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Darboux transformations of lower degree for two-dimensional $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations

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Abstract
For the two-dimensional Toda equations corresponding to the Kac–Moody algebras $C_l^{(1)}$ and $D_{l+1}^{(2)}$, the Darboux transformations with a special choice of spectral parameters are constructed so that the degree of these Darboux transformations is half of that for usual Darboux transformations and the derived solutions become simpler. These Darboux transformations for a Lax pair are constructed from real solutions of itself or a slightly different Lax pair corresponding to the same Toda equation, depending on the parity of l. Exact solutions of these Toda equations are presented by simplifying the results derived from the Darboux transformations.

1. Introduction

It is known that the two-dimensional Toda equations are integrable and have many applications in physics and differential geometry [1–12]. It is also known that for any Kac–Moody algebra g, there is a two-dimensional affine Toda equation

$$w_{k,xt} = \exp\left(\sum_{i=1}^{n} c_{ki} w_i\right) - v_k \exp\left(\sum_{i=1}^{n} c_{0i} w_i\right) \quad (k = 1, \ldots, n), \tag{1}$$

where $C = (c_{ij})_{0\leqslant i,j\leqslant n}$ is the generalized Cartan matrix of g, $v = (v_0, v_1, \ldots, v_n)^T$ is a non-zero vector such that $Cv = 0$ [13, 14].

For $g = C_l^{(1)}$, the equation is

$$\begin{aligned} w_{1,xt} &= e^{2w_1 - w_2} - e^{-2w_1}, \\ w_{j,xt} &= e^{2w_j - w_{j-1} - w_{j+1}} - e^{-2w_1} \quad (j = 2, \ldots, l-1), \\ w_{l,xt} &= e^{2w_l - 2w_{l-1}} - e^{-2w_1}, \end{aligned} \tag{2}$$

or

$$\begin{aligned} u_{1,xt} &= e^{2u_1} - e^{u_2-u_1}, \\ u_{j,xt} &= e^{u_j-u_{j-1}} - e^{u_{j+1}-u_j} \quad (2 \leqslant j \leqslant l-1), \\ u_{l,xt} &= e^{u_l-u_{l-1}} - e^{-2u_l} \end{aligned} \tag{3}$$

if we set $w_j = -(u_1 + \cdots + u_j)(j = 1, \dots, l)$.

For $g = D_{l+1}^{(2)}$, the equation is

$$\begin{aligned} w_{1,xt} &= e^{2w_1-w_2} - e^{-w_1}, \\ w_{j,xt} &= e^{2w_j-w_{j-1}-w_{j+1}} - e^{-w_1} \quad (j = 2, \dots, l-2), \\ w_{l-1,xt} &= e^{2w_{l-1}-w_{l-2}-2w_l} - e^{-w_1}, \\ w_{l,xt} &= \tfrac{1}{2}e^{2w_l-w_{l-1}} - \tfrac{1}{2}e^{-w_1}, \end{aligned} \tag{4}$$

or

$$\begin{aligned} u_{1,xt} &= e^{u_1} - e^{u_2-u_1}, \\ u_{j,xt} &= e^{u_j-u_{j-1}} - e^{u_{j+1}-u_j} \quad (2 \leqslant j \leqslant l-1), \\ u_{l,xt} &= e^{u_l-u_{l-1}} - e^{-u_l} \end{aligned} \tag{5}$$

if we set $w_j = -(u_1 + \cdots + u_j)(j = 1, \dots, l-1)$ and $w_l = -\frac{1}{2}(u_1 + \cdots + u_l)$.

Equations (3) and (5) have similar Lax pairs, in which the matrices are both of even order. Moreover, their Lax pairs have a reality symmetry, a cyclic symmetry and a unitary symmetry simultaneously.

For $g = A_l^{(1)}$, (1) is the periodic Toda equation. The Darboux transformation (in differential form) and binary Darboux transformation (in integral form) were presented [4, 15]. A lot of work has been done since then. For $g = A_{2l}^{(2)}, C_l^{(1)}$ and $D_{l+1}^{(2)}$, the binary Darboux transformation (in integral form) was given by [16] systematically. For $g = A_{2l}^{(2)}$, the order of the matrices in the Lax pair is odd. The usual Darboux transformation in matrix form was given by [17] for the case where all the spectral parameters are complex as well as the case where one spectral parameter is real. The Darboux transformation is simpler when one spectral parameter is real. For $g = C_l^{(1)}$ and $D_{l+1}^{(2)}$, the order of the matrices in the Lax pair is even ($=2n$). The Darboux transformation of degree $2n$ was presented in [18] for the case where all the spectral parameters and the solutions of the Lax pair are complex.

In the present paper, we construct the Darboux transformations of degree n for two-dimensional $C_l^{(1)} (n = l)$ and $D_{l+1}^{(2)} (n = l+1)$ Toda equations. These Darboux transformations depend on a real solution of a Lax pair with a special choice of spectral parameters. When n is odd, the Darboux transformation depends on a real solution of the same Lax pair. However, when n is even, it depends on a real solution of a slightly different Lax pair for the same Toda equation. The Darboux transformation constructed in this paper is simpler than that in [18]. It has degree n and depends on a real solution of the Lax pair, while the Darboux transformation in [18] has degree $2n$ and depends on a complex solution of the Lax pair.

In section 2, the Lax pair for the two-dimensional $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations and its symmetries are discussed. In order to construct Darboux transformations later, two similar but slightly different Lax pairs in (14) (with plus and minus signs) are introduced. In section 3, the Darboux transformation of degree n with special choice of the spectral parameters is constructed so that the Darboux transformation keeps all the reality symmetry, cyclic symmetry and unitary symmetry. Hence the Darboux transformations give the solutions of the corresponding affine Toda equations. To avoid computing the inverse of the matrix (28) of order n^2, the expressions of the solutions are simplified in section 4 so that the final expressions only contain the inverse of a matrix of order n.

2. Two-dimensional $C_l^{(1)}, D_{l+1}^{(2)}$ Toda equations and their Lax pairs

Let us fix some notations used throughout the paper. Let

$$\theta = \exp\left(\frac{\mathrm{i}\pi}{2n}\right), \qquad \omega = \theta^2 = \exp\left(\frac{\mathrm{i}\pi}{n}\right), \tag{6}$$

$$\begin{aligned}\Theta &= \mathrm{diag}(1, \theta^{-1}, \ldots, \theta^{-2n+1}),\\ \Omega &= \Theta^2 = \mathrm{diag}(1, \omega^{-1}, \ldots, \omega^{-2n+1}).\end{aligned} \tag{7}$$

Then $\theta^n = \mathrm{i}$, $\omega^n = -1$.

Let $J_\pm = ((J_\pm)_{ij})_{2n\times 2n}$ where

$$\begin{aligned}(J_+)_{ij} &= \begin{cases}1 & \text{if} \quad i \equiv j-1 \bmod 2n,\\ 0 & \text{otherwise,}\end{cases}\\ (J_-)_{ij} &= \theta^{j-i-1}(J_+)_{ij}.\end{aligned} \tag{8}$$

Note that $(J_+)_{ij} = 0$ unless $j - i - 1 \equiv 0 \bmod 2n$; hence $\theta^{j-i-1} = \pm 1$ if $(J_+)_{ij} \neq 0$. Written explicitly, they are

$$J_\pm = \begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ \pm 1 & & & & 0 \end{pmatrix}_{2n\times 2n}. \tag{9}$$

Let m be an integer which takes only 0 or 1. Let $K_\pm^{(m)} = \left(\left(K_\pm^{(m)}\right)_{ij}\right)_{2n\times 2n}$ where

$$\begin{aligned}\left(K_+^{(m)}\right)_{ij} &= \begin{cases}1 & \text{if} \quad i+j \equiv m \bmod 2n,\\ 0 & \text{otherwise,}\end{cases}\\ (K_-^{(m)})_{ij} &= \theta^{2n+m-i-j}\left(K_+^{(m)}\right)_{ij}.\end{aligned} \tag{10}$$

Written explicitly,

$$K_\pm^{(0)} = \begin{pmatrix} & & & 1 & 0 \\ & & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & & \\ 0 & & & & \pm 1 \end{pmatrix}_{2n\times 2n}, \qquad K_\pm^{(1)} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}_{2n\times 2n}. \tag{11}$$

It was pointed out in [18] that the Toda equations corresponding to the other integer m are the same as that corresponding to $m = 0$ or $m = 1$.

