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Darboux Transformations for Some Two Dimensional Affine Toda Equations

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Abstract

The Lax pairs for the two dimensional $A_{2l}^{(2)}$, $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations have a reality symmetry, a cyclic symmetry and a unitary symmetry. The Darboux transformations for these systems are discussed. These Darboux transformations should be of high degree. Exact solutions are written down by computing the Darboux transformations explicitly.

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1 Introduction

The Toda equation is one of the most important integrable systems. Various kinds of Toda equations have been discussed by a lot of authors after it was proposed. The two dimensional Toda equations have been studied widely [1, 7, 18, 23] and have also important applications in physics [3, 14, 33] as well as in differential geometry [4, 5, 9, 10, 11, 19, 22, 26].

In 1950's, E. Fermi, J. Pasta and S. M. Ulam made a famous numerical experiment (FPU experiment), in which they analyzed the energy distribution on a nonlinear spring with potential like $V(r) = \frac{1}{2}r^2 + \alpha r^3$ rather than the harmonic potential $V(r) = \frac{1}{2}r^2$. They found out the complicated quasi-periodic phenomenon which is completely different from thermalization. This phenomenon attracted many scientists. In 1967, M. Toda introduced the potential $V(r) = \mathrm{e}^{-r} + r - 1 = \frac{1}{2}r^2 - \frac{1}{6}r^3 + o(r^3)$ to emulate the nonlinear spring in the FPU experiment, which led to the celebrated Toda equation

$$u_{j,tt} = \mathrm{e}^{u_{j+1}-u_j} - \mathrm{e}^{u_j-u_{j-1}} \tag{1.1}$$

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which is a system of integrable ordinary differential equations [6].

(1.1) is generalized to the two dimensional Toda equation

$$u_{j,xt} = \mathrm{e}^{u_j - u_{j-1}} - \mathrm{e}^{u_{j+1} - u_j}, \tag{1.2}$$

which is also integrable [20], and has the Lax pair

$$\psi_{j,x} = \lambda\psi_{j+1} + u_{j,x}\psi_j, \quad \psi_{j,t} = \frac{1}{\lambda}\mathrm{e}^{u_j - u_{j-1}}\psi_{j-1}. \tag{1.3}$$

When the two dimensional Toda equation is infinite, the Darboux transformation was presented [17, 18], which could be naturally generalized to the two dimensional periodic Toda equation.

In [25], a kind of Bäcklund transformation for the two dimensional Toda equations corresponding to any semi-simple Lie algebra was obtained in terms of the representation of Lie algebras. Corresponding to any affine Kac-Moody algebra, there is also two dimensional Toda equation [13, 15], which is also called the two dimensional affine Toda equation.

For any affine Kac-Moody algebra, the two dimensional affine Toda equation is [19]

$$w_{j,xt} = \exp\Big(\sum_{i=1}^{l} c_{ji}w_i\Big) - v_j \exp\Big(\sum_{i=1}^{l} c_{0i}w_i\Big) \quad (j = 1, \cdots, l) \tag{1.4}$$

where $C = (c_{ij})_{0\leqslant i,j\leqslant l}$ is the generalized Cartan matrix of the Kac-Moody algebra and $v_0, v_1, \cdots, v_l$ are Coxeter numbers, i.e. they satisfy $C(v_0, v_1, \cdots, v_l)^T = 0$. The simplest two dimensional affine Toda equation, apart from the sinh-Gordon equation, is the one corresponding to $A_2^{(2)}$, which is also called the Tzitzeica equation or Bullough-Dodd-Zhiber-Shabat equation

$$u_{xt} = \mathrm{e}^u - \mathrm{e}^{-2u}. \tag{1.5}$$

Various Kac-Moody algebras correspond to various boundary conditions. It was shown in [15] that these equations are integrable and the Lax pairs were presented. When $\mathfrak{g} = A_l^{(1)}$, the Toda equation is periodic. Its Lax pair has a unitary symmetry and a cyclic symmetry of order l. The Darboux transformation keeping these symmetries was known [17, 18]. Apart from the two dimensional periodic Toda equation, some other integrable systems like Kupershmidt-Wilson hierarchy have also cyclic symmetry, and their Darboux transformation can also be constructed [27].

