, r)^{-1}, \ldots, -\bar{Y}_n(\theta, r)^{-1}). \end{aligned} \tag{71}$$

By lemma 3 of appendix A, there exist $r_2 > 0$ and $c_2 > 0$ such that $\det \widetilde{T} > c_2$ for $\mathrm{e}^{\mathrm{i}\theta} \in \Omega_\delta^{(+1)} \cup \Omega_\delta^{(-1)} \backslash Z$ with $0 < \theta - \theta_0 < \delta$ and $r > r_2$.

Similarly, when $\mathrm{e}^{\mathrm{i}\theta} \in \Omega_\delta^{(+1)} \cup \Omega_\delta^{(-1)} \backslash Z$ with $-\delta < \theta - \theta_0 < 0$ ($\theta_0 \in Z$), there exist $r_3 > 0$ and $c_3 > 0$ such that $\det \widetilde{T} > c_3$ for $r > r_3$.

When $\mathrm{e}^{\mathrm{i}\theta} \in \Omega_\delta^{(0)}$, suppose that $\alpha_j^{(1)}(\theta) > \alpha_j^{(2)}(\theta)$ for $j = 1, \ldots, m$ and $\alpha_j^{(1)}(\theta) < \alpha_j^{(2)}(\theta)$ for $j = m+1, \ldots, n$; then, by (63) and (64),

$$\lim_{r \to +\infty} \det \widetilde{T}(\theta, r) = \begin{vmatrix} A(\theta) & 0 \\ 0 & D(\theta) \\ 0 & \bar{A}(\theta) \\ -\bar{D}(\theta) & 0 \end{vmatrix} = \begin{vmatrix} A(\theta) & 0 \\ \bar{D}(\theta) & 0 \\ 0 & \bar{A}(\theta) \\ 0 & D(\theta) \end{vmatrix} = \left\| \begin{matrix} A(\theta) \\ \bar{D}(\theta) \end{matrix} \right\|^2 \tag{72}$$

holds uniformly, where $A(\theta)$ is an $m \times n$ matrix and $D(\theta)$ is an $(n-m) \times n$ matrix, whose entries are given by

$$\begin{aligned} A_{jk}(\theta) &= \left(\varepsilon_j^{(1)}(\theta)\lambda_j\right)^k, \qquad (j = 1, \ldots, m), \\ \bar{D}_{jk}(\theta) &= \left(-\mathrm{i}\varepsilon_j^{(2)}(\theta)\bar{\lambda}_j\right)^k \qquad (j = m+1, \ldots, n). \end{aligned} \tag{73}$$

Using the condition $\bar{\lambda}_j \neq \pm \mathrm{i}\lambda_l$ and the property of the Vandermonde determinant, we know that there exist $r_4 > 0$ and $c_4 > 0$ such that $\det \widetilde{T} > c_4$ for all $\mathrm{e}^{\mathrm{i}\theta} \in \Omega_\delta^{(0)}$ and $r > r_4$.

Let $r_0 = \max(r_1, r_2, r_3, r_4)$, $c_0 = \min(c_1, c_2, c_3, c_4)$; then for any $\mathrm{e}^{\mathrm{i}\theta} \in S^1$, $\det \widetilde{T} > c_0$ when $r > r_0$.

*Step 3*. Denote $\widetilde{\Pi} = \begin{pmatrix} \hat{T} & \\ & \hat{T} \end{pmatrix}^{-1} \Pi$; then $\lim_{r \to +\infty} \mathrm{Re} \det \widetilde{\Pi} = 0$ for any fixed $\mathrm{e}^{\mathrm{i}\theta} \in S^1$.

When $\mathrm{e}^{\mathrm{i}\theta} \in Z^{(\varepsilon)}$, considering (61), (62) and $\varepsilon_j^{(2)}(\theta) = \varepsilon\varepsilon_j^{(1)}(\theta)$, let $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2)$ with

$$\begin{aligned} \Lambda_1 &= \mathrm{diag}\left(\varepsilon_1^{(1)}(\theta)\lambda_1, \ldots, \varepsilon_n^{(1)}(\theta)\lambda_n\right), \\ \Lambda_2 &= \mathrm{diag}\left(-\mathrm{i}\varepsilon\varepsilon_1^{(1)}(\theta)\bar{\lambda}_1, \ldots, -\mathrm{i}\varepsilon\varepsilon_n^{(1)}(\theta)\bar{\lambda}_n\right), \end{aligned} \tag{74}$$

$$\zeta = \big(\underbrace{1, \ldots, 1}_{n}, -\bar{\xi}_1^{-1}\bar{\eta}_1, \ldots, -\bar{\xi}_n^{-1}\bar{\eta}_n\big)^T; \tag{75}$$

then $\bar{\Lambda}_2 = \mathrm{i}\varepsilon\Lambda_1$, and $\widetilde{\Pi} - \left(\prod_{|\gamma_j(\theta)|>1} |\gamma_j(\theta)|\right)^{-4} \Pi^\Lambda \to 0$ as $r \to +\infty$ where $\gamma_j(\theta)$'s are defined by (65) and $\Pi^\Lambda$ is defined by (A.27). According to lemma 4 of appendix A, $\mathrm{Re} \det \Pi^\Lambda \equiv 0$, which leads to $\lim_{r \to +\infty} \mathrm{Re} \det \widetilde{\Pi} = 0$.

When $\mathrm{e}^{\mathrm{i}\theta} \in S^1 \backslash Z$, suppose that $\alpha_j^{(1)}(\theta) > \alpha_j^{(2)}(\theta)$ for $j = 1, \ldots, m$ and $\alpha_j^{(1)}(\theta) < \alpha_j^{(2)}(\theta)$ for $j = m+1, \ldots, n$. By (63) and (64),

$$\begin{aligned} &\xi_j^{-1}\left(\partial^k \xi_j - \left(\varepsilon_j^{(1)}(\theta)\lambda_j\right)^k \xi_j\right) \to 0, \qquad \xi_j^{-1}\left(\partial^k \eta_j - \left(\mathrm{i}\varepsilon_j^{(1)}(\theta)\lambda_j\right)^k \eta_j\right) \to 0 \\ &\qquad (j = 1, \ldots, m), \\ &\eta_j^{-1}\left(\partial^k \xi_j - \left(\varepsilon_j^{(2)}(\theta)\lambda_j\right)^k \xi_j\right) \to 0, \qquad \eta_j^{-1}\left(\partial^k \eta_j - \left(\mathrm{i}\varepsilon_j^{(2)}(\theta)\lambda_j\right)^k \eta_j\right) \to 0 \\ &\qquad (j = m+1, \ldots, n) \end{aligned} \tag{76}$$

as $r \to +\infty$ since $\xi_j^{-1}\eta_j \to 0$ for $j = 1, \ldots, m$ and $\eta_j^{-1}\xi_j \to 0$ for $j = m+1, \ldots, n$.

