Darboux transformations and exact solutions of two dimensional $A_{2n}^{(2)}$ Toda equation

Zi-Xiang Zhou[a)]
Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China

(Received 7 April 2004; accepted 16 December 2004; published online 16 February 2005)

The Darboux transformations for two dimensional $A_{2n}^{(2)}$ Toda equation are constructed. The lowest degree of the Darboux transformation is $2(2n+1)$ if all the spectral parameters are complex, or $2n+1$ if one spectral parameter is real. Exact solutions are written down by computing the Darboux transformations explicitly. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1857033]

I. INTRODUCTION

The two dimensional Toda equation is an important integrable system which has been studied widely (e.g., Refs. 1–15) and has applications to differential geometry.[4–7] A Toda equation corresponding to a Kac–Moody algebra g of affine type can be written as

$$w_{k,xt} = A_k \exp\left(\sum_{i=1}^{n} c_{ki} w_i\right) - A_0 v_k \exp\left(\sum_{i=1}^{n} c_{0i} w_i\right) \quad (k = 1,\ldots,n), \tag{1.1}$$

where $C=(c_{ij})_{0\leqslant i,j\leqslant n}$ is the generalized Cartan matrix of g, $v=(v_0,v_1,\ldots,v_n)^T$ is a nonzero vector such that $Cv=0$, and $A_0,A_1,\ldots,A_n$ are real constants.[8,9] There have been a lot of works on the two dimensional Toda equations which are infinite, or are periodic (with $g=A_n^{(1)}$),[5,10] or having fixed ends,[11,12] or with finite dimensional Lie algebras (Kac–Moody algebras of finite type).[13–15]

As a special case of $g=A_2^{(2)}$, the Tzitzeica equation is a typical equation in affine geometry describing indefinite affine spheres.[4,16] An expression of Darboux transformation of the Tzitzeica equation, whose spectral parameter is real, was discussed in Ref. 17, and the loop group decomposition was presented in Ref. 18.

In this paper, we consider the Toda equation with Kac–Moody algebra $g=A_{2n}^{(2)}$. It is neither periodic nor with fixed ends. It has an $(2n+1)\times(2n+1)$ Lax pair, and the Lax pair has a unitary symmetry, a reality symmetry and a cyclic symmetry of order $2n+1$. To get the Darboux transformations which generate solutions of the same equation, we need to consider all these symmetries in the construction. Therefore, Darboux transformations of high degree are necessary. We shall consider the case where all the spectral parameters are complex, as well as the case where one spectral parameter is real. The main results are presented in Theorem 1 of Sec. IV and Theorem 2 of Sec. V.

In Sec. II, we discuss the Lax pair of the $A_{2n}^{(2)}$ Toda equation. In Sec. III, the formulas of Darboux transformation are listed in terms of the standard construction. In Sec. IV, the Darboux transformation with complex spectral parameters is written down in an explicit way. Correspondingly, the Darboux transformation with a real spectral parameter is written down explicitly in Sec. V. In Sec. VI, the explicit solutions of the Tzitzeica equation are presented.

II. LAX PAIR AND EVOLUTION EQUATIONS

The $A_2^{(2)}$ Toda equation, or the Tzitzeica equation, is

[a)]Electronic mail: zxzhou@public6.sta.net.cn

$$w_{1,xt} = -A_0 e^{-w_1} + A_1 e^{2w_1}, \tag{2.1}$$

where A_0, A_1 are real constants. It corresponds to the generalized Cartan matrix $C=\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ of $A_2^{(2)}$ with $C(\frac{1}{2},1)^T=0$.

For $n\geqslant 2$, the $A_{2n}^{(2)}$ Toda equation is

$$w_{1,xt} = -A_0 e^{-w_1} + A_1 e^{2w_1-w_2},$$

$$w_{j,xt} = -A_0 e^{-w_1} + A_j e^{2w_j-w_{j-1}-w_{j+1}} \quad (j=2,\ldots,n-1),$$

$$w_{n,xt} = -A_0 e^{-w_1} + A_n e^{2w_n-2w_{n-1}}, \tag{2.2}$$

where $A_0, A_1,\ldots, A_n$ are real constants. It corresponds to the generalized Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & & & & \\ -2 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix} \tag{2.3}$$

of $A_{2n}^{(2)}$ and $C(\frac{1}{2},1,\ldots,1)^T=0$.

For all $n\geqslant 1$, let $w_j=-(u_1+\ldots+u_j)$ $(j=1,\ldots,n)$, then (2.1) becomes

$$u_{1,xt} = A_0 e^{u_1} - A_1 e^{-2u_1}, \tag{2.4}$$

and (2.2) becomes

$$u_{1,xt} = A_0 e^{u_1} - A_1 e^{u_2-u_1},$$

$$u_{j,xt} = A_{j-1} e^{u_j-u_{j-1}} - A_j e^{u_{j+1}-u_j} \quad (2\leqslant j\leqslant n-1),$$

$$u_{n,xt} = A_{n-1} e^{u_n-u_{n-1}} - A_n e^{-2u_n}. \tag{2.5}$$

Hereafter, for any $(2n+1)\times(2n+1)$ matrix A or any $(2n+1)$-vector v, and for any integers i and j, define $A_{ij}=A_{i'j'}$ and $v_i=v_{i'}$ where $i\equiv i' \bmod 2n+1$, $j\equiv j' \bmod 2n+1$, and $1\leqslant i',j'\leqslant 2n+1$. Especially, denote

$$\delta_{ij} = \begin{cases} 1 & \text{if } i-j\equiv 0 \bmod 2n+1, \\ 0 & \text{otherwise}. \end{cases} \tag{2.6}$$

Let $\omega=e^{2\pi i/(2n+1)}$, $\Omega=\mathrm{diag}(1,\omega^{-1},\ldots,\omega^{-2n})$. Let $K=(K_{ij})=(\delta_{i,2n+1-j})_{(2n+1)\times(2n+1)}$ and $J=(J_{ij})=(\delta_{i,j-1})_{(2n+1)\times(2n+1)}$ be two constant matrices and

$$P=(p_i\delta_{ij})_{(2n+1)\times(2n+1)}, \quad Q=(q_i\delta_{i,j+1})_{(2n+1)\times(2n+1)}, \tag{2.7}$$

where

$$p_i = -p_{2n+1-i} = u_{i,x} \quad (1\leqslant i\leqslant n), \quad p_{2n+1}=0,$$

$$q_i = q_{2n-i} = A_i e^{u_{i+1}-u_i} \quad (1\leqslant i\leqslant n-1), \quad q_n = A_n e^{-2u_n}, \quad q_{2n}=q_{2n+1}=A_0 e^{u_1}. \tag{2.8}$$

From (2.7) and (2.8), $\det Q=q_1\ldots q_{2n+1}=A_0^2A_1^2\ldots A_{n-1}^2A_n$ is a constant.

Notice that

$$\Omega^* K = \omega^{-2} K \Omega. \tag{2.9}$$

Here A^* refers to the Hermitian conjugate of a matrix A.