There are also some work for the two dimensional Toda equations with other Kac-Moody algebras [2, 3, 16, 21, 33], although the symmetries are more complicated. When the Kac-Moody algebra is $A_{2l}^{(2)}$, $C_l^{(1)}$ or $D_{l+1}^{(2)}$, the equations have a unitary symmetry, a reality symmetry and a cyclic symmetry of order $N = 2l + 1$ (for $A_{2l}^{(2)}$) or $N = 2l$ (for $C_l^{(1)}$) or $N = 2l + 2$ (for $D_{l+1}^{(2)}$). (In [27], these three symmetries were called $su(p, q)$-reality condition with respect to a nonstandard indefinite matric, $sl(N, \mathbf{R})$-reality condition and KW-reality condition respectively. In the present paper, reality symmetry is only restricted to the condition that a matrix takes real value.) Although the matrices in the Lax pairs are of order $N \times N$, the number of independent functions is only l. This leads to the difficulty in getting explicit solutions. In [21], the binary Darboux transformation for two dimensional $A_{2l}^{(2)}$, $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations were obtained from the periodic reductions of the binary Darboux transformations for A_∞, B_∞ and C_∞ Toda equations. In terms of the binary Darboux transformation, the solutions of the Toda equations were expressed by some integrals of the solutions of the Lax pairs.

In order to get the explicit solutions which are purely algebraically expressed by the solutions of the Lax pair, usual Darboux transformations (without integrals) are necessary. In [12], explicit solutions of the two dimensional $A_2^{(2)}$ Toda equation were presented for real spectral parameters. In [30, 31, 32], the Darboux transformations for the two dimensional $A_{2l}^{(2)}$, $C_l^{(1)}$ and $D_{l+1}^{(2)}$ Toda equations were constructed and explicit solutions of those Toda equations were obtained. Usually, the unitary symmetry is not a difficult restriction in constructing Darboux transformations. However, with the appearance of cyclic symmetry, the unitary symmetry makes the problem much more complicated. In fact, when the system has only the reality and cyclic symmetries as in the $A_l^{(1)}$ case, Theorem 1 below can be used in constructing Darboux transformations. The lowest degree of the Darboux transformations can be 1 (when a spectral parameter is real) or 2 (when all spectral parameters are complex). However, if there is an extra unitary symmetry, the simpler way is to use Theorem 2 so that the lowest degree of the Darboux transformations should be N and $N/2$ (when a spectral parameter is real) depending on whether N is odd or even, or $2N$ and N (when all spectral parameters are complex). Theorem 1 will lead to the same conclusions, but the procedure is more complicated.

In Section 2, two general ways of constructing Darboux transformations are reviewed. Section 3 shows the structure of the Lax pair for two dimensional Toda equations. In Section 4, Darboux transformation for two dimensional periodic Toda equation is reviewed. Section 5 is devoted to our work on the Darboux transformation for two dimensional affine Toda equations mentioned above. Explicit expression of the solutions of the two dimensional $A_{2l}^{(2)}$ Toda equation is presented as an example.

2 General constructions of Darboux transformations

We review the general construction of Darboux transformations for the Lax pair

$$\Phi_x = U(x,t,\lambda)\Phi, \quad \Phi_t = V(x,t,\lambda)\Phi \tag{2.1}$$

where U and V are $N \times N$ matrix-valued rational functions of λ. An $N \times N$ matrix $G(x,t,\lambda)$ is called a Darboux matrix if there exist $\widetilde{U}(x,t,\lambda)$ and $\widetilde{V}(s,t,\lambda)$, which are also $N \times N$ matrix-valued rational functions of λ, such that for any solution

Φ of (2.1), $\widetilde{\Phi} = G\Phi$ satisfies

$$\widetilde{\Phi}_x = \widetilde{U}(x,t,\lambda)\widetilde{\Phi}, \quad \widetilde{\Phi}_t = \widetilde{V}(x,t,\lambda)\widetilde{\Phi}. \tag{2.2}$$

The corresponding transformations of U and V are given by

$$\widetilde{U} = GUG^{-1} + G_x G^{-1}, \quad \widetilde{V} = GVG^{-1} + G_t G^{-1}. \tag{2.3}$$

Without considering any symmetries of the Lax pair, the Darboux transformation is given by the following theorem.