Let $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2)$ with

$$\begin{aligned} \Lambda_1 &= \mathrm{diag}\left(\varepsilon_1^{(1)}(\theta)\lambda_1, \ldots, \varepsilon_m^{(1)}(\theta)\lambda_m, \varepsilon_{m+1}^{(2)}(\theta)\lambda_{m+1}, \ldots, \varepsilon_n^{(2)}(\theta)\lambda_n\right), \\ \Lambda_2 &= \mathrm{diag}\left(-\mathrm{i}\varepsilon_1^{(1)}(\theta)\bar{\lambda}_1, \ldots, -\mathrm{i}\varepsilon_m^{(1)}(\theta)\bar{\lambda}_m, -\mathrm{i}\varepsilon_{m+1}^{(2)}(\theta)\bar{\lambda}_{m+1}, \ldots, -\mathrm{i}\varepsilon_n^{(2)}(\theta)\bar{\lambda}_n\right), \end{aligned} \tag{77}$$

$$\zeta = (\underbrace{1, \ldots, 1}_{m}, \underbrace{0, \ldots, 0}_{n-m}, \underbrace{0, \cdots, 0}_{m}, \underbrace{-1, \ldots, -1}_{n-m})^T; \tag{78}$$

then $\bar{\Lambda}_2 = \mathrm{i}\Lambda_1$, and $\widetilde{\Pi} \to \Pi^\Lambda$ as $r \to +\infty$ where $\Pi^\Lambda$ is defined by (A.27). According to lemma 4 in appendix A, we have $\lim_{r \to +\infty} \mathrm{Re} \det \widetilde{\Pi} = \mathrm{Re} \det \Pi^\Lambda = 0$.

Till now, we have proved that $\det \widetilde{T}$ has a uniform positive lower bound for all $\theta$, and $\lim_{r \to +\infty} \frac{\mathrm{Re} \det \Pi}{(\det T)^2} = 0$ for any fixed $\theta$.

*Step 4*. $\lim_{r \to +\infty} \frac{\mathrm{Re} \det \Pi}{(\det T)^2} = 0$ uniformly for all $\mathrm{e}^{\mathrm{i}\theta} \in S^1$ as $r \to +\infty$.

Note that $\frac{\mathrm{Re} \det \Pi}{(\det T)^2}$ is of form $\frac{f(\theta, r)}{g(\theta, r)}$, where

$$f(\theta, r) = \sum_{j=1}^{m_1} \mathrm{e}^{\widetilde{\alpha}_j(\theta) r + \widetilde{\gamma}_j(\theta)}, \qquad g(\theta, r) = \sum_{j=1}^{m_2} \mathrm{e}^{\widetilde{\beta}_j(\theta) r + \widetilde{\delta}_j(\theta)} \tag{79}$$

are real-valued functions of $(\theta, r)$ and $\widetilde{\alpha}_j(\theta), \widetilde{\beta}_j(\theta), \widetilde{\gamma}_j(\theta), \widetilde{\delta}_j(\theta)$ are complex-valued continuous functions of $\theta$. Let $\widetilde{a}(\theta) = \max_{1\leqslant j\leqslant m_1} \operatorname{Re}\widetilde{\alpha}_j(\theta), \widetilde{b}(\theta) = \max_{1\leqslant j\leqslant m_2} \operatorname{Re}\widetilde{\beta}_j(\theta)$. Since $\lim_{r\to+\infty} \frac{f(\theta,r)}{g(\theta,r)} = 0$ for any fixed $\theta$, the real continuous function $\widetilde{a}(\theta) - \widetilde{b}(\theta) < 0$ achieves its maximum $-\chi < 0$ on the compact set $S^1$. We have known that $\det \widetilde{T}$ has a uniform positive lower bound as $r \geqslant r_0$ ($r_0$ is independent of $\theta$), and so has

$$\sum_{j=1}^{m_2} \mathrm{e}^{(\widetilde{\beta}_j(\theta)-\widetilde{b}(\theta))r+\widetilde{\delta}_j(\theta)}. \tag{80}$$

Hence,

$$\left|\frac{f(\theta,r)}{g(\theta,r)}\right| \leqslant \mathrm{e}^{-\chi r}\frac{\left|\sum_{j=1}^{m_1}\mathrm{e}^{(\widetilde{\alpha}_j(\theta)-\widetilde{a}(\theta))r+\widetilde{\gamma}_j(\theta)}\right|}{\sum_{j=1}^{m_2}\mathrm{e}^{(\widetilde{\beta}_j(\theta)-\widetilde{b}(\theta))r+\widetilde{\delta}_j(\theta)}} \leqslant C\,\mathrm{e}^{-\chi r} \tag{81}$$

as $r \geqslant r_0$ where $C$ is a constant independent of $\theta$. The theorem is proved. □

**Remark 2.** The solution $f$ of the stationary DSII given by the Darboux transformation mentioned in this paper does not tend to zero exponentially at spatial infinity, but the solution $u$ of the hyperbolic NNV equation does. This is possible because $u$ is given by $\mathrm{i}(g - \bar{g})$ as in (14), but not $f$, the solution of the stationary DSII equation.

## 7. Asymptotic behavior of the solutions as $t \to \infty$

In this section, the asymptotic behavior of the multi-soliton solutions as $t \to \infty$ will be discussed. In order to do so, we consider the problem in a moving frame. Let $x = x_0 + \theta_1 t$ and $y = y_0 + \theta_2 t$, where $(\theta_1, \theta_2)$ is the velocity of the moving frame and $(x_0, y_0)$ is the coordinate in the moving frame with this velocity. Then,

$$\begin{aligned} \rho_j^{(1)}(x_0+\theta_1 t, y_0+\theta_2 t, t) &= \varepsilon_j^{(1)}\alpha_j^{(1)}t + \beta_j^{(1)},\\ \rho_j^{(2)}(x_0+\theta_1 t, y_0+\theta_2 t, t) &= \varepsilon_j^{(2)}\alpha_j^{(2)}t + \beta_j^{(2)}, \end{aligned} \tag{82}$$

where $\varepsilon_j^{(k)} = \pm 1 (j = 1, \ldots, n; k = 1, 2)$ so that $\alpha_j^{(k)} \geqslant 0$. Write $\lambda_j = \mu_j + \mathrm{i}\nu_j$ $(j = 1, \ldots, n)$ where $\mu_j$'s and $\nu_j$'s are real; then, according to (33),