Now consider the Lax pair

$$\Phi_x = U(x,t,\lambda)\Phi \equiv (\lambda J + P(x,t))\Phi, \quad \Phi_t = V(x,t,\lambda)\Phi \equiv \lambda^{-1} Q(x,t)\Phi. \tag{2.10}$$

Its integrability condition

$$U_t - V_x + [U,V] = 0 \tag{2.11}$$

is

$$Q_x = [P,Q], \quad P_t + [J,Q] = 0, \tag{2.12}$$

or equivalently

$$q_{i,x} = (p_{i+1} - p_i) q_i, \quad p_{i,t} = q_{i-1} - q_i. \tag{2.13}$$

These are just the $A_{2n}^{(2)}$ Toda equations (2.4) and (2.5) by considering (2.8).

It is easy to check that J, $P(x,t)$, $Q(x,t)$ satisfy the relations

$$\Omega P \Omega^{-1} = P, \quad \Omega J \Omega^{-1} = \omega J, \quad \Omega Q \Omega^{-1} = \omega^{-1} Q,$$

$$KPK^{-1} = -P^T, \quad KJK^{-1} = J^T, \quad KQK^{-1} = Q^T. \tag{2.14}$$

Written equivalently in terms of U and V, (2.14) becomes

$$\overline{U(x,t,\lambda)} = U(x,t,\bar{\lambda}), \quad \overline{V(x,t,\lambda)} = V(x,t,\bar{\lambda}),$$

$$\Omega U(x,t,\lambda)\Omega^{-1} = U(x,t,\omega\lambda), \quad \Omega V(x,t,\lambda)\Omega^{-1} = V(x,t,\omega\lambda),$$

$$KU(x,t,\lambda)K^{-1} = -(U(x,t,-\bar{\lambda}))^*, \quad KV(x,t,\lambda)K^{-1} = -(V(x,t,-\bar{\lambda}))^*. \tag{2.15}$$

Conversely, if (P,Q) satisfies (2.12), then $\det Q$ is independent of x, since

$$(\det Q)_x = \det Q \operatorname{tr}(Q_x Q^{-1}) = \det Q \operatorname{tr}([PQ^{-1}, Q]) = 0. \tag{2.16}$$

Now suppose that $A_0, A_1, \ldots, A_{n-1} \neq 0$ are given constants and (P,Q) is a $(2n+1)\times(2n+1)$ matrix solution of (2.12) satisfying the constraints (2.14) and $\det Q$ is a constant. From (2.14), P and Q must be of form (2.7) with

$$p_i + p_{2n+1-i} = 0, \quad q_i = q_{2n-i} \quad (i = 1,2,\ldots,2n+1). \tag{2.17}$$

In the region where $q_i A_i > 0$ $(i=1,\ldots,n)$, let $q_i = A_i e^{u_{i+1}-u_i}$ $(1 \leqslant i \leqslant n-1)$, $q_{2n} = A_0 e^{u_1}$, then $q_n = A_n e^{-2u_n}$ where $A_n = \det Q / A_0^2 A_1^2 \cdots A_{n-1}^2$. Moreover, from the first equation of (2.12),

$$p_i = -p_{2n+1-i} = u_{i,x} \quad (1 \leqslant i \leqslant n), \quad p_{2n+1} = 0. \tag{2.18}$$

The second equation of (2.12) implies that $(u_1, \ldots, u_n)$ is a solution of the $A_{2n}^{(2)}$ Toda equation (2.4) or (2.5). Therefore, we have the following,

Lemma 1: *Suppose (P,Q) is a $(2n+1)\times(2n+1)$ matrix solution of (2.12) satisfying the constraints (2.14) and $\det Q$ is a constant. Then in the region where $q_i A_i > 0$ $(i=1,\ldots,n)$, $(u_1,\ldots,u_n)$ is a solution of the $A_{2n}^{(2)}$ Toda equation (2.4) or (2.5).*

In order to construct Darboux transformation, we need the following lemma, which is the direct consequence of (2.14).

Lemma 2: *Suppose $\mu \in \mathbf{C}$.*

(i) *If $\Phi(x,t)$ is a solution of (2.10) for $\lambda = \mu$, then $\overline{\Phi}(x,t)$ is a solution of (2.10) for $\lambda = \bar{\mu}$;*

(ii) *If* $\Phi(x,t)$ *is a solution of (2.10) for* $\lambda=\mu$*, then for any integer* k*,* $\Omega^k\Phi(x,t)$ *is a solution of (2.10) for* $\lambda=\omega^k\mu$*.*

(iii) *If* $\Phi(x,t)$ *is a solution of (2.10) for* $\lambda=\mu$*,* $\Psi(x,t)$ *is a solution of (2.10) for* $\lambda=-\bar{\mu}$*, then* $(\Psi^* K\Phi)_x=0$*,* $(\Psi^* K\Phi)_t=0$*.*

III. DARBOUX TRANSFORMATION

The Darboux matrix with unitary reduction can be constructed in the following known procedure.[19,20]

Let $\lambda_1,\dots,\lambda_M$ be M complex numbers such that λ_j, $-\bar{\lambda}_j$ $(j=1,2,\dots,M)$ are distinct. Let H_j be a column solution of the Lax pair (2.10) for $\lambda=\lambda_j$ $(j=1,2,\dots,M)$. Denote

$$\Gamma_{ij}=\frac{H_i^* K H_j}{\bar{\lambda}_i+\lambda_j} \tag{3.1}$$

for $i,j=1,2,\dots,M$, $\Gamma=(\Gamma_{ij})_{1\le i,j\le M}$,

$$G(x,t,\lambda)=\prod_{l=1}^{M}(\lambda+\bar{\lambda}_l)\left(1-\sum_{i,j=1}^{M}\frac{(\Gamma^{-1})_{ij}H_iH_j^*K}{\lambda+\bar{\lambda}_j}\right). \tag{3.2}$$

Then it can be checked directly that

$$G(x,t,\lambda)^{-1}=\prod_{l=1}^{M}(\lambda+\bar{\lambda}_l)^{-1}\left(1+\sum_{i,j=1}^{M}\frac{(\Gamma^{-1})_{ij}H_iH_j^*K}{\lambda-\lambda_i}\right). \tag{3.3}$$

$G(x,t,\lambda)$ is a polynomial of λ of degree M with matrix coefficients. Write

$$G(x,t,\lambda)=\sum_{j=0}^{M}(-1)^{M-j}G_{M-j}(x,t)\lambda^j,\quad G_0(x,t)=I, \tag{3.4}$$

and define

$$\tilde{U}=GUG^{-1}+G_xG^{-1},\quad \tilde{V}=GVG^{-1}+G_tG^{-1}. \tag{3.5}$$

Lemma 3: $\tilde{U}=\lambda J+\tilde{P}$, $\tilde{V}=(1/\lambda)\tilde{Q}$ *where* $\tilde{P}=P+[J,G_1]$, $\tilde{Q}=G_MQG_M^{-1}$.