We have the following relations among Θ, Ω, $K_\pm^{(m)}$ and $J_\pm$:

$$\begin{aligned}&\Theta K_\pm^{(m)}\Theta = -\theta^{2-m}K_\mp^{(m)}, \qquad \Theta^* K_\pm^{(m)} = -\theta^{m-2}K_\mp^{(m)}\Theta,\\ &\Omega K_\pm^{(m)}\Omega = \omega^{2-m}K_\pm^{(m)}, \qquad \Omega^* K_\pm^{(m)} = \omega^{m-2}K_\pm^{(m)}\Omega,\end{aligned} \tag{12}$$

$$\Theta J_\pm \Theta^{-1} = \theta J_\mp, \qquad \Omega J_\pm \Omega^{-1} = \omega J_\pm, \qquad K_\pm^{(m)} J_\pm \left(K_\pm^{(m)}\right)^{-1} = J_\pm^T. \tag{13}$$

Now we consider the Lax pair

$$\begin{aligned}\Phi_x &= U_\pm(x,t,\lambda)\Phi = (\lambda J_\pm + P(x,t))\Phi,\\ \Phi_t &= V_\pm(x,t,\lambda)\Phi = \lambda^{-1}Q_\pm(x,t)\Phi.\end{aligned} \tag{14}$$

Throughout the paper, we will write (14±) for the Lax pair (14) with '+' and '−' signs. In (14), $P(x,t)$ and $Q_\pm(x,t)$ are $2n\times 2n$ matrices of the form

$$P=\begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_{2n}\end{pmatrix},\qquad Q_\pm=\begin{pmatrix} 0 & & & & \pm q_{2n}\\ q_1 & & & & \\ & q_2 & & & \\ & & \ddots & & \\ & & & q_{2n-1} & 0\end{pmatrix}. \tag{15}$$

$p_1,\dots,p_{2n},q_1,\dots,q_{2n}$ are real functions, which are related to the solutions of the two-dimensional Toda equations as follows [18, 19]. (There is a mistake of notations in [18, 19]. The names $C_l^{(1)}$ and $D_{l+1}^{(2)}$ there should be interchanged.)

Case $C_l^{(1)}$: $n=l, m=1$

$$\begin{aligned} &p_i=-p_{2l+1-i}=u_{i,x}\quad (1\leqslant i\leqslant l),\\ &q_i=q_{2l-i}=\mathrm{e}^{u_{i+1}-u_i}\quad (1\leqslant i\leqslant l-1),\\ &q_l=\mathrm{e}^{-2u_l},\qquad q_{2l}=\mathrm{e}^{2u_1}.\end{aligned}\tag{16}$$

Case $D_{l+1}^{(2)}$: $n=l+1, m=0$

$$\begin{aligned} &p_i=-p_{2l+2-i}=u_{i,x}\quad (1\leqslant i\leqslant l),\qquad p_{l+1}=p_{2l+2}=0,\\ &q_i=q_{2l+1-i}=\mathrm{e}^{u_{i+1}-u_i}\quad (1\leqslant i\leqslant l-1),\\ &q_l=q_{l+1}=\mathrm{e}^{-u_l},\qquad q_{2l+1}=q_{2l+2}=\mathrm{e}^{u_1}.\end{aligned}\tag{17}$$

In both cases, $p_i+p_j=0$ if $i+j\equiv m \bmod 2n$, and $q_i=q_j$ if $i+j\equiv m-1 \bmod 2n$. The matrices $J_\pm$, P and $Q_\pm$ satisfy the symmetries

$$\bar J_\pm=J_\pm,\qquad \bar P=P,\qquad \bar Q_\pm=Q_\pm, \tag{18}$$

$$\Omega J_\pm\Omega^{-1}=\omega J_\pm,\qquad \Omega P\Omega^{-1}=P,\qquad \Omega Q_\pm\Omega^{-1}=\omega^{-1}Q_\pm, \tag{19}$$

$$\begin{aligned} &K_\pm^{(m)}J_\pm\big(K_\pm^{(m)}\big)^{-1}=J_\pm^T,\qquad K_\pm^{(m)}P\big(K_\pm^{(m)}\big)^{-1}=-P^T,\\ &K_\pm^{(m)}Q_\pm\big(K_\pm^{(m)}\big)^{-1}=Q_\pm^T.\end{aligned}\tag{20}$$

Moreover, we have

$$\Theta J_\pm\Theta^{-1}=\theta J_\mp,\qquad \Theta P\Theta^{-1}=P,\qquad \Theta Q_\pm\Theta^{-1}=\theta^{-1}Q_\mp. \tag{21}$$

Written in terms of $U_\pm$ and $V_\pm$, the symmetries (18)–(20) become

$$\overline{U_\pm(x,t,\lambda)}=U_\pm(x,t,\bar\lambda),\qquad \overline{V_\pm(x,t,\lambda)}=V_\pm(x,t,\bar\lambda), \tag{22}$$

$$\Omega U_\pm(x,t,\lambda)\Omega^{-1}=U_\pm(x,t,\omega\lambda),\qquad \Omega V_\pm(x,t,\lambda)\Omega^{-1}=V_\pm(x,t,\omega\lambda), \tag{23}$$

$$\begin{aligned} &K_\pm^{(m)}U_\pm(x,t,\lambda)\big(K_\pm^{(m)}\big)^{-1}=-(U_\pm(x,t,-\bar\lambda))^*,\\ &K_\pm^{(m)}V_\pm(x,t,\lambda)\big(K_\pm^{(m)}\big)^{-1}=-(V_\pm(x,t,-\bar\lambda))^*,\end{aligned}\tag{24}$$

and (21) becomes

$$\Theta U_\pm(x,t,\lambda)\Theta^{-1}=U_\mp(x,t,\theta\lambda),\qquad \Theta V_\pm(x,t,\lambda)\Theta^{-1}=V_\mp(x,t,\theta\lambda). \tag{25}$$

The relations (22)–(24) mean that $U_\pm$ and $V_\pm$ satisfy a reality symmetry, a cyclic symmetry of order $2n$ and a unitary symmetry with respect to the indefinite metric given by the real symmetric matrix $K_\pm^{(m)}$.

The integrability conditions of (14) are

$$Q_{\pm,x} = [P, Q_\pm], \qquad P_t + [J_\pm, Q_\pm] = 0. \tag{26}$$

Written in terms of p_j's and q_j's, (26) becomes

$$q_{i,x} = (p_{i+1} - p_i)q_i, \qquad p_{i,t} = q_{i-1} - q_i \quad (i = 1, \cdots, 2n), \tag{27}$$

with $p_{2n+1} = p_1, q_0 = q_{2n}$.

These are the evolution equations which will be discussed in this paper. Written in terms of the solutions of the two-dimensional Toda equations according to (16) and (17), the equations in (27) become (3) and (5) respectively. As mentioned before, the Lax pair satisfying the symmetries (22)–(24) only contains these two cases.

Remark 1. (J_+, P, Q_+) and (J_-, P, Q_-) lead to the same nonlinear evolution equations (27), but lead to two different Lax pairs in (14). These two Lax pairs will be used simultaneously in constructing Darboux matrices of degree n in this paper.