$$\begin{aligned} \varepsilon_j^{(1)}\alpha_j^{(1)} &= \mu_j\theta_1 - \nu_j\theta_2 + 16\big(\nu_j^3 - 3\mu_j^2\nu_j\big),\\ \varepsilon_j^{(2)}\alpha_j^{(2)} &= -\nu_j\theta_1 + \mu_j\theta_2 - 16\big(\mu_j^3 - 3\mu_j\nu_j^2\big) \end{aligned} \tag{83}$$

and

$$\begin{aligned} \varepsilon_j^{(1)}\alpha_j^{(1)} - \varepsilon_j^{(2)}\alpha_j^{(2)} &= (\mu_j+\nu_j)\big(\theta_1 - \theta_2 + 16\big(\mu_j^2 - 4\mu_j\nu_j + \nu_j^2\big)\big),\\ \varepsilon_j^{(1)}\alpha_j^{(1)} + \varepsilon_j^{(2)}\alpha_j^{(2)} &= (\mu_j-\nu_j)\big(\theta_1 + \theta_2 - 16\big(\mu_j^2 + 4\mu_j\nu_j + \nu_j^2\big)\big). \end{aligned} \tag{84}$$

**Theorem 3.** *Suppose that $\lambda_j = \mu_j + \mathrm{i}\nu_j \neq 0$ $(j = 1, \ldots, n)$ are distinct complex numbers where $\mu_j$'s and $\nu_j$'s are real numbers, such that*

$$\begin{aligned} &\mu_j \neq \pm\nu_j && \text{for all } j,\\ &\mu_j^2 + 4\mu_j\nu_j + \nu_j^2 \neq \mu_l^2 + 4\mu_l\nu_l + \nu_l^2 && \text{for all } j \neq l,\\ &\mu_j^2 - 4\mu_j\nu_j + \nu_j^2 \neq \mu_l^2 - 4\mu_l\nu_l + \nu_l^2 && \text{for all } j \neq l. \end{aligned} \tag{85}$$

*$u$ is the multi-soliton solution given by (38). Then for bounded $(x_0, y_0)$, $\lim_{t\to\infty} u(x_0+\theta_1 t, y_0+\theta_2 t, t) = 0$ except when*

$$\begin{aligned} \theta_1 &= 8\big(\mu_l^2 + 4\mu_l\nu_l + \nu_l^2 - \mu_j^2 + 4\mu_j\nu_j - \nu_j^2\big),\\ \theta_2 &= 8\big(\mu_l^2 + 4\mu_l\nu_l + \nu_l^2 + \mu_j^2 - 4\mu_j\nu_j + \nu_j^2\big), \qquad (j, l = 1, 2, \ldots, n). \end{aligned} \tag{86}$$

*Therefore, as $t \to \infty$, $u$ is asymptotic to at most $n^2$ solitons which move in the above velocities $(\theta_1, \theta_2)$ respectively.*

**Proof.** We will always suppose that $(\theta_1, \theta_2)$ does not satisfy (86). Then, by (84), $\alpha_j^{(1)} \neq 0$ whenever $\alpha_j^{(1)} = \alpha_j^{(2)}$. Moreover, we only consider the limit $t \to +\infty$. The conclusion is the same for $t \to -\infty$.

The proof is divided into three steps.

*Step 1.* Obtain the asymptotic behavior of $\xi_j$'s and $\eta_j$'s.

Suppose that $\alpha_j^{(1)} > \alpha_j^{(2)}$ for $j = 1, \ldots, m$; $\alpha_j^{(1)} < \alpha_j^{(2)}$ for $j = m+1, \ldots, p$ and $\alpha_j^{(1)} = \alpha_j^{(2)} \neq 0$ for $j = p+1, \ldots, n$. Then,

$$\begin{aligned}
&\xi_j^{-1}\left(\partial^k \xi_j - \left(\varepsilon_j^{(1)}\lambda_j\right)^k \xi_j\right) \to 0, && \xi_j^{-1}\left(\partial^k \eta_j - \left(\mathrm{i}\varepsilon_j^{(2)} s_j \lambda_j\right)^k \eta_j\right) \to 0 && (j = 1, \ldots, m),\\
&\eta_j^{-1}\left(\partial^k \xi_j - \left(\varepsilon_j^{(1)} s_j \lambda_j\right)^k \xi_j\right) \to 0, && \eta_j^{-1}\left(\partial^k \eta_j - \left(\mathrm{i}\varepsilon_j^{(2)}\lambda_j\right)^k \eta_j\right) \to 0 && (j = m+1, \ldots, p),\\
&\xi_j^{-1}\left(\partial^k \xi_j - \left(\varepsilon_j^{(1)}\lambda_j\right)^k \xi_j\right) \to 0, && \xi_j^{-1}\left(\partial^k \eta_j - \left(\mathrm{i}\varepsilon_j^{(2)}\lambda_j\right)^k \eta_j\right) \to 0 && (j = p+1, \ldots, n)
\end{aligned} \tag{87}$$

as $r \to +\infty$, where $s_1, \ldots, s_p$ are any constants, since $\xi_j^{-1}\eta_j \to 0$ for $j = 1, \ldots, m$ and $\eta_j^{-1}\xi_j \to 0$ for $j = m+1, \ldots, p$.

Now we prove that $p$ can only take $n-1$ or $n$. If $p \leqslant n-2$, then $\varepsilon_j^{(1)}\varepsilon_j^{(2)} = \pm\varepsilon_l^{(1)}\varepsilon_l^{(2)}$ must hold for any $j \neq l$ with $p+1 \leqslant j, l \leqslant n$ since both sides equal $\pm 1$. If $\varepsilon_j^{(1)}\varepsilon_j^{(2)} = \varepsilon_l^{(1)}\varepsilon_l^{(2)}$, then $\varepsilon_j^{(2)}\alpha_j^{(2)} = \varepsilon\varepsilon_j^{(1)}\alpha_j^{(1)}$ and $\varepsilon_l^{(2)}\alpha_l^{(2)} = \varepsilon\varepsilon_l^{(1)}\alpha_l^{(1)}$ hold simultaneously where $\varepsilon = \varepsilon_j^{(1)}\varepsilon_j^{(2)}$. This contradicts condition (85). If $\varepsilon_j^{(1)}\varepsilon_j^{(2)} = -\varepsilon_l^{(1)}\varepsilon_l^{(2)}$, then $\varepsilon_j^{(2)}\alpha_j^{(2)} - \varepsilon\varepsilon_j^{(1)}\alpha_j^{(1)} = 0$ and $\varepsilon_l^{(2)}\alpha_l^{(2)} + \varepsilon\varepsilon_l^{(1)}\alpha_l^{(1)} = 0$ hold simultaneously where $\varepsilon = \varepsilon_j^{(1)}\varepsilon_j^{(2)}$. This contradicts the assumption that $(\theta_1, \theta_2)$ does not satisfy (86). Hence, only $p = n$ or $p = n-1$ is possible.