Proof: Using (2.14) and the fact that H_j is a solution of the Lax pair (2.10) with $\lambda=\lambda_j$, we have

$$\Gamma_{ij,x}=H_i^*KJH_j,\quad \Gamma_{ij,t}=\frac{1}{\bar{\lambda}_i\lambda_j}H_i^*KQH_j. \tag{3.6}$$

Substituting (3.2), (3.3), and (3.6) into (3.5) and using the symmetries (2.14), we get the conclusion of the lemma by direct calculation.

Lemma 4:

$$G(x,t,-\bar{\lambda})^*KG(x,t,\lambda)=\prod_{l=1}^{M}(\bar{\lambda}_l+\lambda)(\lambda_l-\lambda)K. \tag{3.7}$$

Proof: The equality (3.7) follows from

$$G(x,t,-\bar{\lambda})^*=KG(x,t,\lambda)^{-1}K^{-1}\prod_{l=1}^{M}(\bar{\lambda}_l+\lambda)(\lambda_l-\lambda), \tag{3.8}$$

which is a direct result of (3.2), (3.3) and the fact that Γ is Hermitian. The lemma is proved.

The solution $\widetilde{Q}$ has been expressed in terms of G_M. According to (3.2),

$$(-1)^M G_M = \prod_{l=1}^{M} \bar{\lambda}_l \left(1 - \sum_{i,j=1}^{M} \frac{(\Gamma^{-1})_{ij} H_i H_j^* K}{\bar{\lambda}_j} \right) = \prod_{l=1}^{M} \bar{\lambda}_l \big(1 - (2n+1) R^* \Gamma^{-1} S \big), \tag{3.9}$$

where R and S are $M \times (2n+1)$ matrices with

$$R_{ij} = (2n+1)^{-1/2} (\bar{H}_i)_j, \quad S_{ij} = (2n+1)^{-1/2} \bar{\lambda}_i^{-1} \sum_{k=1}^{2n+1} (\bar{H}_i)_{-k}. \tag{3.10}$$

IV. DARBOUX TRANSFORMATION WITH COMPLEX SPECTRAL PARAMETERS

Darboux matrix keeping the reductions (2.14) can be derived by above general construction together with some more constraints on the spectral parameters and the solutions of the Lax pair.

Let μ be a nonzero complex number such that $\arg(\mu) \neq k\pi/(4n+2)$ for any integer k. Let $\lambda_j = \omega^{j-1}\mu$, $\lambda_{2n+1+j} = \omega^{-j+1}\bar{\mu}$ $(j=1,2,\ldots,2n+1)$. Then all λ_j and $-\bar{\lambda}_j$ $(j=1,2,\ldots,2(2n+1))$ are distinct. Let h be a column solution of (2.10) for $\lambda=\mu$, $H_j = \Omega^{j-1} h$, $H_{2n+1+j} = \Omega^{-j+1}\bar{h}$ $(j=1,2,\ldots,2n+1)$. Then H_j is a solution of (2.10) for $\lambda = \lambda_j (j=1,2,\ldots,2(2n+1))$.

We construct Γ, $G(x,t,\lambda)$, $\widetilde{U}(x,t,\lambda)$, $\widetilde{V}(x,t,\lambda)$ according to Sec. III with $M=2(2n+1)$.

More explicit expressions of Γ_{ij} can be written down from their definition (3.1). For $1 \leqslant i,j \leqslant 2n+1$, let

$$A_{ij} = \Gamma_{ij} = \frac{h^* (\Omega^*)^{i-1} K \Omega^{j-1} h}{\omega^{-i+1}\bar{\mu} + \omega^{j-1}\mu} = \omega^{-i+1} \frac{h^* K \Omega^{i+j-2} h}{\bar{\mu} + \omega^{i+j-2}\mu},$$

$$B_{ij} = \Gamma_{i,2n+1+j} = \frac{h^* (\Omega^*)^{i-1} K \Omega^{-j+1} \bar{h}}{\omega^{-i+1}\bar{\mu} + \omega^{-j+1}\bar{\mu}} = \omega^{-i+1} \frac{h^* K \Omega^{i-j} \bar{h}}{\bar{\mu} + \omega^{i-j}\bar{\mu}},$$

$$C_{ij} = \Gamma_{2n+1+i,j} = \frac{\bar{h}^* (\Omega^*)^{-i+1} K \Omega^{j-1} h}{\omega^{i-1}\mu + \omega^{j-1}\mu} = \omega^{i-1} \frac{\bar{h}^* K \Omega^{j-i} h}{\mu + \omega^{j-i}\mu},$$

$$D_{ij} = \Gamma_{2n+1+i,2n+1+j} = \frac{\bar{h}^* (\Omega^*)^{-i+1} K \Omega^{-j+1} \bar{h}}{\omega^{i-1}\mu + \omega^{-j+1}\bar{\mu}} = \omega^{i-1} \frac{\bar{h}^* K \Omega^{-i-j+2} \bar{h}}{\mu + \omega^{-i-j+2}\bar{\mu}}. \tag{4.1}$$

Then Γ is written as a 2×2 block matrix $\Gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where $A=(A_{ij})$, $B=(B_{ij})$, $C=(C_{ij})$, $D=(D_{ij})$ are $(2n+1)\times(2n+1)$ matrices, and $\Gamma^{-1} = \left(\begin{smallmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{smallmatrix}\right)$ where

$$\hat{A} = (A - BD^{-1}C)^{-1}, \quad \hat{B} = -A^{-1}B(D - CA^{-1}B)^{-1},$$

$$\hat{C} = -D^{-1}C(A - BD^{-1}C)^{-1}, \quad \hat{D} = (D - CA^{-1}B)^{-1}. \tag{4.2}$$

From (4.1), we know that $D = \bar{A}$, $C = \bar{B}$. Hence, (4.2) leads to $\hat{D} = \bar{\hat{A}}$, $\hat{C} = \bar{\hat{B}}$. Then, from (3.2), we have the following.

Lemma 5: *$\overline{G(x,t,\bar{\lambda})} = G(x,t,\lambda)$. Hence the coefficients of each power of λ in $G(x,t,\lambda)$ are all real matrices.*

From (4.1), we get

$$A_{i+1,j-1} = \omega^{-1} A_{ij}, \quad B_{i+1,j+1} = \omega^{-1} B_{ij}, \quad C_{i+1,j+1} = \omega C_{ij}, \quad D_{i+1,j-1} = \omega D_{ij}. \tag{4.3}$$

Written equivalently, they are

$$JAJ=\omega^{-1}A, \quad JBJ^{-1}=\omega^{-1}B, \quad J^{-1}CJ=\omega^{-1}C, \quad J^{-1}DJ^{-1}=\omega^{-1}D. \tag{4.4}$$

According to (4.2),

$$J\hat{A}J=\omega^{-1}\hat{A}, \quad J\hat{B}J^{-1}=\omega^{-1}\hat{B}, \quad J^{-1}\hat{C}J=\omega^{-1}\hat{C}, \quad J^{-1}\hat{D}J^{-1}=\omega^{-1}\hat{D}. \tag{4.5}$$

Hence

$$\hat{A}_{i+1,j-1}=\omega^{-1}\hat{A}_{ij}, \quad \hat{B}_{i+1,j+1}=\omega^{-1}\hat{B}_{ij}, \quad \hat{C}_{i+1,j+1}=\omega\hat{C}_{ij}, \quad \hat{D}_{i+1,j-1}=\omega\hat{D}_{ij}. \tag{4.6}$$

Lemma 6: $\Omega G(x,t,\lambda)\Omega^{-1}=G(x,t,\omega\lambda)$.