3. Darboux transformation

When unitary symmetry is considered, the general construction of Darboux transformation is as follows [20, 21]. Let $\lambda_1, \ldots, \lambda_M$ be M complex numbers such that $\lambda_j, -\bar\lambda_j (j = 1, \ldots, M)$ are distinct. Let H_i be a $2n \times r$ solution of rank $r (1 \leqslant r \leqslant 2n - 1)$ to the Lax pair (14±) with $\lambda = \lambda_i (i = 1, \ldots, M)$. Let

$$\Gamma_{ij} = \frac{H_i^* K_\pm^{(m)} H_j}{\bar\lambda_i + \lambda_j} \tag{28}$$

be $M \times M$ matrices $(i, j = 1, 2, \ldots, M)$, $\Gamma = (\Gamma_{ij})_{1\leqslant i,j\leqslant M}$. Denote $\Gamma^{-1} = (\check\Gamma_{ij})_{1\leqslant i,j\leqslant M}$ where $\check\Gamma_{ij}$'s are $M \times M$ matrices. Let

$$G(x, t, \lambda) = \prod_{s=1}^{M}(\lambda + \bar\lambda_s) \left(1 - \sum_{i,j=1}^{M} \frac{H_i \check\Gamma_{ij} H_j^* K_\pm^{(m)}}{\lambda + \bar\lambda_j}\right), \tag{29}$$

then $G(x, t, \lambda)$ is a polynomial of λ of degree M with $2n \times 2n$ matrix coefficients. Write

$$G(x, t, \lambda) = \sum_{j=0}^{M} (-1)^j G_j(x, t)\lambda^{M-j}, \qquad G_0(x, t) = I_{2n}, \tag{30}$$

and define

$$\begin{aligned} \widetilde U_\pm(x, t, \lambda) &= G(x, t, \lambda)U_\pm(x, t, \lambda)G(x, t, \lambda)^{-1} + G_x(x, t, \lambda)G(x, t, \lambda)^{-1}, \\ \widetilde V_\pm(x, t, \lambda) &= G(x, t, \lambda)V_\pm(x, t, \lambda)G(x, t, \lambda)^{-1} + G_t(x, t, \lambda)G(x, t, \lambda)^{-1}. \end{aligned} \tag{31}$$

Then for any solution Φ of (14±), $\widetilde\Phi = G\Phi$ satisfies $\widetilde\Phi_x = \widetilde U_\pm \widetilde\Phi$, $\widetilde\Phi_t = \widetilde V_\pm \widetilde\Phi$. G is called a Darboux matrix and the transformation $(U_\pm, V_\pm, \Phi) \to (\widetilde U_\pm, \widetilde V_\pm, \widetilde\Phi)$ is called a Darboux transformation.

Owing to this construction of Darboux transformation with unitary symmetry, we have the following two lemmas by direct computation [18].

Lemma 1. $\widetilde U_\pm = \lambda J_\pm + \widetilde P$, $\widetilde V_\pm = \lambda^{-1}\widetilde Q_\pm$ *where* $\widetilde P = P + [J_\pm, G_1]$, $\widetilde Q_\pm = G_M Q_\pm G_M^{-1}$.

Lemma 2.

$$G(x,t,-\bar{\lambda})^* K_{\pm}^{(m)} G(x,t,\lambda) = \prod_{s=1}^{M} (\bar{\lambda}_s + \lambda)(\lambda_s - \lambda) K_{\pm}^{(m)}. \tag{32}$$

When $U_{\pm}$ and $V_{\pm}$ have extra symmetries other than the unitary symmetry, the spectral parameter λ_j's and the solutions H_j's of the Lax pairs in constructing Darboux matrices should be chosen suitably so that the derived $\widetilde{U}_{\pm}$ and $\widetilde{V}_{\pm}$ keep those symmetries. To do so, we need the following lemma.

Lemma 3. *Suppose* $\mu \in \boldsymbol{C}\backslash\{0\}$.

(i) If $\Phi(x,t)$ *is a solution of* (14±) *for* $\lambda = \mu$, *then* $\bar{\Phi}(x,t)$ *is a solution of* (14±) *for* $\lambda = \bar{\mu}$;
(ii) If $\Phi(x,t)$ *is a solution of* (14±) *for* $\lambda = \mu$, *then* $\Theta\Phi(x,t)$ *is a solution of* (14∓) *for* $\lambda = \theta\mu$.
(iii) If $\Phi(x,t)$ *is a solution of* (14±) *for* $\lambda = \mu$, *then* $\Omega\Phi(x,t)$ *is a solution of* (14±) *for* $\lambda = \omega\mu$.
(iv) If $\Phi(x,t)$ *is a solution of* (14±) *for* $\lambda = \mu$, *then* $\Omega^n\bar{\Phi}$ *is a solution of* (14±) *for* $\lambda = -\bar{\mu}$.
(v) If $\Phi(x,t)$ *is a solution of* (14±) *for* $\lambda = \mu$, *then* $\Psi = K_{\pm}^{(m)}\Omega^n\Phi$ *is a solution of the adjoint Lax pair*

$$\Psi_x = -U_{\pm}(\mu)^T\Psi, \qquad \Psi_t = -V_{\pm}(\mu)^T\Psi. \tag{33}$$

Therefore, $(\Phi^T K_{\pm}^{(m)}\Omega^n\Phi)_x = 0$, $\left(\Phi^T K_{\pm}^{(m)}\Omega^n\Phi\right)_t = 0$.

Proof. (i), (ii) and (iii) follow from (22), (25) and (23) respectively. (iv) follows from (22) and (23). (v) follows from (22) and (24). The lemma is proved. □

Now take $M = n$ and $r = n$ in the general construction of Darboux matrix shown above. According to the parity of n, the spectral parameters and the solutions of the Lax pair in constructing the Darboux transformation are chosen in different ways.

When n is odd, let $\mu \in \mathbf{R}\backslash\{0\}$, H be a $2n \times n$ real solution of the Lax pair (14±) for $\lambda = \mu$ such that H is of rank n and $H^T K_{\pm}^{(m)}\Omega^n H = 0$ holds at a certain point. Then according to (v) of lemma 3, $H^T K_{\pm}^{(m)}\Omega^n H = 0$ holds everywhere. Take $\lambda_j = \omega^{2j-2}\mu$, $H_j = \Omega^{2j-2}H$. Then $\lambda_j, -\bar{\lambda}_j (j = 1,\ldots,n)$ are distinct, and $H_j^T K_{\pm}^{(m)}\Omega^n H_j = 0$ holds for each j.

When n is even, let $\mu \in \mathbf{R}\backslash\{0\}$, H be a $2n \times n$ real solution of the Lax pair (14∓) for $\lambda = \mu$ such that H is of rank n and $H^T K_{\mp}^{(m)}\Omega^n H = 0$ holds everywhere. Take $\lambda_j = \theta\omega^{2j-2}\mu$, $H_j = \Theta\Omega^{2j-2}H$. Then $\lambda_j, -\bar{\lambda}_j (j = 1,\ldots,n)$ are distinct, and $H_j^T K_{\pm}^{(m)}\Omega^n H_j = 0$ holds for each j. According to (ii) of lemma 3, H_j is a solution of (14±) for $\lambda = \lambda_j$.

λ_j's and H_j's being chosen, the Darboux matrix $G(x,t,\lambda)$ is constructed from (29).

Remark 2. The choice of λ_j's and H_j's above is based on the following facts.

According to lemma 3, if λ is an eigenvalue, so are $\omega^{k-1}\lambda$ and $\overline{\omega^{k-1}\lambda} (k = 1,2,\ldots,2n)$. The corresponding solutions of the Lax pair are $\Omega^{k-1}H$ and $\overline{\Omega^{k-1}H} (k = 1,2,\ldots,2n)$.

Let Σ be the set of all the spectral parameters λ_j's in constructing Darboux transformation, then $\Sigma \cap (-\bar{\Sigma}) = \emptyset$ should hold. Otherwise, the Darboux transformation will not exist.