Theorem 1. [8, 24] *Let $\lambda_1, \cdots, \lambda_N$ be N complex numbers such that λ_j ($j = 1, \cdots, N$) are not all the same. Let h_j be a column solution of the Lax pair* (2.1) *for $\lambda = \lambda_j$ ($j = 1, \cdots, N$). Let $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, $H = (h_1, \cdots, h_N)$, then when $\det H \neq 0$, $G(x,t,\lambda) = \lambda I - H\Lambda H^{-1}$ is a Darboux matrix.*

Now we consider the system with a unitary symmetry, i.e., there is an invertible real symmetric matrix K such that U and V satisfy

$$KU(x,t,\lambda)K^{-1} = -(U(x,t,-\bar{\lambda}))^*, \; KV(x,t,\lambda)K^{-1} = -(V(x,t,-\bar{\lambda}))^*. \tag{2.4}$$

In this case, for any solution Φ of (2.1) with $\lambda = \mu$ and any solution Ψ of (2.1) with $\lambda = -\bar{\mu}$, $(\Psi^* K\Phi)_x = (\Psi^* K\Phi)_t = 0$ holds. Hence $\Psi^* K\Phi = 0$ identically if it holds at one point.

For the system with a unitary symmetry, the most useful way to construct Darboux transformation is given by the following theorem.

Theorem 2. [29] *Let $\lambda_1, \cdots, \lambda_M$ be M complex numbers such that λ_j, $-\bar{\lambda}_j$ ($j = 1, 2, \cdots, M$) are distinct. Let H_j be an $N \times r$ solution of the Lax pair* (2.1) *for $\lambda = \lambda_j$ ($j = 1, 2, \cdots, M$). Let $\Gamma = (\Gamma_{ij})_{1 \leqslant i,j \leqslant M}$ where*

$$\Gamma_{ij} = \frac{H_i^* K H_j}{\bar{\lambda}_i + \lambda_j} \quad (i, j = 1, 2, \cdots, M) \tag{2.5}$$

are $r \times r$ matrices. Then

$$G(x,t,\lambda) = \prod_{l=1}^{M} (\lambda + \bar{\lambda}_l) \left(1 - \sum_{i,j=1}^{M} \frac{H_i (\Gamma^{-1})_{ij} H_j^* K}{\lambda + \bar{\lambda}_j} \right) \tag{2.6}$$

is a Darboux matrix.

In this construction, $G(x,t,\lambda)$ is a polynomial of λ of degree M with matrix coefficients. Usually we write

$$G(x,t,\lambda) = \sum_{j=0}^{M} (-1)^{M-j} G_{M-j}(x,t)\lambda^j, \quad G_0(x,t) = I. \tag{2.7}$$

Remark 1. For the simplest case $M = 1$, the Darboux transformation (2.7) is the same as the one given by Theorem 1 if Λ and H in Theorem 1 are chosen as $\Lambda = \mathrm{diag}(\overbrace{\lambda_1, \cdots, \lambda_1}^{r}, \overbrace{-\bar{\lambda}_1, \cdots, -\bar{\lambda}_1}^{N-r})$, $H = (h_1, \cdots, h_r, w_{r+1}, \cdots, w_N)$ where $(h_1, \cdots, h_r) = H_1$, $w_{r+1}, \cdots, w_N$ are linearly independent solutions of the Lax pair (2.1) for $\lambda = -\bar{\lambda}_1$ such that $w_j^* K h_k = 0$ identically ($j = r+1, \cdots, N$; $k = 1, \cdots, r$). For general M, the Darboux transformation in Theorem 2 can also be constructed by the composition of M Darboux transformations in Theorem 1, but the construction is more complicated.

3 Structure of the Lax pairs for two dimensional Toda equations

For any $N \times N$ matrix A or any N dimensional vector v, and for any integers i and j, define $A_{ij} = A_{i'j'}$ and $v_i = v_{i'}$ where $i \equiv i' \mod N$, $j \equiv j' \mod N$ and $1 \leqslant i', j' \leqslant N$. Especially, δ_{ij} equals 1 if $i \equiv j \mod N$ and equals 0 otherwise.