Proof:

$$\Omega\frac{\Omega^{i-1}h(\Omega^{j-1}h)^*K}{\lambda+\overline{\omega^{j-1}\mu}}\Omega^{-1}=\frac{\omega\Omega^{i}h(\Omega^{j-2}h)^*\Omega^*K\Omega^{-1}}{\omega\lambda+\overline{\omega^{j-2}\mu}}=\frac{\omega^{-1}\Omega^{i}h(\Omega^{j-2}h)^*K}{\omega\lambda+\overline{\omega^{j-2}\mu}}. \tag{4.7}$$

Hence,

$$\Omega\sum_{i,j=1}^{2n+1}\hat{A}_{ij}\frac{\Omega^{i-1}h(\Omega^{j-1}h)^*K}{\lambda+\overline{\omega^{j-1}\mu}}\Omega^{-1}=\sum_{i,j=1}^{2n+1}\hat{A}_{i+1,j-1}\frac{\Omega^{i}h(\Omega^{j-2}h)^*K}{\omega\lambda+\overline{\omega^{j-2}\mu}}=\sum_{i,j=1}^{2n+1}\hat{A}_{ij}\frac{\Omega^{i-1}h(\Omega^{j-1}h)^*K}{\omega\lambda+\overline{\omega^{j-1}\mu}} \tag{4.8}$$

by (4.6). Similar relations hold for the terms in G with B_{ij}, C_{ij}, and D_{ij}. Moreover, $\Pi_{l=1}^{2(2n+1)}(\lambda+\overline{\lambda}_l)=\Pi_{l=1}^{2(2n+1)}(\omega\lambda+\overline{\lambda}_l)$. Hence, by (3.2), the lemma is true.

From Lemma 4, Lemma 5, and Lemma 6, $\widetilde{U}$ and $\widetilde{V}$ defined by (3.5) are real and satisfy the relations (2.15). Moreover, after the Darboux transformation, they must satisfy the integrability condition (2.11). Hence, by Lemma 1, we have the following.

Proposition 1: *$(\widetilde{P},\widetilde{Q})$ generated by the Darboux transformation constructed above is a real solution of (2.12) satisfying the relations (2.14). Therefore, it gives a solution of the $A_{2n}^{(2)}$ Toda equation (2.4) or (2.5).*

In order to get explicit expressions of the solutions we should derive the explicit expression of $G_{2(2n+1)}$. Equation (3.2) is too complicated to be computed directly even by computer. Therefore, we need to represent the matrix Γ, regarded as a linear transformation, in another basis, so that its inverse can be computed explicitly.

Let $\xi_k=(2n+1)^{-1/2}\Omega^{k-1}(1,1,\ldots,1)^T$, then the ith component of ξ_k is

$$(\xi_k)_i=(2n+1)^{-1/2}\omega^{-(k-1)(i-1)}. \tag{4.9}$$

Using the fact

$$\sum_{j=1}^{2n+1}\omega^{jk}=\begin{cases}0 & \text{if } k\not\equiv 0 \bmod 2n+1,\\ 2n+1 & \text{if } k\equiv 0 \bmod 2n+1,\end{cases} \tag{4.10}$$

we know that $(\xi_1,\ldots,\xi_{2n+1})$ is an orthonormal basis of $\mathbf{C}^{2n+1}$.

From (4.1),

$$A\xi_k=\alpha_k\xi_{3-k}, \quad B\xi_k=\overline{\gamma}_{2-k}\xi_{k+1}, \quad C\xi_k=\gamma_k\xi_{k-1}, \quad D\xi_k=\overline{\alpha}_{2-k}\xi_{1-k}, \tag{4.11}$$

where

$$\alpha_k=\sum_{j=1}^{2n+1}\frac{h^*K\Omega^jh}{\overline{\mu}+\omega^j\mu}\omega^{-(k-1)j}, \quad \gamma_k=\sum_{j=1}^{2n+1}\frac{\overline{h}^*K\Omega^jh}{\mu+\omega^j\mu}\omega^{-(k-1)j}. \tag{4.12}$$

Using (2.9), we have $\overline{h^*K\Omega^jh}=(h^*K\Omega^jh)^*=\omega^{-2j}h^*K\Omega^jh$, $\overline{h}^*K\Omega^jh=(\overline{h}^*K\Omega^jh)^T=\omega^{2j}\overline{h}^*K\Omega^{-j}h$. Hence

$$\bar{\alpha}_k = \alpha_{3-k}, \quad \gamma_k = \gamma_{3-k}. \tag{4.13}$$

From (4.2) and (4.11), we obtain

$$\hat{A}\xi_k = \hat{\alpha}_k \xi_{3-k}, \quad \hat{B}\xi_k = \bar{\hat{\gamma}}_{2-k}\xi_{k+1}, \quad \hat{C}\xi_k = \hat{\gamma}_k \xi_{k-1}, \quad \hat{D}\xi_k = \bar{\hat{\alpha}}_{2-k}\xi_{1-k}, \tag{4.14}$$

where

$$\hat{\alpha}_k = \bar{\alpha}_{3-k}(|\alpha_{3-k}|^2 - |\gamma_{3-k}|^2)^{-1}, \quad \hat{\gamma}_k = -\gamma_{3-k}(|\alpha_{3-k}|^2 - |\gamma_{3-k}|^2)^{-1}. \tag{4.15}$$

Let $h=(h_1,\ldots,h_{2n+1})^T$, then the entries of $\bar{H}_i$, $\bar{H}_{2n+1+i}$ $(1\leqslant i\leqslant 2n+1)$ are $(\bar{H}_i)_j=\omega^{(i-1)(j-1)}\bar{h}_j$, $(\bar{H}_{2n+1+i})_j=\omega^{-(i-1)(j-1)}h_j$. Hence, from (3.10), for $1\leqslant i,j\leqslant 2n+1$,

$$R_{ij} = (2n+1)^{-1/2}\omega^{(i-1)(j-1)}\bar{h}_j, \quad R_{2n+1+i,j} = (2n+1)^{-1/2}\omega^{-(i-1)(j-1)}h_j,$$

$$S_{ij} = (2n+1)^{-1/2}\bar{\mu}^{-1}\omega^{-(i-1)j}\bar{h}_{-j}, \quad S_{2n+1+i,j} = (2n+1)^{-1/2}\mu^{-1}\omega^{(i-1)j}h_{-j}. \tag{4.16}$$

Written in matrices, the ith column of R is $(\bar{h}_i\xi_{2-i},h_i\xi_i)^T$ and the jth column of S is $(\bar{\mu}^{-1}\bar{h}_{-j}\xi_{j+1},\mu^{-1}h_{-j}\xi_{1-j})^T$.