When n is odd and μ is real, $\overline{\omega^{2n-k+1}\mu} = \omega^{k-1}\mu$, so the set $\{\omega^{k-1}\mu | k = 1,2,\ldots,2n\}$ and $\{\overline{\omega^{k-1}\mu} | k = 1,2,\ldots,2n\}$ are the same. Hence $\Sigma \subset \{\omega^{k-1}\mu | k = 1,2,\ldots,2n\}$. However, since $-\overline{\omega^{n-k+1}\mu} = \omega^{k-1}\mu$ and $\Sigma \cap (-\bar{\Sigma}) = \emptyset$, $\omega^{k-1}\mu \in \Sigma$ implies $\omega^{n-k+1}\mu \notin \Sigma$. Hence we choose $\Sigma = \{\omega^{2k-2}\mu | k = 1,\ldots,n\}$ so that $\Sigma \cap (-\bar{\Sigma}) = \emptyset$ holds (see figure 1 for $n = 3$). Moreover, according to another equivalent construction of Darboux transformation

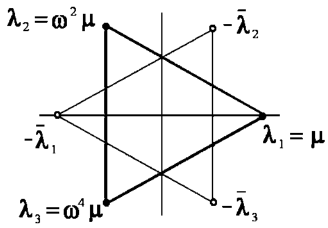

Figure 1.

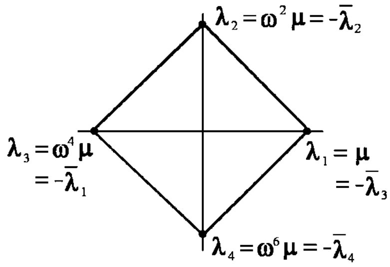

Figure 2.

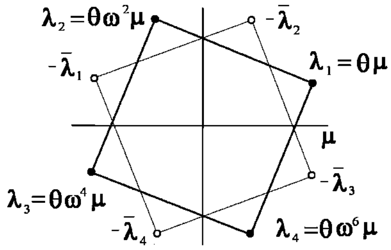

Figure 3.

with unitary symmetry (see p 48 of [9]), $\Omega^n\bar{H} = \Omega^n H$, the solution of the Lax pair for $\lambda = -\bar{\mu}$, should be orthogonal to H, the solution of the Lax pair for $\lambda = \mu$, in the metric defined by $K_\pm^{(m)}$, i.e. $H^T K_\pm^{(m)}\Omega^n H = 0$.

When n is even, if we still take $\lambda_j = \omega^{2j-2}\mu$, then $\lambda_{n/2+1} = -\lambda_1 = -\bar{\lambda}_1$. This is contradictory to $\Sigma \cap (-\bar{\Sigma}) = \emptyset$. (see figure 2 for $n = 4$), which means that λ_1 cannot be real. In order to derive the Darboux transformation from a real solution of a Lax pair, we need the solution of (14∓) in considering the Darboux transformation for the Lax pair (14±) (see figure 3 for $n = 4$). Then the choice of λ_j's and H_j's is similar to the case when n is odd.

In figures 1–3, the solid dots represent the set Σ and the small circles represent the set $-\bar{\Sigma}$.

In order to prove that $\widetilde{U}_\pm$ and $\widetilde{V}_\pm$ satisfy the reality symmetry and the cyclic symmetry, we need the following lemma. The motivation follows from another equivalent construction of Darboux transformation [22, 23].

Lemma 4. *Suppose* $(\Gamma_{ij}(x,t))_{n\times n}$ *defined by* (28) *is non-degenerate at* (x,t). *Suppose*

$$\Delta(x,t,\lambda)=\sum_{j=0}^{n-1}\Delta_{n-j}(x,t)\lambda^j \tag{34}$$

is a $2n\times 2n$ *matrix which is a polynomial of* λ *of degree* $\leqslant n-1$ *such that*

$$\Delta(x,t,\lambda_k)H_k=0,\qquad \Delta(x,t,-\bar\lambda_k)\Omega^n\bar H_k=0,\qquad (k=1,\ldots,n). \tag{35}$$

Then $\Delta(x,t,\lambda)=0$.

Proof. (35) is equivalent to

$$(\Delta_n,\Delta_{n-1},\ldots,\Delta_1)\mathcal{M}=0 \tag{36}$$

where $\mathcal{M}=(\mathcal{M}_{ij})_{1\leqslant i\leqslant n,1\leqslant j\leqslant 2n}$ with

$$\mathcal{M}_{ij}=\lambda_j^{i-1}H_j,\qquad \mathcal{M}_{i,n+j}=(-\bar\lambda_j)^{i-1}\Omega^n\bar H_j \tag{37}$$

for $i,j=1,\ldots,n$. Each $\mathcal{M}_{ij}$ is a $2n\times n$ matrix.

Define the block matrix $\mathcal{N}=(\mathcal{N}_{ij})_{1\leqslant i\leqslant 2n,1\leqslant j\leqslant n}$ with

$$\mathcal{N}_{ij}=(-\bar\lambda_i)^{-j+1}H_i^*K_\pm^{(m)},\qquad \mathcal{N}_{n+i,j}=\lambda_i^{-j+1}H_i^TK_\pm^{(m)}\Omega^n \tag{38}$$

for $i,j=1,\ldots,n$. Each $\mathcal{N}_{ij}$ is an $n\times 2n$ matrix.

When n is odd, we have

$$\begin{aligned}\sum_{k=1}^n\mathcal{N}_{ik}\mathcal{M}_{kj}&=\sum_{k=1}^n(-\bar\lambda_i)^{-k+1}\lambda_j^{k-1}H_i^*K_\pm^{(m)}H_j\\&=\frac{1-(-\lambda_j/\bar\lambda_i)^n}{1+\lambda_j/\bar\lambda_i}H_i^*K_\pm^{(m)}H_j=\bar\lambda_i(1-(-\omega^{2(i+j-2)})^n)\Gamma_{ij}=2\bar\lambda_i\Gamma_{ij},\end{aligned} \tag{39}$$

$$\sum_{k=1}^n\mathcal{N}_{ik}\mathcal{M}_{k,n+j}=\sum_{k=1}^n(\bar\lambda_j/\bar\lambda_i)^{k-1}H_i^*K_\pm^{(m)}\Omega^n\bar H_j=n\delta_{ij}\overline{H_i^TK_\pm^{(m)}\Omega^nH_j}=0 \tag{40}$$

since $\omega^n=-1$, $(\bar\lambda_j/\bar\lambda_i)^n=\omega^{2n(i-j)}=1$ and

$$\sum_{k=1}^n(\bar\lambda_j/\bar\lambda_i)^{k-1}=\begin{cases}\dfrac{1-(\bar\lambda_j/\bar\lambda_i)^n}{1-\bar\lambda_j/\bar\lambda_i}=0, & i\neq j,\\ n, & i=j.\end{cases} \tag{41}$$

Likewise, we have

$$\sum_{k=1}^n\mathcal{N}_{n+i,k}\mathcal{M}_{kj}=0,\qquad \sum_{k=1}^n\mathcal{N}_{n+i,k}\mathcal{M}_{k,n+j}=2\lambda_i\bar\Gamma_{ij}. \tag{42}$$

Hence

$$\mathcal{N}\mathcal{M}=\begin{pmatrix}2\bar\Lambda\Gamma & 0\\ 0 & 2\Lambda\bar\Gamma\end{pmatrix} \tag{43}$$

where $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$. Since $\det\Gamma\neq 0$, $\det\mathcal{M}\neq 0$ holds.