Let $\omega = \mathrm{e}^{2\pi \mathrm{i}/N}$, $\Omega = \mathrm{diag}(1, \omega^{-1}, \cdots, \omega^{-N+1})$. Let m be a fixed integer. Let $K = (K_{ij}) = (\delta_{i,m-j})_{N\times N}$, $J = (J_{ij}) = (\delta_{i,j-1})_{N\times N}$ be constant matrices. Then K is symmetric and $\Omega^* K = \omega^{m-2} K \Omega$ where Ω^* refers to the Hermitian conjugate of Ω.

The Lax pair for the Toda equations is

$$\begin{aligned} &\Phi_x = U(x,t,\lambda)\Phi = (\lambda J + P(x,t))\Phi, \\ &\Phi_t = V(x,t,\lambda)\Phi = \lambda^{-1} Q(x,t)\Phi \end{aligned} \tag{3.1}$$

and its integrability conditions are

$$Q_x = [P, Q], \quad P_t + [J, Q] = 0. \tag{3.2}$$

There will be different symmetries in the coefficients (P, Q) corresponding to different Kac-Moody algebras.

For the Lax pair (3.1), the Darboux transformation (2.7) gives $\widetilde{P} = P + [J, G_1]$, $\widetilde{Q} = G_M Q G_M^{-1}$, which is derived from (2.3).

For $g = A_l^{(1)}$, the corresponding $N = l$, and the coefficients U and V satisfy a reality symmetry and a cyclic symmetry of order N as

$$\overline{U(x,t,\lambda)} = U(x,t,\bar{\lambda}), \quad \overline{V(x,t,\lambda)} = V(x,t,\bar{\lambda}), \tag{3.3}$$

$$\Omega U(x,t,\lambda)\Omega^{-1} = U(x,t,\omega\lambda), \quad \Omega V(x,t,\lambda)\Omega^{-1} = V(x,t,\omega\lambda). \tag{3.4}$$

Under these symmetries, $P = (p_i \delta_{ij})_{1\leqslant i,j\leqslant N}$, $Q = (q_j \delta_{i,j+1})_{1\leqslant i,j\leqslant N}$ where $p_i(x,t)$'s and $q_i(x,t)$'s are real functions. The integrability conditions (3.2) become

$$q_{j,x} = (p_{j+1} - p_j) q_j, \quad p_{j,t} = q_{j-1} - q_j \quad (j = 1, \cdots, N). \tag{3.5}$$

For $g = A_{2l}^{(2)}$ $(N = 2l+1)$, $C_l^{(1)}$ $(N = 2l)$ and $D_{l+1}^{(2)}$ $(N = 2l+2)$, there is an extra unitary symmetry (2.4). Under this symmetry, we have

$$p_{m-j} = -p_j, \quad q_{m-1-j} = q_j \quad (j = 1, 2, \cdots, N). \tag{3.6}$$

The symmetries of U and V lead to the symmetries of the spectrum.

Lemma 1. Suppose $\mu \in \mathbf{C}\backslash\{0\}$, $\Phi(x,t)$ is a solution of (3.1) for $\lambda = \mu$.

If (U,V) satisfies (3.3) and (3.4), then $\bar{\Phi}(x,t)$ is a solution of (3.1) for $\lambda = \bar{\mu}$, and $\Omega\Phi(x,t)$ is a solution of (3.1) for $\lambda = \omega\mu$.

If, moreover, $N = 2n$ is even and (U,V) also satisfies (2.4), then $\Omega^n\bar{\Phi}$ is a solution of (3.1) for $\lambda = -\bar{\mu}$, and $\Psi = K\Omega^n\Phi$ is a solution of the adjoint Lax pair $\Psi_x = -U(\mu)^T\Psi$, $\Psi_t = -V(\mu)^T\Psi$.