*Step 2.* There exist $t_0 > 0$ and $c > 0$ such that $\det\widetilde{T} > c$ for $t \geqslant t_0$.

When $p = n-1$,

$$\frac{|\xi_n(\theta, r)|}{\max(|\xi_n(\theta, r)|, |\eta_n(\theta, r)|)} \to \gamma_0 \equiv \frac{\left|\kappa_n^{(1)}\right| \mathrm{e}^{\varepsilon_n^{(1)}\beta_n^{(1)}}}{\max\left(\left|\kappa_n^{(1)}\right| \mathrm{e}^{\varepsilon_n^{(1)}\beta_n^{(1)}}, \left|\kappa_n^{(2)}\right| \mathrm{e}^{\varepsilon_n^{(2)}\beta_n^{(2)}}\right)}. \tag{88}$$

Denote

$$V(\lambda_j, \ldots, \lambda_l) = \begin{pmatrix} \lambda_j^{n-1} & \cdots & 1 \\ \vdots & & \vdots \\ \lambda_l^{n-1} & \cdots & 1 \end{pmatrix}_{(l-j+1)\times n} \tag{89}$$

for $j \leqslant l$; then, for $p = n-1$, (87) leads to

$$\det\widetilde{T} = \gamma_0^2 \begin{vmatrix} V\left(\varepsilon_1^{(1)}\lambda_1, \ldots, \varepsilon_m^{(1)}\lambda_m\right) & 0 \\ 0 & V\left(\mathrm{i}\varepsilon_{m+1}^{(2)}\lambda_{m+1}, \ldots, \mathrm{i}\varepsilon_{n-1}^{(2)}\lambda_{n-1}\right) \\ V\left(\varepsilon_n^{(1)}\lambda_n\right) & \xi_n^{-1}\eta_n V\left(\mathrm{i}\varepsilon_n^{(2)}\lambda_n\right) \\ 0 & V\left(\varepsilon_1^{(1)}\bar{\lambda}_1, \ldots, \varepsilon_m^{(1)}\bar{\lambda}_m\right) \\ -V\left(-\mathrm{i}\varepsilon_{m+1}^{(2)}\bar{\lambda}_{m+1}, \ldots, -\mathrm{i}\varepsilon_{n-1}^{(2)}\bar{\lambda}_{n-1}\right) & 0 \\ -\bar{\xi}_n^{-1}\bar{\eta}_n V\left(-\mathrm{i}\varepsilon_n^{(2)}\bar{\lambda}_n\right) & V\left(\varepsilon_n^{(1)}\bar{\lambda}_n\right) \end{vmatrix} + o(1)$$

$$= \gamma_0^2 \begin{vmatrix} V\big(\varepsilon_1^{(1)}\lambda_1,\dots,\varepsilon_m^{(1)}\lambda_m\big) & 0 \\ V\big(-\mathrm{i}\varepsilon_{m+1}^{(2)}\bar\lambda_{m+1},\dots,-\mathrm{i}\varepsilon_{n-1}^{(2)}\bar\lambda_{n-1}\big) & 0 \\ V\big(\varepsilon_n^{(1)}\lambda_n\big) & \xi_n^{-1}\eta_n V\big(\mathrm{i}\varepsilon_n^{(2)}\lambda_n\big) \\ 0 & V\big(\varepsilon_1^{(1)}\bar\lambda_1,\dots,\varepsilon_m^{(1)}\bar\lambda_m\big) \\ 0 & V\big(\mathrm{i}\varepsilon_{m+1}^{(2)}\lambda_{m+1},\dots,\mathrm{i}\varepsilon_{n-1}^{(2)}\lambda_{n-1}\big) \\ -\bar\xi_n^{-1}\bar\eta_n V\big(-\mathrm{i}\varepsilon_n^{(2)}\bar\lambda_n\big) & V\big(\varepsilon_n^{(1)}\bar\lambda_n\big) \end{vmatrix} + o(1) \qquad (90)$$

as $t \to +\infty$. Let

$$\begin{aligned} &\Lambda = \mathrm{diag}\big(\varepsilon_1^{(1)}\lambda_1,\dots,\varepsilon_m^{(1)}\lambda_m,-\mathrm{i}\varepsilon_{m+1}^{(2)}\bar\lambda_{m+1},\dots,-\mathrm{i}\varepsilon_{n-1}^{(2)}\bar\lambda_{n-1},\varepsilon_n^{(1)}\lambda_n\big),\\ &\Gamma = \mathrm{diag}\big(0,\dots,0,\xi_n^{-1}\eta_n\big), \qquad \varepsilon = \varepsilon_n^{(1)}\varepsilon_n^{(2)}; \end{aligned} \qquad (91)$$

then we get $\liminf_{t\to+\infty}\det\widetilde T > 0$ by lemma 3 of appendix A.

Similarly, when $p = n$,

$$\lim_{t\to+\infty}\det\widetilde T = \begin{vmatrix} V\big(\varepsilon_1^{(1)}\lambda_1,\dots,\varepsilon_m^{(1)}\lambda_m\big) & 0 \\ V\big(-\mathrm{i}\varepsilon_{m+1}^{(2)}\bar\lambda_{m+1},\dots,-\mathrm{i}\varepsilon_n^{(2)}\bar\lambda_n\big) & 0 \\ 0 & V\big(\varepsilon_1^{(1)}\bar\lambda_1,\dots,\varepsilon_m^{(1)}\bar\lambda_m\big) \\ 0 & V\big(\mathrm{i}\varepsilon_{m+1}^{(2)}\lambda_{m+1},\dots,\mathrm{i}\varepsilon_n^{(2)}\lambda_n\big) \end{vmatrix}.$$

$$= \big|\det V\big(\varepsilon_1^{(1)}\lambda_1,\dots,\varepsilon_m^{(1)}\lambda_m,-\mathrm{i}\varepsilon_{m+1}^{(2)}\bar\lambda_{m+1},\dots,-\mathrm{i}\varepsilon_n^{(2)}\bar\lambda_n\big)\big|^2. \qquad (92)$$

Using the condition $\bar\lambda_j \neq \pm\mathrm{i}\lambda_l$, we get $\liminf_{t\to+\infty}\det\widetilde T > 0$ since the Vandermonde determinant is non-zero.