Let

$$\mathcal{A}_l = \bar{h}_l h_{-l}, \quad \mathcal{C}_l = h_l h_{-l}. \tag{4.17}$$

Lemma 7: *$|\mu|^{-2(2n+1)}G_{2(2n+1)}=1-(2n+1)R^*\Gamma^{-1}S=\mathrm{diag}(g_1,\ldots,g_{2n+1})$ where*

$$g_j = 1 - 2(2n+1)\mathrm{Re}\left(\frac{1}{\mu}\frac{\mathcal{A}_j\alpha_{2-j} - \mathcal{C}_j\bar{\gamma}_{2-j}}{|\alpha_{2-j}|^2 - |\gamma_{2-j}|^2}\right). \tag{4.18}$$

Proof: Using the expressions (3.9) and (4.14), we have $|\mu|^{-2(2n+1)}G_{2(2n+1)}=1-(2n+1)R^*\Gamma^{-1}S$ and

$$(R^*\Gamma^{-1}S)_{ij} = \left(h_i\xi_{2-i}^*,\bar{h}_i\xi_i^*\right)\begin{pmatrix}\hat{A} & \hat{B}\\ \hat{C} & \hat{D}\end{pmatrix}\begin{pmatrix}\bar{\mu}^{-1}\bar{h}_{-j}\xi_{j+1}\\ \mu^{-1}h_{-j}\xi_{1-j}\end{pmatrix} = f_i\delta_{ij}, \tag{4.19}$$

where

$$f_j = \bar{\mu}^{-1}\bar{h}_{-j}(h_j\hat{\alpha}_{j+1} + \bar{h}_j\hat{\gamma}_{j+1}) + \mu^{-1}h_{-j}(\bar{h}_j\bar{\hat{\alpha}}_{j+1} + h_j\bar{\hat{\gamma}}_{j+1}) = \frac{1}{|\alpha_{2-j}|^2 - |\gamma_{2-j}|^2}\Big(\bar{\mu}^{-1}\bar{\mathcal{A}}_j\bar{\alpha}_{2-j} - \bar{\mu}^{-1}\bar{\mathcal{C}}_j\gamma_{2-j} - \mu^{-1}\mathcal{C}_j\bar{\gamma}_{2-j} + \mu^{-1}\mathcal{A}_j\alpha_{2-j}\Big). \tag{4.20}$$

The lemma is proved.

According to Lemma 3, we get the entries $(\tilde{q}_1,\ldots,\tilde{q}_{2n+1})$ of $\tilde{Q}$ in (2.7) as

$$\tilde{q}_j = \frac{g_{j+1}}{g_j}q_j. \tag{4.21}$$

However, the expression of g_j in (4.18) is still not simple enough. To simplify the expressions of the solutions, we need the following two lemmas.

Lemma 8: *Suppose $k-2=(2n+1)s+r$ where s, r are integers, $1\leqslant r\leqslant 2n+1$, then*

$$\alpha_k = \frac{2n+1}{\bar{\mu}^{2n+1}+\mu^{2n+1}}\left(\sum_{l=1}^{r}\mathcal{A}_l(-\mu)^{r-l}\bar{\mu}^{2n+l-r} + \sum_{l=r+1}^{2n+1}\mathcal{A}_l(-\mu)^{2n+1+r-l}\bar{\mu}^{l-r-1}\right),$$

$$\gamma_k = \frac{2n+1}{2\mu^{2n+1}}\left(\sum_{l=1}^{r}\mathcal{C}_l(-\mu)^{r-l}\mu^{2n+l-r}+\sum_{l=r+1}^{2n+1}\mathcal{C}_l(-\mu)^{2n+1+r-l}\mu^{l-r-1}\right). \tag{4.22}$$

Proof: According to (4.12),

$$\alpha_k = \sum_{j=1}^{2n+1}\sum_{l=1}^{2n+1}\frac{\mathcal{A}_l\omega^{-j(k-l-2)}}{\overline{\mu}+\omega^j\mu}. \tag{4.23}$$

Let θ be a constant with $|\theta|<1$. Let

$$\alpha_k^{(\theta)} = \sum_{j=1}^{2n+1}\sum_{l=1}^{2n+1}\frac{\mathcal{A}_l\omega^{-j(k-l-2)}}{\overline{\mu}+\theta\omega^j\mu} = \sum_{j=1}^{2n+1}\sum_{l=1}^{2n+1}\sum_{\rho=0}^{+\infty}\overline{\mu}^{-1}\mathcal{A}_l\omega^{j(\rho+2+l-k)}(-\theta\mu\overline{\mu}^{-1})^{\rho}. \tag{4.24}$$

Using (4.10) we have $\sum_{j=1}^{2n+1}\omega^{j(\rho+2+l-k)}=0$ unless $\rho+2+l-k\equiv 0 \bmod 2n+1$. Let $\rho=k-l-2+(2n+1)\sigma$, then, using $k-2=(2n+1)s+r$,

$$\begin{aligned}(2n+1)^{-1}\alpha_k^{(\theta)} &= \sum_{l=1}^{2n+1}\sum_{\sigma\geq\frac{1}{2n+1}(l-k+2)}\overline{\mu}^{-1}\mathcal{A}_l(-\theta\mu\overline{\mu}^{-1})^{k-l-2+(2n+1)\sigma}\\ &= \sum_{l=1}^{r}\sum_{\sigma=-s}^{+\infty}\overline{\mu}^{-1}\mathcal{A}_l(-\theta\mu\overline{\mu}^{-1})^{k-l-2+(2n+1)\sigma}+\sum_{l=r+1}^{2n+1}\sum_{\sigma=-s+1}^{+\infty}\overline{\mu}^{-1}\\ &\quad\times\mathcal{A}_l(-\theta\mu\overline{\mu}^{-1})^{k-l-2+(2n+1)\sigma} = \sum_{l=1}^{r}\frac{\mathcal{A}_l(-\theta\mu)^{r-l}\overline{\mu}^{2n+l-r}}{\overline{\mu}^{2n+1}+(\theta\mu)^{2n+1}}+\sum_{l=r+1}^{2n+1}\frac{\mathcal{A}_l(-\theta\mu)^{2n+1+r-l}\overline{\mu}^{l-r-1}}{\overline{\mu}^{2n+1}+(\theta\mu)^{2n+1}}.\end{aligned} \tag{4.25}$$

Here a sum is zero if the lower bound is greater than the upper bound. Since $\arg(\mu)\notin\{k\pi/(4n+2)|k\in\mathbf{Z}\}$, by taking $\theta\to 1$, we get the expression of α_k in (4.22). The expression of γ_k in (4.22) is obtained similarly. The lemma is proved.