Now suppose n is even. Then

$$\begin{aligned}\sum_{k=1}^n\mathcal{N}_{ik}\mathcal{M}_{kj}&=\frac{1-(-\lambda_j/\bar\lambda_i)^n}{1+\lambda_j/\bar\lambda_i}H_i^*K_\pm^{(m)}H_j\\&=\bar\lambda_i(1-(-\omega^{2i+2j-3})^n)\Gamma_{ij}=\bar\lambda_i(1-(\omega^n)^{2i+2j-3})\Gamma_{ij}=2\bar\lambda_i\Gamma_{ij},\end{aligned} \tag{44}$$

$$\sum_{k=1}^{n} \mathcal{N}_{ik}\mathcal{M}_{k,n+j} = n\delta_{ij}\overline{H_i^T K_\pm^{(m)}\Omega^n H_j} = 0 \tag{45}$$

$$\sum_{k=1}^{n} \mathcal{N}_{n+i,k}\mathcal{M}_{kj} = 0, \qquad \sum_{k=1}^{n} \mathcal{N}_{n+i,k}\mathcal{M}_{k,n+j} = 2\lambda_i\bar{\Gamma}_{ij}. \tag{46}$$

We have also

$$\mathcal{N}\mathcal{M} = \begin{pmatrix} 2\bar{\Lambda}\Gamma & 0 \\ 0 & 2\Lambda\bar{\Gamma} \end{pmatrix}. \tag{47}$$

Hence $\det\mathcal{M} \neq 0$ when $\det\Gamma \neq 0$.

Therefore, for both odd and even n, (36) has a unique solution $\Delta_n = \cdots = \Delta_1 = 0$. The lemma is proved. □

Now we prove that the Darboux transformation constructed above keeps the reality symmetry and the cyclic symmetry.

Lemma 5. $G(x,t,\lambda)$ *satisfies*

$$\overline{G(x,t,\bar{\lambda})} = G(x,t,\lambda). \tag{48}$$

$$\Omega G(x,t,\lambda)\Omega^{-1} = -G(x,t,\omega\lambda). \tag{49}$$

Proof. For both odd and even n,

$$G(x,t,\lambda_k)H_k = \prod_{s=1}^{n}(\lambda_k+\bar{\lambda}_s)\left(H_k - \sum_{i,j=1}^{n}\frac{H_i\check{\Gamma}_{ij}H_j^*K_\pm^{(m)}H_k}{\bar{\lambda}_j+\lambda_k}\right) = 0, \tag{50}$$

$$\begin{aligned} G(x,t,-\bar{\lambda}_k)\Omega^n\bar{H}_k &= -\prod_{s\neq k}(\bar{\lambda}_s-\bar{\lambda}_k)\sum_{i=1}^{n}H_i\check{\Gamma}_{ik}H_k^*K_\pm^{(m)}\Omega^n\bar{H}_k \\ &= -\prod_{s\neq k}(\bar{\lambda}_s-\bar{\lambda}_k)\sum_{i=1}^{n}H_i\check{\Gamma}_{ik}\overline{H_k^TK_\pm^{(m)}\Omega^nH_k} = 0. \end{aligned} \tag{51}$$

Now suppose n is odd, then

$$\begin{aligned} \overline{G(x,t,\bar{\lambda}_k)}H_k &= \overline{G(x,t,\lambda_{2-k})H_{2-k}} = 0, \\ \overline{G(x,t,-\lambda_k)}\Omega^n\bar{H}_k &= \overline{G(x,t,-\bar{\lambda}_{2-k})\Omega^n\bar{H}_{2-k}} = 0. \end{aligned} \tag{52}$$

Let $\Delta(x,t,\lambda) = \overline{G(x,t,\bar{\lambda})} - G(x,t,\lambda)$, then $\Delta(x,t,\lambda)$ is a matrix polynomial of λ of degree $\leqslant n-1$. By lemma 4, (48) holds for odd n.

On the other hand,

$$\begin{aligned} G(x,t,\omega\lambda_k)\Omega H_k &= G(x,t,-\bar{\lambda}_{(n+3)/2-k})\Omega^n\bar{H}_{(n+3)/2-k} = 0, \\ G(x,t,-\omega\bar{\lambda}_k)\Omega\Omega^n\bar{H}_k &= G(x,t,\lambda_{(n+5)/2-k})H_{(n+5)/2-k} = 0. \end{aligned} \tag{53}$$

Let $\Delta(x,t,\lambda) = G(x,t,\omega\lambda)\Omega + \Omega G(x,t,\lambda)$, then $\Delta(x,t,\lambda)$ is a matrix polynomial of λ of degree $\leqslant n-1$. Lemma 4 implies that (49) holds for odd n.

When n is even, (48) follows from lemma 4 and

$$\begin{aligned} \overline{G(x,t,\bar{\lambda}_k)}H_k &= \overline{G(x,t,-\bar{\lambda}_{k+n/2})\Omega^n\bar{H}_{k+n/2}} = 0, \\ \overline{G(x,t,-\lambda_k)}\Omega^n\bar{H}_k &= \overline{G(x,t,\lambda_{k+n/2})H_{k+n/2}} = 0, \end{aligned} \tag{54}$$

and (49) follows from Lemma 4 and

$$
\begin{aligned}
&G(x,t,\omega\lambda_k)\Omega H_k = G(x,t,-\bar\lambda_{(n+2)/2-k})\Omega^n \bar H_{(n+2)/2-k} = 0,\\
&G(x,t,-\omega\bar\lambda_k)\Omega\Omega^n \bar H_k = G(x,t,\lambda_{(n+4)/2-k})H_{(n+4)/2-k} = 0.
\end{aligned} \tag{55}
$$

The lemma is proved. □

Finally, from lemma 2, lemma 5 and equality (31), we have

Theorem 1. *Suppose $p_1,\ldots,p_n$ and $q_1,\ldots,q_n$'s satisfy (27). Let μ be a non-zero real number, $H = \begin{pmatrix} h_1\\ \vdots \\ h_{2n}\end{pmatrix}$ be a $2n\times n$ real solution of the Lax pair (14ϵ) with $\lambda=\mu$ such that H is of rank n and $H^T K_\varepsilon^{(m)}\Omega^n H = 0$ holds. Here $\varepsilon = \pm$ if n is odd and $\varepsilon=\mp$ if n is even. Take $\lambda_j = \omega^{2j-2}\mu$, $H_j = \Omega^{2j-2}H (j=1,2,\ldots,n)$ when n is odd, and $\lambda_j = \theta\omega^{2j-2}\mu$, $H_j = \Theta\Omega^{2j-2}H$ when n is even. Then after the action of Darboux matrix $G(x,t,\lambda)$ given by (29), $\widetilde U_\pm$ and $\widetilde V_\pm$ in (31) satisfy*

$$
\begin{aligned}
&\overline{\widetilde U_\pm(x,t,\lambda)} = \widetilde U_\pm(x,t,\bar\lambda), && \overline{\widetilde V_\pm(x,t,\lambda)} = \widetilde V_\pm(x,t,\bar\lambda),\\
&\Omega\widetilde U_\pm(x,t,\lambda)\Omega^{-1} = \widetilde U_\pm(x,t,\omega\lambda), && \Omega\widetilde V_\pm(x,t,\lambda)\Omega^{-1} = \widetilde V_\pm(x,t,\omega\lambda),\\
&K_\pm^{(m)}\widetilde U_\pm(x,t,\lambda)\big(K_\pm^{(m)}\big)^{-1} = -(\widetilde U_\pm(x,t,-\bar\lambda))^*,\\
&K_\pm^{(m)}\widetilde V_\pm(x,t,\lambda)\big(K_\pm^{(m)}\big)^{-1} = -(\widetilde V_\pm(x,t,-\bar\lambda))^*.
\end{aligned} \tag{56}
$$

Therefore, the Darboux transformation transforms a solution of a two-dimensional $C_l^{(1)}$ or $D_{l+1}^{(2)}$ Toda equation to a solution of the same equation.