*Step 3.* Denote $\widetilde\Pi = \begin{pmatrix} \hat T & \\ & \hat T\end{pmatrix}^{-1}\Pi$. If the velocity $(\theta_1,\theta_2)$ does not satisfy (86), then $\lim_{t\to+\infty}\mathrm{Re}\det\widetilde\Pi = 0$.

From (87), let $\Lambda = \mathrm{diag}(\Lambda_1,\Lambda_2)$ with

$$\begin{aligned} \Lambda_1 &= \mathrm{diag}\big(\varepsilon_1^{(1)}\lambda_1,\dots,\varepsilon_m^{(1)}\lambda_m,\varepsilon_{m+1}^{(1)}s_{m+1}\lambda_{m+1},\dots,\varepsilon_p^{(1)}s_p\lambda_p,\varepsilon_{p+1}^{(1)}\lambda_{p+1},\dots,\varepsilon_n^{(1)}\lambda_n\big),\\ \Lambda_2 &= \mathrm{diag}\big(-\mathrm{i}\varepsilon_1^{(2)}\bar s_1\bar\lambda_1,\dots,-\mathrm{i}\varepsilon_m^{(2)}\bar s_m\bar\lambda_m,-\mathrm{i}\varepsilon_{m+1}^{(2)}\bar\lambda_{m+1},\dots,-\mathrm{i}\varepsilon_p^{(2)}\bar\lambda_p,\\ &\quad -\mathrm{i}\varepsilon_{p+1}^{(2)}\bar\lambda_{p+1},\dots,-\mathrm{i}\varepsilon_n^{(2)}\bar\lambda_n\big), \end{aligned} \qquad (93)$$

$$\zeta = \big(\underbrace{1,\dots,1}_{m},\underbrace{0,\dots,0}_{p-m},\underbrace{1,\dots,1}_{n-p},\underbrace{0,\dots,0}_{m},\underbrace{-1,\dots,-1}_{p-m},-\bar\xi_{p+1}^{-1}\bar\eta_{p+1},\dots,-\bar\xi_n^{-1}\bar\eta_n\big)^T; \qquad (94)$$

then $\widetilde\Pi - \gamma_0^4\Pi^\Lambda \to 0$ for $p = n-1$ and $\widetilde\Pi - \Pi^\Lambda \to 0$ for $p = n$ as $t \to +\infty$ where $\gamma_0$ is defined by (88) and $\Pi^\Lambda$ is defined by (A.27). $\bar\Lambda_2 = \mathrm{i}\varepsilon\Lambda_1$ holds for $\varepsilon = \pm 1$ if and only if $s_j = \varepsilon\varepsilon_j^{(1)}\varepsilon_j^{(2)}$ for $j = 1,\dots,p$ and $\varepsilon_j^{(2)} = \varepsilon\varepsilon_j^{(1)}$ for $j = p+1,\dots,n$. Since $s_j$ $(j = 1,\dots,p)$ can be arbitrary, $\varepsilon$ can be taken as $\pm1$ arbitrarily, and $p = n$ or $n-1$, we have $\bar\Lambda_2 = \mathrm{i}\varepsilon\Lambda_1$ by taking $\varepsilon = \varepsilon_n^{(1)}\varepsilon_n^{(2)}$ for $p = n-1$, and either $\varepsilon = 1$ or $\varepsilon = -1$ for $p = n$.

According to lemma 4 of appendix A, $\mathrm{Re}\det\Pi^\Lambda \equiv 0$. Hence, $\lim_{r\to+\infty}\mathrm{Re}\det\widetilde\Pi = 0$ if $(\theta_1,\theta_2)$ does not satisfy (86). The theorem is proved. □

**Remark 3.** If $(\theta_1,\theta_2)$ satisfies (86), then $\alpha_j^{(1)} = \alpha_j^{(2)}, \alpha_l^{(1)} = \alpha_l^{(2)}, \varepsilon_j^{(2)} = \varepsilon_j^{(1)}, \varepsilon_l^{(2)} = -\varepsilon_l^{(1)}$ for $j \neq l$ with $p+1 \leqslant j,l \leqslant n$. Hence, there is no common $\varepsilon = \pm1$ such that $\varepsilon_i^{(2)} = \varepsilon\varepsilon_i^{(1)}$ holds for all $i = p+1,\dots,n$, which contradicts the condition $\bar\Lambda_2 = \mathrm{i}\varepsilon\Lambda_1$ in lemma 4 of

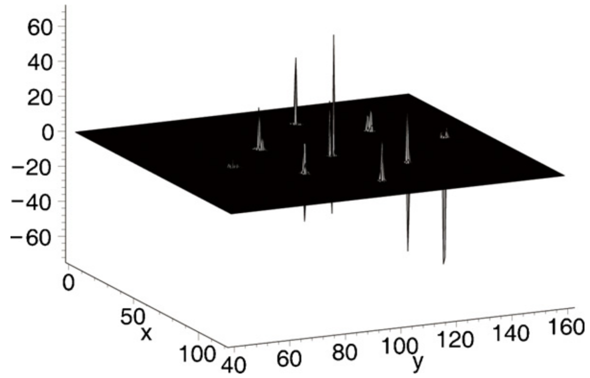


**Figure 4.** $3 \times 3$ soliton solution $u$: $\lambda_1 = 2 + 0.5\mathrm{i}$, $\lambda_2 = 2.5 + 0.4\mathrm{i}$, $\lambda_3 = 3 + 0.3\mathrm{i}$, $\kappa_j^{(1)} = 1$, $\kappa_j^{(2)} = 1.2$, $\rho_{j0}^{(1)} = 0$, $\rho_{j0}^{(2)} = 0$, $\sigma_{j0}^{(1)} = 0$, $\sigma_{j0}^{(2)} = 0 (j = 1, 2, 3)$, $t = 1$.

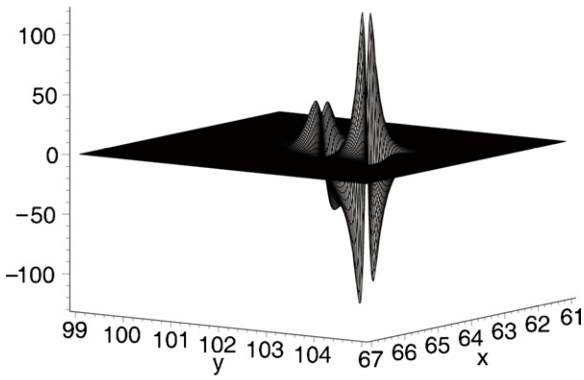


**Figure 5.** Local behavior of one soliton in the $3 \times 3$ soliton solution.

appendix A. In fact, the solution does not tend to zero in this case, which can be seen in the following example.