According to Lemma 1, if μ is an eigenvalue of the Lax pair, so are $\omega^j\mu$ and $\overline{\omega^j\mu}$ $(j = 1,\cdots,N)$. Let

$$\begin{array}{l} S_1(\mu) = \{\omega^j\mu \,|\, j = 1,\cdots,N\}, \quad S_2(\mu) = \{\overline{\omega^j\mu} \,|\, j = 1,\cdots,N\}, \\ S_j^*(\mu) = \{-\bar{\lambda} \,|\, \lambda \in S_k(\mu)\}, \ (k = 1,2). \end{array} \tag{3.7}$$

For given $\mu \neq 0$, if the Darboux transformation given by Theorem 2 keeps all the symmetries of the Lax pair, then $\lambda_1,\cdots,\lambda_M$ can not be arbitrary. We denote $SP(\mu) = \{\lambda_1,\cdots,\lambda_M\}$ and $SP^*(\mu) = \{-\bar{\lambda}_1,\cdots,-\bar{\lambda}_M\}$. The relation between $SP(\mu)$, $S_j(\mu)$ and $SP^*(\mu)$, $S_j^*(\mu)$ are determined by the relation among $S_j(\mu)$ and $S_j^*(\mu)$ $(j = 1,2)$, which depends on the parity of N. This will be specified in the following two sections. It is clear that if Γ in (2.5) exists, then $SP(\mu) \cap SP^*(\mu) = \emptyset$.

4 Darboux transformation for two dimensional periodic ($A_l^{(1)}$) Toda equation

For the two dimensional $A_l^{(1)}$ equation, $N = l$, $p_j = u_{j,x}$, $q_j = \mathrm{e}^{u_{j+1}-u_j}$. The equation is

$$u_{j,xt} = \mathrm{e}^{u_j - u_{j-1}} - \mathrm{e}^{u_{j+1}-u_j}. \tag{4.1}$$

The Darboux transformation can be obtained according to Theorem 1. Here we take $M = l$, $\lambda_j = \omega^j\mu$, $H_j = \Omega^j H$ where $\mu \neq 0$ is a real number and H is a column solution of the Lax pair (3.1) with $\lambda = \mu$. Let $H = (h_1,\cdots,h_N)^T$, then the Darboux matrix is given by

$$G_{ij} = \lambda\delta_{ij} - \frac{\mu h_i}{h_{i-1}}\delta_{i-1,j} \tag{4.2}$$

which leads to the transformation

$$\widetilde{\phi}_j = \lambda\phi_j - \mu\frac{h_j}{h_{j-1}}\phi_{j-1} \tag{4.3}$$

and a new solution of the two dimensional $A_l^{(1)}$ Toda equation

$$\widetilde{q}_j = \frac{h_{j+1}h_{j-1}}{h_j^2}q_{j-1}, \tag{4.4}$$

or equivalently

$$\widetilde{u}_j = u_{j-1} + \ln \frac{h_j}{h_{j-1}} + c \tag{4.5}$$

where c is a constant. This construction of Darboux transformation is essentially the same as that given in [18]

The Darboux transformation with complex spectral parameters can be constructed similarly, although its degree should be 2 rather than 1.

5 Darboux transformation for two dimensional $A_{2l}^{(2)}$, $C_l^{(1)}$, $D_{l+1}^{(2)}$ Toda equations

5.1 Explicit form of the evolution equations

It was known [31] that there are only three sets of non-equivalent equations in the system (3.1) with symmetries (2.4), (3.3) and (3.4). They are

(1) $A_{2l}^{(2)}$ ($N = 2l+1$, $m = 0$): $p_j = -p_{2l+1-j} = u_{j,x}$ $(1 \leqslant j \leqslant l)$, $p_{2l+1} = 0$; $q_j = q_{2l-j} = \mathrm{e}^{u_{j+1}-u_j}$ $(1 \leqslant j \leqslant l-1)$, $q_l = \mathrm{e}^{-2u_l}$, $q_{2l} = q_{2l+1} = \mathrm{e}^{u_1}$. The evolution equations are

$$\begin{aligned} &u_{j,xt} = \mathrm{e}^{u_j - u_{j-1}} - \mathrm{e}^{u_{j+1}-u_j} \quad (2 \leqslant j \leqslant l-1),\\ &u_{1,xt} = \mathrm{e}^{u_1} - \mathrm{e}^{u_2 - u_1}, \quad u_{l,xt} = \mathrm{e}^{u_l - u_{l-1}} - \mathrm{e}^{-2u_l}. \end{aligned} \tag{5.1}$$