Lemma 9: *For any integer k,*

$$|\alpha_{2-k}|^2-|\alpha_{1-k}|^2-|\gamma_{2-k}|^2+|\gamma_{1-k}|^2 = 2(2n+1)\mathrm{Re}\left(\mu^{-1}(\mathcal{A}_k\alpha_{2-k}-\mathcal{C}_k\overline{\gamma}_{2-k})\right). \tag{4.26}$$

Proof: First suppose $-k-1=(2n+1)s+r$, $1\leq r\leq 2n$, then $(2-k)-2=(2n+1)s+r+1$, $(1-k)-2=(2n+1)s+r$. According to Lemma 8,

$$\alpha_{2-k} = \frac{2n+1}{\overline{\mu}^{2n+1}+\mu^{2n+1}}\left(\sum_{l=1}^{r+1}\mathcal{A}_l(-\mu)^{r+1-l}\overline{\mu}^{2n-1+l-r}+\sum_{l=r+2}^{2n+1}\mathcal{A}_l(-\mu)^{2n+2+r-l}\overline{\mu}^{l-r-2}\right),$$

$$\alpha_{1-k} = \frac{2n+1}{\overline{\mu}^{2n+1}+\mu^{2n+1}}\left(\sum_{l=1}^{r}\mathcal{A}_l(-\mu)^{r-l}\overline{\mu}^{2n+l-r}+\sum_{l=r+1}^{2n+1}\mathcal{A}_l(-\mu)^{2n+1+r-l}\overline{\mu}^{l-r-1}\right). \tag{4.27}$$

Hence

$$\alpha_{1-k} = -\mu^{-1}\overline{\mu}(\alpha_{2-k}-(2n+1)\overline{\mu}^{-1}\mathcal{A}_{r+1}). \tag{4.28}$$

This implies

$$|\alpha_{2-k}|^2-|\alpha_{1-k}|^2 = 2(2n+1)\mathrm{Re}\left(\mu^{-1}\overline{\mathcal{A}}_{r+1}\alpha_{2-k}\right)-(2n+1)^2|\mu|^{-2}|\mathcal{A}_{r+1}|^2. \tag{4.29}$$

Similarly, we have

$$-|\gamma_{2-k}|^2+|\gamma_{1-k}|^2=-2(2n+1)\mathrm{Re}(\bar{\mu}^{-1}\bar{\mathcal{C}}_{r+1}\gamma_{2-k})+(2n+1)^2|\mu|^{-2}|\mathcal{C}_{r+1}|^2. \quad (4.30)$$

Since $|\mathcal{C}_{r+1}|^2=|\mathcal{A}_{r+1}|^2$,

$$|\alpha_{2-k}|^2-|\alpha_{1-k}|^2-|\gamma_{2-k}|^2+|\gamma_{1-k}|^2=2(2n+1)\mathrm{Re}(\mu^{-1}\bar{\mathcal{A}}_{r+1}\alpha_{2-k})-2(2n+1)\mathrm{Re}(\mu^{-1}\mathcal{C}_{r+1}\bar{\gamma}_{2-k}). \quad (4.31)$$

According to (4.17), $\mathcal{A}_{r+1}=\bar{\mathcal{A}}_k$, $\mathcal{C}_{r+1}=\mathcal{C}_k$. Hence (4.26) holds.

If $-k-1\equiv 0 \mod 2n+1$, (4.28) still holds. The same result is obtained. This proves the lemma.

From (4.18) and (4.26),

$$g_k=1-\frac{2(2n+1)}{|\alpha_{2-k}|^2-|\gamma_{2-k}|^2}\mathrm{Re}\left(\frac{\mathcal{A}_k\alpha_{2-k}-\mathcal{C}_k\bar{\gamma}_{2-k}}{\mu}\right)=\frac{|\alpha_{1-k}|^2-|\gamma_{1-k}|^2}{|\alpha_{2-k}|^2-|\gamma_{2-k}|^2}. \quad (4.32)$$

Using (4.21), we get the transformation of the solution of (2.12),

$$\tilde{q}_k=\frac{\eta_k\eta_{k+2}}{\eta_{k+1}^2}q_k, \quad (4.33)$$

Where $\eta_k=|\alpha_{2-k}|^2-|\gamma_{2-k}|^2$.

Remark 1: *From (4.13), the equality $\eta_{1-k}=\eta_k$ always holds. Hence $\tilde{q}_{-k-1}=\tilde{q}_k$ holds provided that $q_{-k-1}=q_k$ holds. This means that the Darboux transformation does not change the natural relation (2.17).*

η_k's can be written down explicitly. For $k=1,2,\ldots,n+1$, we have $(2-k)-2=-(2n+1)+2n+1-k$. The first equation of (4.27) can be written as

$$\alpha_{2-k}=\frac{2n+1}{\bar{\mu}^{2n+1}+\mu^{2n+1}}\left(\sum_{l=1}^{2n+1-k}\mathcal{A}_l(-\mu)^{2n+1-k-l}\bar{\mu}^{k+l-1}+\sum_{l=2n+2-k}^{2n+1}\mathcal{A}_l(-\mu)^{4n+2-k-l}\bar{\mu}^{k+l-2n-2}\right). \quad (4.34)$$

Similarly,

$$\gamma_{2-k}=\frac{2n+1}{2\mu^{2n+1}}\left(\sum_{l=1}^{2n+1-k}\mathcal{C}_l(-\mu)^{2n+1-k-l}\mu^{k+l-1}+\sum_{l=2n+2-k}^{2n+1}\mathcal{C}_l(-\mu)^{4n+2-k-l}\mu^{k+l-2n-2}\right)$$
$$=-\frac{2n+1}{2\mu}\left(\sum_{l=1}^{2n+1-k}\mathcal{C}_l(-1)^{k+l}-\sum_{l=2n+2-k}^{2n+1}\mathcal{C}_l(-1)^{k+l}\right). \quad (4.35)$$

Equation (4.33) leads to the transformation of the solution $(\tilde{u}_1,\ldots,\tilde{u}_n)$ of (2.4) or (2.5) as

$$\tilde{u}_k=u_k+\ln\frac{\eta_{k+1}}{\eta_k} \quad (1\le k\le n) \quad (4.36)$$

If $\eta_1,\ldots,\eta_n$ have the same sign.

In summary, we have the following theorem.

Theorem 1: *Suppose $(u_1,\ldots,u_n)$ is a solution of (2.4) or (2.5). Let $\mu\in\mathbf{C}\setminus\{0\}$ such that $\arg(\mu)\neq k\pi/(4n+2)$ for any inter k. Let $h=(h_1,\ldots,h_{2n+1})^T$ be a column solution of (2.10) for $\lambda=\mu$. Let $\lambda_j=\omega^{j-1}\mu$, $\lambda_{2n+1+j}=\omega^{-j+1}\bar{\mu}$, $H_j=\Omega^{j-1}h$, $H_{2n+1+j}=\Omega^{-j+1}\bar{h}(j=1,2,\ldots,2n+1)$. Define $\Gamma_{ij}=H_i^*KH_j/(\bar{\lambda}_i+\lambda_j)$ $(i,j=1,\ldots,2(2n+1))$. Let $G(x,t,\lambda)$ be defined by (3.2) with $M=2(2n+1)$. Then G is a Darboux matrix for (2.10) in the sense that for any solution Φ of (2.10), $\tilde{\Phi}=G\Phi$ satisfies*

$$\tilde{\Phi}_x=(\lambda J+\tilde{P})\tilde{\Phi}, \quad \tilde{\Phi}_t=\lambda^{-1}\tilde{Q}\tilde{\Phi}, \quad (4.37)$$

where $\tilde{P}=P+[J,G_1]$, $\tilde{Q}=G_{2(2n+1)}QG_{2(2n+1)}^{-1}$.