As an example, a $3 \times 3$ soliton is shown in figure 4, in which there are nine solitons. The local behavior of each soliton is still complicated, and one of them is shown in figure 5.

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## Appendix A. Some linear algebraic lemmas

**Lemma 1.** *Suppose that X and Y are* $2n \times r$ *and* $2n \times (2n - r)$ *matrices respectively; then,*

$$\overline{|X \quad K_n \bar{Y}|} = |Y \quad K_n \bar{X}|, \tag{A.1}$$

*where* $K_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

**Proof.**

$$K_n\overline{(X \quad K_n\bar{Y})}K_n^{-1} = (K_n\bar{X} \quad -Y)\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = (Y \quad K_n\bar{X}). \tag{A.2}$$

The lemma is obtained by taking the determinants on both sides. □

**Lemma 2.** *Suppose that* $T = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ *where A and B are* $n \times n$ *matrices; then* $\det T \geqslant 0$.

**Proof.** By lemma 1, $\det T$ is real. First, suppose that both $A$ and $B$ are invertible; then,

$$\det T = \det A \det(\bar{A} + \bar{B}A^{-1}B) = |\det A|^2 \det(I + \overline{A^{-1}B}A^{-1}B). \tag{A.3}$$

Let $N = A^{-1}B$. Suppose that $\lambda$ is an eigenvalue of $\bar{N}N$, $v \in \mathbf{R}^{2n}$ is a vector in the corresponding root space $\mathcal{R}_\lambda$, i.e. $(\bar{N}N - \lambda I)^m v = 0$ for a certain positive integer $m$. Then,

$$(\bar{N}N - \bar{\lambda}I)^m(\bar{N}\bar{v}) = \bar{N}\overline{(\bar{N}N - \lambda I)^m v} = 0. \tag{A.4}$$

Hence, $\bar{N}\bar{v} \in \mathcal{R}_{\bar{\lambda}}$. If $\lambda$ is a non-real eigenvalue of $\bar{N}N$ of multiplicity $k$, the multiplicity of $\bar{\lambda}$ is also $k$.

Now suppose that $\lambda < 0$ is an eigenvalue of $\bar{N}N$, $\mathcal{R}_\lambda = V_1 \oplus \cdots \oplus V_l$, where $V_1, \ldots, V_l$ are irreducible invariant subspaces. Suppose that $V_1 = \mathrm{span}\{\zeta, (\bar{N}N - \lambda I)\zeta, \ldots, (\bar{N}N - \lambda I)^{m-1}\zeta\}$ with $(\bar{N}N - \lambda I)^{m-1}\zeta \neq 0$ and $(\bar{N}N - \lambda I)^m\zeta = 0$. Then $\zeta \notin \mathrm{Image}(\bar{N}N - \lambda I)$, and $(\bar{N}N - \lambda I)^m\bar{N}\bar{\zeta} = \bar{N}\overline{(\bar{N}N - \lambda I)^m\zeta} = 0$, $(\bar{N}N - \lambda I)^{m-1}\bar{N}\bar{\zeta} = \bar{N}\overline{(\bar{N}N - \lambda I)^{m-1}\zeta} \neq 0$ since $\det N \neq 0$. We will prove that

$$\zeta, (\bar{N}N - \lambda I)\zeta, \ldots, (\bar{N}N - \lambda I)^{m-1}\zeta, \bar{N}\bar{\zeta}, (\bar{N}N - \lambda I)\bar{N}\bar{\zeta}, \ldots, (\bar{N}N - \lambda I)^{m-1}\bar{N}\bar{\zeta} \tag{A.5}$$

are linearly independent. Suppose that

$$\sum_{j=1}^{m} \alpha_j(\bar{N}N - \lambda I)^{j-1}\zeta + \sum_{j=1}^{m} \beta_j(\bar{N}N - \lambda I)^{j-1}\bar{N}\bar{\zeta} = 0, \tag{A.6}$$

where $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_m$ are complex numbers. Acting $(\bar{N}N - \lambda I)^{m-1}$ on both sides of (A.6), we get

$$(\bar{N}N - \lambda I)^{m-1}(\alpha_1\zeta + \beta_1\bar{N}\bar{\zeta}) = 0. \tag{A.7}$$

Then,

$$\begin{aligned}
(|\alpha_1|^2 - \lambda|\beta_1|^2)(\bar{N}N - \lambda I)^{m-1}\zeta &= -\bar{\alpha}_1\beta_1(\bar{N}N - \lambda I)^{m-1}\bar{N}\bar{\zeta} - \lambda|\beta_1|^2(\bar{N}N - \lambda I)^{m-1}\zeta \\
&= -\bar{\alpha}_1\beta_1\bar{N}\overline{(\bar{N}N - \lambda I)^{m-1}\zeta} - \lambda|\beta_1|^2(\bar{N}N - \lambda I)^{m-1}\zeta \\
&= \beta_1\bar{N}\overline{\beta_1(\bar{N}N - \lambda I)^{m-1}\bar{N}\bar{\zeta}} - \lambda|\beta_1|^2(\bar{N}N - \lambda I)^{m-1}\zeta \\
&= |\beta_1|^2\bar{N}N(\bar{N}N - \lambda I)^{m-1}\zeta - \lambda|\beta_1|^2(\bar{N}N - \lambda I)^{m-1}\zeta \\
&= |\beta_1|^2(\bar{N}N - \lambda I)^m\zeta = 0.
\end{aligned} \tag{A.8}$$

Since $\lambda < 0$ and $(\bar{N}N - \lambda I)^{m-1}\zeta \neq 0$, we have $\alpha_1 = \beta_1 = 0$. Continuing this process by acting $(\bar{N}N - \lambda I)^{m-2}, \ldots, (\bar{N}N - \lambda I)^0$ on both sides of (A.6) respectively, we get $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_n = 0$. This proves the linear independence of the vectors in (A.5). Let $\widetilde{V}_1 = \mathrm{span}\{\bar{N}\bar{\zeta}, (\bar{N}N - \lambda I)\bar{N}\bar{\zeta}, \ldots, (\bar{N}N - \lambda I)^{m-1}\bar{N}\bar{\zeta}\}$. If $\bar{N}\bar{\zeta} = (\bar{N}N - \lambda I)\zeta' \in \mathrm{Image}(\bar{N}N - \lambda I)$, then $\zeta = (\bar{N}N - \lambda I)N^{-1}\bar{\zeta}' \in \mathrm{Image}(\bar{N}N - \lambda I)$, which contradicts the choice of $\zeta$. Hence, $\bar{N}\bar{\zeta} \notin \mathrm{Image}(\bar{N}N - \lambda I)$. Moreover, $\widetilde{V}_1$ is invariant and irreducible under the action of $\bar{N}N - \lambda I$. Hence, it must be one of $V_j$ with $2 \leqslant j \leqslant l$, which means that $l$ is even ($l = 2k$) and $\mathcal{R}_\lambda = (W_1 \oplus \widetilde{W}_1) \oplus \cdots \oplus (W_k \oplus \widetilde{W}_k)$ where $(W_1, \ldots, W_k, \widetilde{W}_1, \ldots, \widetilde{W}_k)$ is a permutation of $(V_1, \ldots, V_{2k})$. Therefore, the multiplicity of each negative eigenvalue of $\bar{N}N$ must be even.