4. Explicit expressions of the solutions

Theorem 1 has already presented the algebraic expressions of new solutions. However, these expressions concern the inverse of the matrix Γ, which is of order n^2. Usually it is a difficult task to compute it explicitly. In this section, the expressions of these solutions will be simplified. Before that, we first prove the following lemma.

Lemma 6. *Let n be a positive integer, $\omega = \exp\left(\frac{\pi \mathrm{i}}{n}\right)$. Let p be an integer, $[p]$ and $\{p\}$ be the quotient and the remainder of p divided by n respectively. Then, when n is odd,*

$$\sum_{s=1}^{n}\frac{\omega^{-2sp}}{1+\omega^{2s}} = \frac{n}{2}(-1)^{\{p\}}; \tag{57}$$

and when n is even,

$$\sum_{s=1}^{n}\frac{\omega^{-(2s-1)p}}{1+\omega^{2s-1}} = \frac{n}{2}(-1)^{\{p\}+[p]}. \tag{58}$$

Proof. We only prove the lemma for even n. The odd case is similar.

Let $\kappa\in\mathbf{C}$ such that $|\kappa|<1$, then

$$\sum_{s=1}^{n}\frac{\omega^{-(2s-1)p}}{1+\kappa\omega^{2s-1}} = \sum_{s=1}^{n}\sum_{l=0}^{\infty}(-\kappa)^l\omega^{(2s-1)(l-p)}. \tag{59}$$

Note that

$$\sum_{s=1}^{n} \omega^{2s(l-p)} = \begin{cases} n, & \text{if} \quad l-p \equiv 0 \bmod n, \\ 0, & \text{otherwise}, \end{cases} \tag{60}$$

we only need to consider the terms with $l - p \equiv 0 \bmod n$. Let $l = n\sigma + \{p\}$ where σ is an integer. Since $p = n[p] + \{p\}$, $\omega^n = -1$, we have

$$\sum_{s=1}^{n} \frac{\omega^{-(2s-1)p}}{1+\rho\omega^{2s-1}} = n \sum_{\sigma=0}^{\infty} (-\kappa)^{n\sigma+\{p\}} \omega^{-n\sigma+n[p]}$$

$$= n \sum_{\sigma=0}^{\infty} (-\kappa\omega^{-1})^{n\sigma} (-\kappa)^{\{p\}} (-1)^{[p]} = \frac{n(-\kappa)^{\{p\}}}{1-(-\kappa\omega^{-1})^n} (-1)^{[p]}. \tag{61}$$

Let $\kappa \to 1$, we get (58). The lemma is proved. □

Now we consider the solutions of the Toda equations. According to lemma 1, the solution $Q_\pm$ of the Toda equations is fully determined by G_n. Therefore, it is crucial to simplify the expression of G_n.

According to (29),

$$G_n = (-1)^n G(x,t,0) = (-1)^n \prod_{s=1}^{n} \bar{\lambda}_s \left(1 - \sum_{i,j=1}^{n} \frac{H_i \check{\Gamma}_{ij} H_j^* K_\pm^{(m)}}{\bar{\lambda}_j} \right)$$

$$= \rho(1 - R^* \Gamma^{-1} S). \tag{62}$$

Here R and S are block matrices

$$R = \begin{pmatrix} H_1^* \\ \vdots \\ H_n^* \end{pmatrix}, \qquad S = \begin{pmatrix} \bar{\lambda}_1^{-1} H_1^* K_\pm^{(m)} \\ \vdots \\ \bar{\lambda}_n^{-1} H_n^* K_\pm^{(m)} \end{pmatrix}, \tag{63}$$

$\rho = -\mu^n$ if n is odd and $\rho = -\mathrm{i}\mu^n$ if n is even.

Let

$$\xi_k = \begin{pmatrix} (\xi_k)_1 \\ \vdots \\ (\xi_k)_n \end{pmatrix} = \left(\Omega^{2k-2} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right) \otimes I_n \tag{64}$$

($k = 0, \pm 1, \pm 2, \ldots$) be $n^2 \times n$ matrices with $(\xi_k)_i = \omega^{-2(k-1)(i-1)} I_n$, then $\xi_{n+k} = \xi_k$ for any integer k. The columns of $\xi_1, \ldots, \xi_n$ are linearly independent vectors in $\mathbf{R}^{n^2}$ and satisfy

$$\xi_i^* \xi_j = \begin{cases} n I_n & \text{if} \quad i-j \equiv 0 \bmod n, \\ 0 & \text{otherwise}. \end{cases} \tag{65}$$

The expression of G_n is given by the following two lemmas for odd and even n separately.

Lemma 7. *When n is odd,*

$$\mu^{-n} G_n = 2 \begin{pmatrix} & & & f_1 & & \\ & & & & \ddots & \\ & & & & & f_n \\ \pm f_{n+1} & & & & & \\ & \ddots & & & & \\ & & \pm f_{2n} & & & \end{pmatrix} \tag{66}$$

where

$$f_j = \begin{cases} h_j \left(\sum_{k=1}^{2n} (-1)^{\{k-j\}} \left(w_k^{(m)}\right)^T h_k \right)^{-1} \left(w_{n+j}^{(m)}\right)^T, & (1 \leqslant j \leqslant n), \\ \pm h_j \left(\sum_{k=1}^{2n} (-1)^{\{k-j\}} \left(w_k^{(m)}\right)^T h_k \right)^{-1} \left(w_{j-n}^{(m)}\right)^T, & (n+1 \leqslant j \leqslant 2n). \end{cases} \tag{67}$$

Proof. From (28),

$$\Gamma \xi_k = \xi_{3-k-m} \alpha_k, \qquad \Gamma^{-1} \xi_k = \xi_{3-k-m} \alpha_{3-k-m}^{-1} \tag{68}$$

where

$$\alpha_k = \sum_{j=1}^{n} \frac{H^T K_{\pm}^{(m)} \Omega^{2j} H}{(1+\omega^{2j})\mu} \omega^{-2(k-1)j} \tag{69}$$

with $\alpha_k^* = \alpha_{3-k-m}$ and $\alpha_{n+k} = \alpha_k$. Write

$$H = \begin{pmatrix} h_1 \\ \vdots \\ h_{2n} \end{pmatrix}, \qquad K_{\pm}^{(m)} H = \begin{pmatrix} w_1^{(m)} \\ \vdots \\ w_{2n}^{(m)} \end{pmatrix}, \tag{70}$$

where $h_1, \ldots, h_{2n}, w_1^{(m)}, \ldots, w_{2n}^{(m)}$ are $1 \times n$ real matrices, then

$$\begin{pmatrix} w_1^{(0)} \\ \vdots \\ w_{2n}^{(0)} \end{pmatrix} = \begin{pmatrix} h_{2n-1} \\ \vdots \\ h_1 \\ \pm h_{2n} \end{pmatrix}, \qquad \begin{pmatrix} w_1^{(1)} \\ \vdots \\ w_{2n}^{(1)} \end{pmatrix} = \begin{pmatrix} h_{2n} \\ h_{2n-1} \\ \vdots \\ h_1 \end{pmatrix}. \tag{71}$$

According to lemma 6,

$$\begin{aligned} \alpha_{2-j} &= \sum_{s=1}^{n} \frac{H^T K_{\pm}^{(m)} \Omega^{2s} H}{(1+\omega^{2s})\mu} \omega^{2(j-1)s} = \sum_{s=1}^{n} \sum_{k=1}^{2n} \frac{\omega^{2(j-k)s}}{(1+\omega^{2s})\mu} \left(w_k^{(m)}\right)^T h_k \\ &= \frac{n}{2\mu} \sum_{k=1}^{2n} (-1)^{\{k-j\}} \left(w_k^{(m)}\right)^T h_k, \end{aligned} \tag{72}$$

which is a real matrix.