(2) $C_l^{(1)}$ ($N = 2l$, $m = 1$): $p_j = -p_{2l+1-j} = u_{j,x}$, $(1 \leqslant j \leqslant l)$; $q_j = q_{2l-j} = \mathrm{e}^{u_{j+1}-u_j}$ $(1 \leqslant j \leqslant l-1)$, $q_l = \mathrm{e}^{-2u_l}$, $q_{2l} = \mathrm{e}^{2u_1}$. The evolution equations are

$$\begin{aligned} &u_{j,xt} = \mathrm{e}^{u_j - u_{j-1}} - \mathrm{e}^{u_{j+1}-u_j} \quad (2 \leqslant j \leqslant l-1),\\ &u_{1,xt} = \mathrm{e}^{2u_1} - \mathrm{e}^{u_2 - u_1}, \quad u_{l,xt} = \mathrm{e}^{u_l - u_{l-1}} - \mathrm{e}^{-2u_l}. \end{aligned} \tag{5.2}$$

(3) $D_{l+1}^{(2)}$ ($N = 2l+2$, $m = 0$): $p_j = -p_{2l+2-j} = u_{j,x}$, $(1 \leqslant j \leqslant l)$, $p_{l+1} = p_{2l+2} = 0$; $q_j = q_{2l+1-j} = \mathrm{e}^{u_{j+1}-u_j}$ $(1 \leqslant j \leqslant l-1)$, $q_l = q_{l+1} = \mathrm{e}^{-u_l}$, $q_{2l+1} = q_{2l+2} = \mathrm{e}^{u_1}$. The evolution equations are

$$\begin{aligned} &u_{j,xt} = \mathrm{e}^{u_j - u_{j-1}} - \mathrm{e}^{u_{j+1}-u_j} \quad (2 \leqslant j \leqslant l-1),\\ &u_{1,xt} = \mathrm{e}^{u_1} - \mathrm{e}^{u_2 - u_1}, \quad u_{l,xt} = \mathrm{e}^{u_l - u_{l-1}} - \mathrm{e}^{-u_l}. \end{aligned} \tag{5.3}$$

These equations can be changed to (1.4) by a simple linear transformation of the unknowns $(u_1, \cdots, u_l)$. (There is a mistake of notations in [31] where the names $C_l^{(1)}$ and $D_{l+1}^{(2)}$ there should be interchanged.)

5.2 Constructions of Darboux transformations

The spectrum is different for odd and even N. Hence we will construct the Darboux transformations for odd and even N separately.

When $N = 2n+1$, all the points in $S_1(\mu)$, $S_2(\mu)$, $S_1^*(\mu)$, $S_2^*(\mu)$ are distinct if $\arg(\mu) \neq \dfrac{k\pi}{2(2n+1)}$ for any integer k (See Figure 1). Hence we can choose

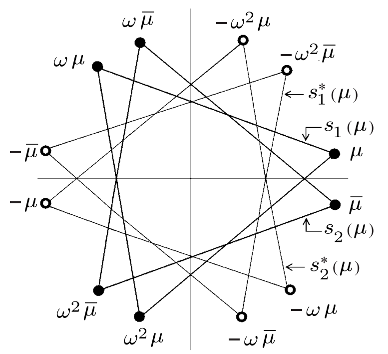

Figure 1 The spectrum $S_j(\mu)$ and $S_j^*(\mu)$ for $N = 3$ (odd N). A dark dot refers to a point in $SP(\mu)$ and a circle refers to a point in $SP^*(\mu)$. The vertices of each triangle refer to one $S_j(\mu)$ or $S_j^*(\mu)$.