Let

$$\zeta_k = \frac{1}{|\overline{\mu}^{2n+1}+\mu^{2n+1}|^2}\left|\sum_{l=1}^{2n+1-k}\overline{h}_l h_{-l}(-\mu)^{2n+1-k-l}\overline{\mu}^{k+l-1}+\sum_{l=2n+2-k}^{2n+1}\overline{h}_l h_{-l}(-\mu)^{4n+2-k-l}\overline{\mu}^{k+l-2n-2}\right|^2 - \frac{1}{4|\mu|^2}\left|\sum_{l=1}^{2n+1-k}h_l h_{-l}(-1)^{k+l}-\sum_{l=2n+2-k}^{2n+1}h_l h_{-l}(-1)^{k+l}\right|^2, \tag{4.38}$$

then in the region where $\zeta_1,\ldots,\zeta_n$ *have the same sign, the new solution* $(\tilde{u}_1,\ldots,\tilde{u}_n)$ *of (2.4) or (2.5) is given by*

$$\tilde{u}_k = u_k + \ln\frac{\zeta_{k+1}}{\zeta_k} \quad (k=1,2,\ldots,n). \tag{4.39}$$

V. DARBOUX TRANSFORMATION WITH A REAL SPECTRAL PARAMETER

Now suppose a spectral parameter in constructing Darboux matrix is real. Considering the symmetries in Lemma 2, the Darboux matrix can be derived as follows.

Let μ be a nonzero real number, $\lambda_j=\omega^{j-1}\mu(j=1,2,\ldots,2n+1)$, then all λ_j, $-\overline{\lambda}_j$ $(j=1,\ldots,2n+1)$ are distinct. Let h be a column solution of (2.10) for $\lambda=\mu$, $H_j=\Omega^{j-1}h$, $(j=1,2,\ldots,2n+1)$. Then H_j is a solution of (2.10) for $\lambda=\lambda_j$ $(j=1,2,\ldots,2n+1)$.

According to Sec. III for $M=2n+1$, we can also construct Γ and $G(x,t,\lambda)$. From (3.1),

$$\Gamma_{ij} = \frac{h^*(\Omega^*)^{i-1}K\Omega^{j-1}h}{\omega^{-i+1}\mu+\omega^{j-1}\mu} = \omega^{-i+1}\frac{h^*K\Omega^{i+j-2}h}{\mu+\omega^{i+j-2}\mu}. \tag{5.1}$$

Now

$$\Gamma\xi_k = \alpha_k\xi_{3-k}, \quad \alpha_k = \sum_{j=1}^{2n+1}\frac{h^*K\Omega^j h}{\mu+\omega^j\mu}\omega^{-(k-1)j}, \quad \Gamma^{-1}\xi_k = \hat{\alpha}_k\xi_{3-k}, \quad \hat{\alpha}_k = \alpha_{3-k}^{-1}. \tag{5.2}$$

Lemma 10: $\overline{G(x,t,\overline{\lambda})}=G(x,t,\lambda)$.

Proof: Since $\lambda_{2-i}=\overline{\lambda}_i$, $H_{2-i}=\overline{H}_i$, we have

$$\overline{\Gamma}_{2-i,2-j} = \frac{\overline{H}^*_{2-i}K\overline{H}_{2-j}}{\overline{\lambda}_{2-j}+\lambda_{2-i}} = \frac{H_i^*KH_j}{\overline{\lambda}_i+\lambda_j} = \Gamma_{ij}. \tag{5.3}$$

Let L be a constant matrix such that $L_{11}=1$, $L_{i+1,1-i}=1$ $(i=1,2,\ldots,2n)$ and $L_{ij}=0$ for other (i,j). Then Γ satisfies $L\Gamma L=\overline{\Gamma}$. This leads to $L\Gamma^{-1}L=\overline{\Gamma}^{-1}$. Therefore,

$$\sum_{i,j=1}^{2n+1}\overline{\left(\frac{(\Gamma^{-1})_{ij}H_iH_j^*K}{\overline{\lambda}+\overline{\lambda}_j}\right)} = \sum_{i,j=1}^{2n+1}\frac{(\Gamma^{-1})_{2-i,2-j}H_{2-i}H_{2-j}^*K}{\lambda+\overline{\lambda}_{2-j}} = \sum_{i,j=1}^{2n+1}\frac{(\Gamma^{-1})_{ij}H_iH_j^*K}{\lambda+\overline{\lambda}_j}. \tag{5.4}$$

The lemma is proved.

Hence, the coefficients of each power of λ in $G(x,t,\lambda)$ are all real matrices.

Similar to Lemma 6, we have

$$\Omega G(x,t,\lambda)\Omega^{-1} = G(x,t,\omega\lambda). \tag{5.5}$$

From Lemma 4, Lemma 10, and (5.5), $\tilde{U}$ and $\tilde{V}$ defined by (3.5) satisfy the relations (2.15).

Let $h=(h_1,\ldots,h_{2n+1})^T$, then

$$R_{ij} = (2n+1)^{-1/2}\omega^{(i-1)(j-1)}h_j, \quad S_{ij} = (2n+1)^{-1/2}\mu^{-1}\omega^{-(i-1)j}h_{-j}. \tag{5.6}$$

Written in matrices, the ith column of R is $h_i\xi_{2-i}$, and the jth column of S is $\mu^{-1}h_{-j}\xi_{j+1}$.

Therefore, $-\mu^{-2n-1}G_{2n+1}=1-(2n+1)R^{*}\Gamma^{-1}S=\mathrm{diag}(g_1,\ldots,g_{2n+1})$ where

$$g_j = 1-(2n+1)\mu^{-1}h_jh_{-j}/\alpha_{2-j}. \tag{5.7}$$

Similar to Lemma 8, α_k can be expressed as

$$\begin{aligned}\alpha_k &= \frac{2n+1}{2\mu^{2n+1}}\left(\sum_{l=1}^{r}\mathcal{A}_l(-\mu)^{r-l}\mu^{2n+l-r}+\sum_{l=r+1}^{2n+1}\mathcal{A}_l(-\mu)^{2n+1+r-l}\mu^{l-r-1}\right)\\ &= \frac{2n+1}{2\mu}\left(\sum_{l=1}^{r}(-1)^{r-l}\mathcal{A}_l-\sum_{l=r+1}^{2n+1}(-1)^{r-l}\mathcal{A}_l\right),\end{aligned} \tag{5.8}$$

where $\mathcal{A}_l=h_lh_{-l}$ and $k-2=(2n+1)s+r$ with s, $r\in\mathbf{Z}$ and $1\leqslant r\leqslant 2n+1$.