Now the ith column of R is $\xi_{2-i} h_i^T$ and the j th column of S_n is $\mu^{-1} \xi_{j-m+1} \left(w_j^{(m)}\right)^T$.

$$\begin{aligned} (R^* \Gamma^{-1} S)_{ij} &= \mu^{-1} h_i \xi_{2-i}^* \Gamma^{-1} \xi_{j-m+1} \left(w_j^{(m)}\right)^T \\ &= \mu^{-1} h_i \xi_{2-i}^* \xi_{2-j} \alpha_{2-j}^{-1} \left(w_j^{(m)}\right)^T. \end{aligned} \tag{73}$$

Hence for $1 \leqslant i \leqslant n$, the possibly non-zero components of $R^* \Gamma^{-1} S$ are

$$\begin{aligned} (R^* \Gamma^{-1} S)_{ii} &= n \mu^{-1} h_i \alpha_{2-i}^{-1} \left(w_i^{(m)}\right)^T, \\ (R^* \Gamma^{-1} S)_{i,n+i} &= n \mu^{-1} h_i \alpha_{2-i}^{-1} \left(w_{n+i}^{(m)}\right)^T, \\ (R^* \Gamma^{-1} S)_{n+i,i} &= n \mu^{-1} h_{n+i} \alpha_{2-i}^{-1} \left(w_i^{(m)}\right)^T, \\ (R^* \Gamma^{-1} S)_{n+i,n+i} &= n \mu^{-1} h_{n+i} \alpha_{2-i}^{-1} \left(w_{n+i}^{(m)}\right)^T. \end{aligned} \tag{74}$$

However, we can prove that actually $(R^*\Gamma^{-1}S)_{ii} = 0$ for $i = 1, \ldots, 2n$. In fact, according to (49), $\Omega G_n = -G_n\Omega$. Written in components, it is $(\omega^{n+i-j} - 1)(G_n)_{ij} = 0$. Hence $(G_n)_{ij} = 0$ unless $i - j \equiv n \bmod 2n$. The only non-zero components of $G_n = -\mu^n(I - R^*\Gamma^{-1})$ are

$$(G_n)_{i,n+i} = n\mu^{n-1}h_i\alpha_{2-i}^{-1}\big(w_{n+i}^{(m)}\big)^T, \qquad (G_n)_{n+i,i} = n\mu^{n-1}h_{n+i}\alpha_{2-i}^{-1}\big(w_i^{(m)}\big)^T. \tag{75}$$

The lemma is proved. □

Lemma 8. *When n is even,*

$$\mu^{-n}G_n = 2\begin{pmatrix} & & & f_1 & & \\ & & & & \ddots & \\ & & & & & f_n \\ \pm f_{n+1} & & & & & \\ & \ddots & & & & \\ & & \pm f_{2n} & & & \end{pmatrix} \tag{76}$$

where

$$f_j = \begin{cases} h_j\left(\sum_{k=1}^{2n}(-1)^{\{k-j\}+[k-j]}\big(w_k^{(m)}\big)^T h_k\right)^{-1}\big(w_{j+n}^{(m)}\big)^T, & (1 \leqslant j \leqslant n),\\ \mp h_j\left(\sum_{k=1}^{2n}(-1)^{\{k-j\}+[k-j]}\big(w_k^{(m)}\big)^T h_k\right)^{-1}\big(w_{j-n}^{(m)}\big)^T, & (n+1 \leqslant j \leqslant 2n). \end{cases} \tag{77}$$

Proof. From (28),

$$\Gamma\xi_k = \xi_{3-k-m}\beta_k, \qquad \Gamma^{-1}\xi_k = \xi_{3-k-m}\beta_{3-k-m}^{-1} \tag{78}$$

where

$$\beta_k = -\theta^{m-1}\sum_{j=1}^{n}\frac{H^T K_{\mp}^{(m)}\Omega^{2j-1}H}{(1+\omega^{2j-1})\mu}\omega^{-2(k-1)(j-1)} \tag{79}$$

with $\beta_k^* = \beta_{3-k-m}$ and $\beta_{n+k} = \beta_k$. Write

$$H = \begin{pmatrix} h_1 \\ \vdots \\ h_{2n} \end{pmatrix}, \qquad K_{\mp}^{(m)}H = \begin{pmatrix} w_1^{(m)} \\ \vdots \\ w_{2n}^{(m)} \end{pmatrix}, \tag{80}$$

where $h_1, \ldots, h_{2n}, w_1^{(m)}, \ldots, w_{2n}^{(m)}$ are $1 \times n$ real matrices, then

$$\begin{pmatrix} w_1^{(0)} \\ \vdots \\ w_{2n}^{(0)} \end{pmatrix} = \begin{pmatrix} h_{2n-1} \\ \vdots \\ h_1 \\ \mp h_{2n} \end{pmatrix}, \qquad \begin{pmatrix} w_1^{(1)} \\ \vdots \\ w_{2n}^{(1)} \end{pmatrix} = \begin{pmatrix} h_{2n} \\ h_{2n-1} \\ \vdots \\ h_1 \end{pmatrix}. \tag{81}$$

According to lemma 6,

$$\begin{aligned}\beta_{2-j} &= -\theta^{m-1}\sum_{s=1}^{n}\frac{H^T K_{\mp}^{(m)}\Omega^{2s-1}H}{(1+\omega^{2s-1})\mu}\omega^{2(j-1)(s-1)}\\ &= -\theta^{m-1}\omega^{-j+1}\sum_{s=1}^{n}\sum_{k=1}^{2n}\frac{\omega^{(j-k)(2s-1)}}{(1+\omega^{2s-1})\mu}\big(w_k^{(m)}\big)^T h_k\\ &= -\theta^{m-1}\omega^{-j+1}\sum_{k=1}^{2n}\frac{n}{2\mu}(-1)^{\{k-j\}+[k-j]}\big(w_k^{(m)}\big)^T h_k\\ &\equiv -\theta^{m-1}\omega^{-j+1}\gamma_{2-j},\end{aligned}\tag{82}$$

where γ_{2-j} is a real matrix with $\gamma_j^T = \gamma_{3-k-m}$.

The ith column of R is $\theta^{i-1}\xi_{2-i}h_i^T$ and the jth column of S is $-\theta^{m-j}\mu^{-1}\xi_{j-m+1}\big(w_j^{(m)}\big)^T$.

$$\begin{aligned}(R^*\Gamma^{-1}S)_{ij} &= -\mu^{-1}\theta^{m-i-j+1}h_i\xi_{2-i}^*\Gamma^{-1}\xi_{j-m+1}\big(w_j^{(m)}\big)^T\\ &= -\mu^{-1}\theta^{m-i-j+1}h_i\xi_{2-i}^*\xi_{2-j}\beta_{2-j}^{-1}\big(w_j^{(m)}\big)^T\\ &= \mu^{-1}\theta^{j-i}h_i\xi_{2-i}^*\xi_{2-j}\gamma_{2-j}^{-1}\big(w_j^{(m)}\big)^T.\end{aligned}\tag{83}$$

As in the proof of the last lemma, $(G_n)_{ij} = 0$ unless $i + j \equiv n \bmod 2n$. Hence the only non-zero components of $G_n = \mathrm{i}\mu^n(I - R^*\Gamma^{-1}S)$ are

$$(G_n)_{i,n+i} = n\mu^{n-1}h_i\gamma_{2-i}^{-1}\big(w_{n+i}^{(m)}\big)^T, \qquad (G_n)_{n+i,i} = -n\mu^{n-1}h_{n+i}\gamma_{2-i}^{-1}\big(w_i^{(m)}\big)^T.\tag{84}$$

This proves the lemma. □

Note that

$$(-1)^{p+[p]} = \begin{cases}(-1)^{\{p\}} & \text{if } n \text{ is odd,}\\ (-1)^{\{p\}+[p]} & \text{if } n \text{ is even,}\end{cases}\tag{85}$$

hence (67) and (77) can be written uniformly as

$$f_j = \begin{cases} h_j\left(\displaystyle\sum_{k=1}^{2n}(-1)^{k-j+[k-j]}\big(w_k^{(m)}\big)^T h_k\right)^{-1}\big(w_{j+n}^{(m)}\big)^T, & (1\leqslant j\leqslant n),\\ \epsilon h_j\left(\displaystyle\sum_{k=1}^{2n}(-1)^{k-j+[k-j]}\big(w_k^{(m)}\big)^T h_k\right)^{-1}\big(w_{j-n}^{(m)}\big)^T, & (n+1\leqslant j\leqslant 2n)\end{cases}\tag{86}$$

where $\varepsilon = \pm$ if n is odd and $\varepsilon = \mp$ if n is even.