$SP(\mu) = S_1(\mu) \cup S_2(\mu)$. The Darboux transformation is constructed as follows. Let $\lambda_j = \omega^{j-1}\mu$, $\lambda_{2n+1+j} = \bar{\lambda}_j$ $(j = 1, 2, \cdots, 2n{+}1)$. Let H be a column solution of (3.1) for $\lambda = \mu$, $H_j = \Omega^{j-1}H$, $H_{2n+1+j} = \bar{H}_j$ $(j = 1, 2, \cdots, 2n+1)$. The Darboux matrix $G(x, t, \lambda)$ is constructed according to Theorem 2 with $M = 2(2n + 1)$.

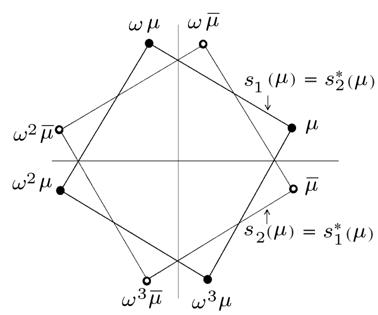

Figure 2 The spectrum $S_j(\mu)$ and $S_j^*(\mu)$ for $N = 4$ (even N). A dark dot refers to a point in $SP(\mu)$ and a circle refers to a point in $SP^*(\mu)$. The vertices of each square refer to one $S_j(\mu)$ or $S_j^*(\mu)$.

When $N = 2n$, $S_1^*(\mu) = S_2(\mu)$, $S_2^*(\mu) = S_1(\mu)$. Moreover, when $\arg(\mu) \neq \dfrac{k\pi}{2n}$ for any integer k, $S_1(\mu) \cap S_2(\mu) = \emptyset$ (See Figure 2). Clearly, we can not choose $SP(\mu) = S_1(\mu) \cup S_2(\mu)$, since otherwise $SP^*(\mu) = SP(\mu)$ would hold, which contradicts the existence of Γ. Hence we choose $SP(\mu) = S_1(\mu)$. Now $S_2(\mu)$ is not contained in $SP(\mu)$, but contained in $SP^*(\mu)$. According to Remark 1, in order to keep the symmetry between $S_1(\mu)$ and $S_2(\mu)$, there should be a symmetry between the solution $h_1, \cdots, h_r$ and $w_{r+1}, \cdots w_{2n}$ where $w_{r+1}, \cdots w_{2n}$ are defined in Remark 1. Moreover, according to (c) of Lemma 1, $r = n$, and $w_j = \Omega^n \bar{h}_j$ should hold. Therefore, the Darboux transformation is constructed as follows. Let $\lambda_j = \omega^{j-1}\mu$ $(j = 1, 2, \cdots, 2n)$. Let H be an $2n \times n$ matrix solution of (3.1) for $\lambda = \mu$ such that $H^T K \Omega^n H = (\Omega^n \bar{H})^* K H = 0$ at certain point (x_0, t_0). Then (c) of Lemma 1 implies that $H^T K \Omega^n H = 0$ holds identically. Let $H_j = \Omega^{j-1} H$ $(j = 1, 2, \cdots, 2n)$. The Darboux matrix $G(x, t, \lambda)$ is constructed according to Theorem 2 with $M = 2n$.

After the Darboux transformation, the coefficients of the Lax pair are changed to $\widetilde{U} = \lambda J + \widetilde{P}$, $\widetilde{V} = \frac{1}{\lambda}\widetilde{Q}$ where $\widetilde{P} = P + [J, G_1]$, $\widetilde{Q} = G_M Q G_M^{-1}$.

Moreover, we have

Theorem 3. [30, 31] *The Darboux transformation satisfies the following symmetries:*

$$\textit{Unitary:}\quad G(x,t,-\bar{\lambda})^* K G(x,t,\lambda) = \prod_{l=1}^{M} (\bar{\lambda}_l + \lambda)(\lambda_l - \lambda) K, \tag{5.4}$$

$$\textit{Cyclic:}\quad \Omega G(x,t,\omega^{-1}\lambda)\Omega^{-1} = G(x,t,\lambda), \tag{5.5}$$

$$\textit{Reality:}\quad \overline{G(x,t,\bar{\lambda})} = G(x,t,\lambda) \tag{5.6}$$

where $M = 2(2n+1)$ for $N = 2n+1$ and $M = 2n$ for $N = 2n$.