From (5.8), we get $\alpha_{2-k}+\alpha_{1-k}=(2n+1)\mu^{-1}\mathcal{A}_k$ corresponding to Lemma 9. From (5.7), $g_k=1-(2n+1)\mu^{-1}\mathcal{A}_k\alpha_{2-k}^{-1}=-\alpha_{1-k}\alpha_{2-k}^{-1}$. Hence, $\tilde{q}_k=(\eta_k\eta_{k+2}/\eta_{k+1}^2)q_k$ where $\eta_k=\alpha_{2-k}$.

Since $(2-k)-2=-(2n+1)s+2n+1-k$, (5.8) leads to

$$\eta_k=\alpha_{2-k}=-\frac{2n+1}{2\mu}\left(\sum_{l=1}^{2n+1-k}(-1)^{k+l}\mathcal{A}_l-\sum_{l=2n+2-k}^{2n+1}(-1)^{k+l}\mathcal{A}_l\right) \tag{5.9}$$

$(k=1,2,\ldots,n+1)$. Finally,

$$\tilde{u}_k=u_k+\ln\frac{\eta_{k+1}}{\eta_k}. \tag{5.10}$$

If $\eta_1,\ldots,\eta_n$ have the same sign.

The above results are summarized in the following theorem.

Theorem 2: *Suppose $(u_1,\ldots,u_n)$ is a solution of (2.4) or (2.5). Let $\mu\in\mathbf{R}\backslash\{0\}$. Let $h=(h_1,\ldots,h_{2n+1})^T$ be a real column solution of (2.10) for $\lambda=\mu$. Let $\lambda_j=\omega^{j-1}\mu$, $H_j=\Omega^{j-1}h$ $(j=1,2,\ldots,2n+1)$. Define $\Gamma_{ij}=H_i^{*}KH_j/(\bar{\lambda}_i+\lambda_j)$ $(i,j=1,\ldots,2n+1)$. Let $G(x,t,\lambda)$ be defined by (3.2) with $M=2n+1$. Then G is a Darboux matrix for (2.10) in the sense that for any solution Φ of (2.10), $\tilde{\Phi}=G\Phi$ satisfies*

$$\tilde{\Phi}_x=(\lambda J+\tilde{P})\tilde{\Phi},\quad \tilde{\Phi}_t=\lambda^{-1}\tilde{Q}\tilde{\Phi}, \tag{5.11}$$

Where $\tilde{P}=P+[J,G_1]$, $\tilde{Q}=G_{2n+1}QG_{2n+1}^{-1}$.

Let

$$\zeta_k=\sum_{l=1}^{2n+1-k}(-1)^{k+l}h_lh_{-l}-\sum_{l=2n+2-k}^{2n+1}(-1)^{k+l}h_lh_{-l} \tag{5.12}$$

then in the region where $\zeta_1,\ldots,\zeta_n$ have the same sign, the new solution $(\tilde{u}_1,\ldots,\tilde{u}_n)$ of (2.4) or (2.5) is given by

$$\tilde{u}_k=u_k+\ln\frac{\zeta_{k+1}}{\zeta_k}\quad(k=1,2,\ldots,n). \tag{5.13}$$

VI. APPLICATION TO THE TZITZEICA EQUATION

The Tzitzeica equation

$$u_{xt}=e^u-e^{-2u} \tag{6.1}$$

is a special two dimensional Toda equation with $n=1$.

Suppose u is a solution of (6.1) and $h=(h_1,h_2,h_3)^T$ is a column solution of its Lax pair for $\lambda=\mu$. Using Theorem 1 and Theorem 2, we get the new solution of (6.1). When μ is taken as a complex number,

$$\tilde{u} = u + \ln\frac{4|\mu|^2|\bar{\mu}^2\bar{h}_1h_2 + \mu^2h_1\bar{h}_2 - |\mu|^2|h_3|^3|^2 - (\mu^3+\bar{\mu}^3)^2|2h_1h_2 - h_3^2|^2}{\left(2|\mu|^2\bar{\mu}h_1\bar{h}_2 - 2|\mu|^2\mu\bar{h}_1h_2 + (\mu^3-\bar{\mu}^3)|h_3|^2\right)^2}. \quad (6.2)$$

When μ is a real number,

$$\tilde{u} = u + \ln\frac{2h_1h_2 - h_3^2}{h_3^2}. \quad (6.3)$$

Equation (6.3) is similar to that given by Ref. 17.

ACKNOWLEDGMENTS

The author is grateful to Professor C. H. Gu and Professor H. S. Hu for their helpful suggestions and discussions on constructing the Darboux transformations. He is also grateful to Dr. J. Inoguchi for his suggestion on studying the Tzitzeica equation together with helpful discussions. This work was supported by the special funds for Chinese major state basic research projects and the trans-century training program foundation for the talents by the Ministry of Education of China.

[1] A. V. Mikhailov, JETP Lett. **30**, 414 (1979).
[2] A. P. Fordy and J. Gibbons, Commun. Math. Phys. **77**, 21 (1980).
[3] A. P. Fordy and J. C. Wood, *Harmonic Maps and Integrable System* (Vieweg, Braunschweig, 1994).
[4] C. Rogers and W. K. Schief, *Bäcklund and Darboux Transformations, Geometry and Modern Applications in Soliton Theory* (Cambridge University Press, Cambridge, 2002).
[5] H. S. Hu, Contemp. Math. **308**, 179 (2002).
[6] H. S. Hu, Lett. Math. Phys. **57**, 19 (2001).
[7] A. V. Razumov and M. V. Saveliev, Commun. Anal. Geom. **2**, 461 (1994).
[8] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).
[9] I. McIntosh, Nonlinearity **7**, 85 (1994).
[10] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons* (Springer, Heidelberg, 1991).
[11] A. N. Leznov and E. A. Yuzbashjan, Lett. Math. Phys. **35**, 345 (1995).
[12] A. N. Leznov and A. Sorin, Phys. Lett. B **402**, 87 (1997).
[13] C. L. Terng and K. Uhlenbeck, Commun. Pure Appl. Math. **53**, 1 (2000).
[14] M. Adler and P. Van Moerbeke, Duke Math. J. **112**, 1 (2002).
[15] M. V. Savaliev, Phys. Lett. A **122**, 312 (1987).
[16] C. L. Terng, *Integrable Systems in Differential Geometry*, Vol. 1, Advanced Studies in Pure Mathematics (Mathematical Society of Japan, 2005).
[17] A. R. Its, J. Sov. Math. **31**, 3330 (1985).
[18] J. Dorfmeister and U. Eitner, Abh. Math. Sem. Univ. Hamburg **71**, 225 (2001).
[19] V. B. Zakharov and A. V. Mikhailov, Commun. Math. Phys. **74**, 21 (1980).
[20] J. Cieśliński, J. Math. Phys. **36**, 5670 (1995).