Thus, if the eigenvalues of $\bar{N}N$ are $\lambda_1, \ldots, \lambda_{2n}$ (multiple eigenvalues are listed repeatedly), then

$$\begin{aligned}\det T &= |\det A|^2 \det(I + \bar{N}N)\\ &= |\det A|^2 \left( \prod_{\mathrm{Re}\lambda_j \neq 0} |1+\lambda_j|^2 \prod_{\lambda_j<0} (1+\lambda_j)^2 \right)^{1/2} \prod_{\lambda_j \geqslant 0} (1+\lambda_j) \geqslant 0. \quad \text{(A.9)}\end{aligned}$$

If $A$ or $B$ is not invertible, it is a limit of invertible matrices, and the conclusion is also true. The lemma is proved. □

**Lemma 3.** *Suppose that* $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ *where* $\lambda_1, \ldots, \lambda_n$ *are distinct complex numbers such that* $\bar{\lambda}_j \neq \pm \mathrm{i}\lambda_l$ *for all* $j, l = 1, \ldots, n$, $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$. *Denote*

$$V = \begin{pmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \cdots & 1 \end{pmatrix}, \qquad E = \mathrm{diag}((\mathrm{i}\varepsilon)^{n-1}, (\mathrm{i}\varepsilon)^{n-2}, \ldots, 1), \quad \text{(A.10)}$$

*where* $\varepsilon = \pm 1$. *Let*

$$T = \begin{pmatrix} V & \Gamma V E \\ -\bar{\Gamma}\bar{V}\bar{E} & \bar{V} \end{pmatrix}; \quad \text{(A.11)}$$

*then there is a positive number C depending on* $\Lambda$ *only, such that* $\det T > C$.

**Proof.** Denote

$$S_k^{(\hat{m})} = \sum_{\substack{j_1 < \cdots < j_k \\ j_1, \cdots, j_k \neq m}} \lambda_{j_1} \cdots \lambda_{j_k}; \quad \text{(A.12)}$$

then

$$\sum_{k=1}^{n} (-1)^{k-1} S_{k-1}^{(\hat{m})} x^{n-k} = \prod_{\substack{s=1 \\ s \neq m}}^{n} (x - \lambda_s). \quad \text{(A.13)}$$

Since the entries of $V$ are $V_{jk} = \lambda_j^{n-k}$, we have

$$(V^{-1})_{jk} = \frac{(-1)^{j-1} S_{j-1}^{(\hat{k})}}{\prod_{\substack{s=1 \\ s \neq k}}^{n} (\lambda_k - \lambda_s)}. \quad \text{(A.14)}$$

Hence,

$$\begin{aligned}\det T &= \det(V) \det(\bar{V} + \bar{\Gamma}\bar{V}\bar{E}V^{-1}\Gamma V E)\\ &= |\det(V)|^2 \det(I + \bar{\Gamma}\bar{V}\bar{E}V^{-1}\Gamma V E \bar{V}^{-1}). \quad \text{(A.15)}\end{aligned}$$

For $j = 1, \ldots, n$, let $\lambda_j = \mathrm{e}^{\mathrm{i}\pi/4}\delta r_j$ where $\delta = 1$ if $\varepsilon = 1$ and $\delta = \mathrm{i}$ if $\varepsilon = -1$; then $r_j$'s are distinct and $\bar{r}_j \pm r_l \neq 0$ for all $j, l = 1, \ldots, n$:

$$\begin{aligned}(\bar{V}\bar{E}V^{-1})_{jk} &= \sum_{l=1}^{n} (-\mathrm{i}\varepsilon\bar{\lambda}_j)^{n-l} \frac{(-1)^{l-1} S_{l-1}^{(\hat{k})}}{\prod_{\substack{s=1 \\ s \neq k}}^{n} (\lambda_k - \lambda_s)} = \prod_{\substack{s=1 \\ s \neq k}}^{n} \frac{-\mathrm{i}\varepsilon\bar{\lambda}_j - \lambda_s}{\lambda_k - \lambda_s}\\ &= (-1)^{n-1} \prod_{\substack{s=1 \\ s \neq k}}^{n} \frac{\bar{r}_j + r_s}{r_k - r_s} = (-1)^{n-1} (AMB^{-1})_{jk}, \quad \text{(A.16)}\end{aligned}$$

where

$$A = (a_j\delta_{jk}), \qquad B = (b_j\delta_{jk}), \qquad M = (M_{jk}), \tag{A.17}$$

$$a_j = \prod_{s=1}^{n}(\bar{r}_j + r_s), \qquad b_k = \prod_{\substack{s=1\\ s\neq k}}^{n}(r_k - r_s), \qquad M_{jk} = (\bar{r}_j + r_k)^{-1}. \tag{A.18}$$

Since $\overline{\bar{V}\bar{E}V^{-1}}\bar{V}\bar{E}V^{-1} = I$, we have $\overline{AMB^{-1}} = (AMB^{-1})^{-1}$. Hence,

$$\begin{aligned}\det T &= |\det(V)|^2 \det(I + \bar{\Gamma}AMB^{-1}\Gamma BM^{-1}A^{-1})\\ &= |\det(V)|^2(\det M)^{-1}\det(M + \bar{\Gamma}M\Gamma)\end{aligned} \tag{A.19}$$

since $\Gamma$, $A$ and $B$ are diagonal matrices.