Having the expression of G_n, we can write the expression of $\widetilde{Q}_\pm = G_n Q_\pm G_n^{-1}$. The entries of $\widetilde{Q}_\pm$ are

$$\widetilde{q}_j = \frac{f_{j+1}}{f_j}q_{j+n} \qquad (j = 1,\ldots,2n),\tag{87}$$

with $f_{2n+k} = f_k$ and $q_{2n+k} = q_k (k = 0, \pm 1, \ldots)$.

Remark 3. Since $q_{m-1-j} = q_j$ and $\widetilde{q}_{m-1-j} = \widetilde{q}_j$ hold, $f_j f_{m-j}$ is the same for all $j = 0, \pm 1, \ldots$.

In summary, we have

Theorem 2. *Suppose* $\widetilde{U}_\pm = \lambda J_\pm + \widetilde{P}$ *and* $\widetilde{V}_\pm = \lambda^{-1}\widetilde{Q}_\pm$ *are derived from Darboux transformation given by theorem 1. All the notations here are the same as in theorem 1. Let*

$$f_j = \begin{cases} h_j \left(\displaystyle\sum_{k=1}^{2n} (-1)^{k-j+[k-j]} \left(w_k^{(m)}\right)^T h_k\right)^{-1} \left(w_{j+n}^{(m)}\right)^T, & (1 \leqslant j \leqslant n), \\ \varepsilon h_j \left(\displaystyle\sum_{k=1}^{2n} (-1)^{k-j+[k-j]} \left(w_k^{(m)}\right)^T h_k\right)^{-1} \left(w_{j-n}^{(m)}\right)^T, & (n+1 \leqslant j \leqslant 2n), \end{cases} \tag{88}$$

where $w_k^{(m)}$*'s are given by (71) and (81) for odd and even n respectively. Then, after the Darboux transformation,*

$$\widetilde{q}_j = \frac{f_{j+1}}{f_j} q_{j+n} \quad (j = 1, \ldots, 2n). \tag{89}$$

Corresponding to $m = 1$ and $m = 0$, the results in theorem 2 can be written more explicitly in terms of u_j's for the two-dimensional $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations.

Theorem 3. *If* $(u_1, \ldots, u_l)$ *is a solution of the two-dimensional* $C_l^{(1)}$ *Toda equation (3), then*

$$\widetilde{u}_j = -u_{l+1-j} + \ln \frac{f_j}{\sqrt{f_1 f_{2l}}} = -u_{l+1-j} + \frac{1}{2} \ln \frac{f_j}{f_{2l+1-j}} \quad (j = 1, \ldots, l) \tag{90}$$

gives a new solution of the two-dimensional $C_l^{(1)}$ *Toda equation (3). Here*

$$f_j = \begin{cases} h_j \left(\displaystyle\sum_{k=1}^{2l} (-1)^{k-j+[k-j]} h_{2l+1-k}^T h_k\right)^{-1} h_{l+1-j}^T, & (1 \leqslant j \leqslant l), \\ \epsilon h_j \left(\displaystyle\sum_{k=1}^{2l} (-1)^{k-j+[k-j]} h_{2l+1-k}^T h_k\right)^{-1} h_{3l+1-j}^T, & (l+1 \leqslant j \leqslant 2l). \end{cases} \tag{91}$$

If $(u_1, \ldots, u_l)$ *is a solution of the two-dimensional* $D_{l+1}^{(2)}$ *Toda equation (5), then*

$$\widetilde{u}_j = -u_{l+1-j} + \ln \frac{f_j}{f_{2l+2}} = -u_{l+1-j} + \frac{1}{2} \ln \frac{f_j}{f_{2l+2-j}} \quad (j = 1, \ldots, l) \tag{92}$$

gives a new solution of the two-dimensional $D_{l+1}^{(2)}$ *Toda equation (5). Here*

$$f_j = \begin{cases} h_j \left(\displaystyle\sum_{k=1}^{2l+1} (-1)^{k-j+[k-j]} h_{2l+2-k}^T h_k + (-1)^{j+1} \varepsilon h_{2l+2}^T h_{2l+2}\right)^{-1} h_{l+1-j}^T \\ \qquad (1 \leqslant j \leqslant l+1), \\ \epsilon h_j \left(\displaystyle\sum_{k=1}^{2l+1} (-1)^{k-j+[k-j]} h_{2l+2-k}^T h_k + (-1)^{j} \varepsilon h_{2l+2}^T h_{2l+2}\right)^{-1} h_{3l+3-j}^T \\ \qquad (l+2 \leqslant j \leqslant 2l+2). \end{cases} \tag{93}$$

Proof. For the two-dimensional $C_l^{(1)}$ Toda equation, $n = l$ and $m = 1$. From remark 3, $f_j f_{2l+1-j}$ is the same for all j, i.e.

$$f_1 f_{2l} = f_2 f_{2l-1} = \cdots = f_l f_{l+1}, \tag{94}$$

hence

$$\left(\frac{f_j}{\sqrt{f_1 f_{2l}}}\right)^2 = \frac{f_j}{f_{2l+1-j}} \tag{95}$$

holds in (90). By using the relations (16) between q_j's and u_j's, the Darboux transformation (89) becomes

$$2\widetilde{u}_1 = -2u_l + \ln\frac{f_1}{f_{2l}}, \qquad \widetilde{u}_{j+1} - \widetilde{u}_j = u_{l-j+1} - u_{l-j} + \ln\frac{f_{j+1}}{f_j}, \tag{96}$$

and

$$-2\widetilde{u}_l = 2u_l + \ln\frac{f_{l+1}}{f_l}, \qquad \widetilde{u}_{l-j+1} - \widetilde{u}_{l-j} = u_{j+1} - u_j + \ln\frac{f_{l+j+1}}{f_{l+j}} \tag{97}$$

with $j = 1, 2, \ldots, l-1$. By the recursion relations (96), (90) is derived by induction as follows. First,

$$\widetilde{u}_1 = -u_l + \frac{1}{2}\ln\frac{f_1}{f_{2l}}. \tag{98}$$

Assume

$$\widetilde{u}_j = -u_{l+1-j} + \frac{1}{2}\ln\frac{f_j}{f_{2l+1-j}} \tag{99}$$

holds, then from (94), (96) and (99), we have

$$\begin{aligned}\widetilde{u}_{j+1} = \widetilde{u}_j + u_{l-j+1} - u_{l-j} + \ln\frac{f_{j+1}}{f_j} &= -u_{l-j} + \frac{1}{2}\ln\frac{f_{j+1}^2}{f_j f_{2l+1-j}}\\ = -u_{l-j} + \frac{1}{2}\ln\frac{f_{j+1}^2}{f_{j+1} f_{2l-j}} &= -u_{l-j} + \frac{1}{2}\ln\frac{f_{j+1}}{f_{2l-j}}.\end{aligned} \tag{100}$$

Moreover, it can be checked directly that all the relations in (97) hold. This proves (90). (92) for the two-dimensional $D_{l+1}^{(2)}$ Toda equation is derived similarly, with $n = l+1$ and $m = 0$. The theorem is proved. □

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