(5.4) is a direct conclusion of the construction of the Darboux transformation (2.6). (5.5) can be verified directly. (5.6) is also obtained directly when N is odd. However, when N is even, since the spectrum does not have the reality symmetry explicitly, direct verification of (5.6) is very difficult. Instead, let $\Delta(x,t,\lambda) = \overline{G(x,t,\bar{\lambda})} - G(x,t,\lambda)$, then it can be proved that $\Delta(x,t,\lambda_i)H_i = 0$, $\Delta(x,t,-\bar{\lambda}_i)\Omega^n \bar{H}_i = 0$ hold. Then we can get $\Delta(x,t,\lambda) \equiv 0$ after proving the invertibility of a matrix composing λ_i's and H_i's.

Theorem 3 implies that the Darboux transformation keeps all the symmetries (2.4), (3.3) and (3.4).

5.3 Explicit expressions of the solutions

In order to get explicit expressions of the solutions we should derive the explicit expression of $G_{2(2n+1)}$. (2.6) is too complicated to be computed directly even by computer. Therefore, we need to represent the matrix Γ, regarded as a linear transformation, in another basis, so that the computation can be simplified. For example, when N is odd, the solution of the two dimensional $A_{2l}^{(2)}$ Toda equation can be obtained explicitly as follows.

Theorem 4. [30] *Suppose $(u_1, \cdots, u_l)$ be a solution of the two dimensional $A_{2l}^{(2)}$ Toda equation (5.1), μ is a complex number with $\arg(\mu) \neq \dfrac{k\pi}{2(2l+1)}$ for any integer k, $H = (h_1, \cdots, h_{2l+1})^T$ is a column solution of the Lax pair for $\lambda = \mu$, then*

$$\widetilde{u}_k = u_k + \ln \frac{\zeta_{k+1}}{\zeta_k} \quad (k = 1, 2, \cdots, l) \tag{5.7}$$

is a new solution of the two dimensional $A_{2l}^{(2)}$ Toda equation (5.1) where

$$\begin{aligned}\zeta_k = \frac{1}{|\bar{\mu}^{2l+1}+\mu^{2l+1}|^2}&\left|\sum_{s=1}^{2l+1-k}\bar{h}_s h_{-s}(-\mu)^{2l+1-k-s}\bar{\mu}^{k+s-1}\right.\\&\left.+\sum_{s=2l+2-k}^{2l+1}\bar{h}_s h_{-s}(-\mu)^{4l+2-k-s}\bar{\mu}^{k+s-2l-2}\right|^2\\&-\frac{1}{4|\mu|^2}\left|\sum_{s=1}^{2l+1-k}h_s h_{-s}(-1)^{k+s}-\sum_{s=2l+2-k}^{2l+1}h_s h_{-s}(-1)^{k+s}\right|^2.\end{aligned}\tag{5.8}$$

When $N = 2n$ is even, the solutions for $C_l^{(1)}$ $(n = l)$ and $D_{l+1}^{(2)}$ $(n = l+1)$ cannot be written down in such a simple form without matrix operation. However, they can be simplified so that they only depends on the the inverse of an $n \times n$ matrix rather than the original $2n \times 2n$ matrices.

Till now, we need the condition $\arg(\mu) \neq \dfrac{k\pi}{2(2n+1)}$ when $N = 2n+1$ and $\arg(\mu) \neq \dfrac{k\pi}{2n}$ when $N = 2n$ for any integer k. However, if $\arg(\mu) = \dfrac{k\pi}{2(2n+1)}$ when $N = 2n+1$ or $\arg(\mu) = \dfrac{k\pi}{2n}$ when $N = 2n$ for certain integer k, the relations among the sets $S_1(\mu)$, $S_2(\mu)$, $S_1^*(\mu)$ and $S_2^*(\mu)$ are changed. In fact, when N is odd, we have either $S_2(\mu) = S_1(\mu)$, $S_2^*(\mu) = S_1^*(\mu)$ or $S_1^*(\mu) = S_1(\mu)$, $S_2^*(\mu) = S_2(\mu)$. When N is even, we have $S_1^*(\mu) = S_2^*(\mu) = S_1(\mu) = S_2(\mu)$. The Darboux transformations can be constructed differently [30, 32].

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