Suppose that $\mathrm{Re}(r_1), \dots, \mathrm{Re}(r_m) < 0$, $\mathrm{Re}(r_{m+1}), \dots, \mathrm{Re}(r_n) > 0$ since $\bar{r}_j + r_j \neq 0$ for all $j$. Write

$$\begin{aligned} M &= \left(\frac{1}{\bar{r}_j + r_k}\right)_{n\times n} = \begin{pmatrix} M_{11} & M_{12}\\ M_{12}^* & M_{22}\end{pmatrix},\\ M + \bar{\Gamma}M\Gamma &= \left(\frac{1 + \bar{\gamma}_j\gamma_k}{\bar{r}_j + r_k}\right)_{n\times n} = \begin{pmatrix} N_{11} & N_{12}\\ N_{12}^* & N_{22}\end{pmatrix},\end{aligned} \tag{A.20}$$

where $M_{11}$ and $N_{11}$, $M_{12}$ and $N_{12}$, and $M_{22}$ and $N_{22}$ are $m\times m$, $m\times(n-m)$, and $(n-m)\times(n-m)$ matrices respectively. Then $M_{11}$ and $N_{11}$ are negative definite Hermitian matrices, and so are $M_{11}^{-1}$ and $N_{11}^{-1}$; $M_{22}$ and $N_{22}$ are positive definite Hermitian matrices.

Let

$$\Gamma_1 = \mathrm{diag}(\gamma_1, \dots, \gamma_m), \qquad \Gamma_2 = \mathrm{diag}(\gamma_{m+1}, \dots, \gamma_n); \tag{A.21}$$

then

$$\begin{aligned}&\det(-N_{11}) = \det(-M_{11} + \Gamma_1^*(-M_{11})\Gamma_1) \geqslant \det(-M_{11}),\\ &\det(N_{22}) = \det(M_{22} + \Gamma_2^* M_{22}\Gamma_2) \geqslant \det(M_{22}).\end{aligned} \tag{A.22}$$

Hence,

$$\begin{aligned}(-1)^m \det(M + \bar{\Gamma}M\Gamma) &= \det(-N_{11})\det(N_{22} + N_{12}^*(-N_{11})^{-1}N_{12})\\ &\geqslant |\det M_{11}|\cdot|\det M_{22}|.\end{aligned} \tag{A.23}$$

On the other hand,

$$\begin{aligned}(-1)^m \det M &= \det(-M_{11})\det(M_{22} + M_{12}^*(-M_{11})^{-1}M_{12})\\ &\geqslant |\det M_{11}|\cdot|\det M_{22}| > 0;\end{aligned} \tag{A.24}$$

hence

$$\det T \geqslant |\det V|^2\frac{|\det M_{11}|\cdot|\det M_{22}|}{|\det M|}. \tag{A.25}$$

The lemma is proved. □

**Lemma 4.** *Let* $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2)$ *where* $\Lambda_1$ *and* $\Lambda_2$ *are* $n\times n$ *diagonal matrices satisfying* $\bar{\Lambda}_2 = \mathrm{i}\varepsilon\Lambda_1$ *where* $\varepsilon = \pm 1$. *Let* $\zeta$ *be a* $2n$ *dimensional column vector. For* $j \geqslant k$, *denote*

$$R_{j\cdots k}^{\Lambda} = (\Lambda^j\zeta \quad \Lambda^{j-1}\zeta, \dots, \Lambda^k\zeta). \tag{A.26}$$

*Let*

$$\Pi^{\Lambda} = \begin{pmatrix} \Lambda^n\zeta & \Lambda^{n-1}\zeta & \Lambda^{n-2}\zeta & R_{n-3\cdots 0}^{\Lambda} & K_n\bar{R}_{n-1\cdots 0}^{\Lambda} & 0 & 0\\ \Lambda^{n+1}\zeta & \Lambda^n\zeta & \Lambda^{n-1}\zeta & 0 & 0 & R_{n-2\cdots 0}^{\Lambda} & K_n\bar{R}_{n-1\cdots 0}^{\Lambda}\end{pmatrix}, \tag{A.27}$$

*where* $K_n = \begin{pmatrix} & -I_n\\ I_n & \end{pmatrix}$; *then* $\det\Pi^{\Lambda}$ *is purely imaginary.*

**Proof.** $\bar{\Lambda}_2 = \mathrm{i}\varepsilon\Lambda_1$ is equivalent to $\Lambda K_n = -\mathrm{i}\varepsilon K_n\bar{\Lambda}$. Denote $d^\Lambda = \det \Delta^\Lambda$. Multiplying row $j$ of $d^\Lambda$ by $-\lambda_j$ and adding it to row $2n + j (j = 1, \ldots, 2n)$, we get

$$d^\Lambda = \begin{vmatrix} \Lambda^n\zeta & \Lambda^{n-1}\zeta & \Lambda^{n-2}\zeta & R^\Lambda_{n-3\cdots 0} & K_n\bar{R}^\Lambda_{n-1\cdots 0} & 0 & 0 \\ 0 & 0 & 0 & -R^\Lambda_{n-2\cdots 1} & \mathrm{i}\varepsilon K_n\bar{R}^\Lambda_{n\cdots 0} & R^\Lambda_{n-2\cdots 0} & K_n\bar{R}^\Lambda_{n-1\cdots 0} \end{vmatrix} \tag{A.28}$$

by using $\Lambda K_n = -\mathrm{i}\varepsilon K_n\bar{\Lambda}$. Adding columns $2n + 2, \ldots, 3n - 1$ to columns $4, \ldots, n + 1$, multiplying columns $3n+1, \ldots, 4n-1$ by $-\mathrm{i}\varepsilon$ and adding them to columns $n+3, \ldots, 2n+1$, we get

$$d^\Lambda = \begin{vmatrix} R^\Lambda_{n\cdots 0} & K_n\bar{\Lambda}^{n-1}\bar{\zeta} & K_n\bar{R}^\Lambda_{n-2\cdots 0} & 0 & 0 \\ 0 & \mathrm{i}\varepsilon K_n\bar{\Lambda}^n\bar{\zeta} & 0 & R^\Lambda_{n-2\cdots 0} & K_n\bar{R}^\Lambda_{n-1\cdots 0} \end{vmatrix}, \tag{A.29}$$

By moving the columns,

$$\begin{aligned} d^\Lambda &= \begin{vmatrix} R^\Lambda_{n\cdots 0} & K_n\bar{R}^\Lambda_{n-2\cdots 0} & 0 & K_n\bar{\Lambda}^{n-1}\bar{\zeta} & 0 \\ 0 & 0 & R^\Lambda_{n-2\cdots 0} & \mathrm{i}\varepsilon K_n\bar{\Lambda}^n\bar{\zeta} & K_n\bar{R}^\Lambda_{n-1\cdots 0} \end{vmatrix} \\ &= \mathrm{i}\varepsilon \left| R^\Lambda_{n\cdots 0} \quad K_n\bar{R}^\Lambda_{n-2\cdots 0} \right| \cdot \left| R^\Lambda_{n-2\cdots 0} \quad K_n\bar{R}^\Lambda_{n\cdots 0} \right|, \end{aligned} \tag{A.30}$$

which is purely imaginary according to lemma 1. The lemma is proved. □

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