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Liouville integrability of the finite dimensional Hamiltonian systems derived from principal chiral field

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For finite dimensional Hamiltonian systems derived from $1+1$ dimensional integrable systems, if they have Lax representations, then the Lax operator creates a set of conserved integrals. When these conserved integrals are in involution, it is believed quite popularly that there will be enough functionally independent ones among them to guarantee the Liouville integrability of the Hamiltonian systems, at least for those derived from physical problems. In this article, we give a counterexample based on the $U(2)$ principal chiral field. It is proved that the finite dimensional Hamiltonian systems derived from the $U(2)$ principal chiral field are Liouville integrable. Moreover, their Lax operator gives a set of involutive conserved integrals, but they are not enough to guarantee the integrability of the Hamiltonian systems. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1501446]

I. INTRODUCTION

For many $1+1$ dimensional integrable systems, the nonlinearization method can be applied to get finite dimensional $(1+0$ dimensional) Hamiltonian systems.[1] Usually these Hamiltonian systems have Lax representations so that the involutive conserved integrals can be obtained. In this way the original nonlinear partial differential equations are changed to systems of nonlinear ordinary differential equations.[2–8] Many interesting exact solutions, especially quasi-periodic solutions, were obtained in this way.

For a finite dimensional Hamiltonian system, if it can be written in the Lax form as

$$\frac{\mathrm{d}}{\mathrm{d}t}L(\lambda)=[M(\lambda),L(\lambda)], \tag{1.1}$$

then the conserved integrals are easily derived from the coefficients of $\mathrm{tr}(L^k(\lambda))$'s $(k\geqslant 1)$ when they are expanded as Laurent series of λ. Usually the number of these coefficients is infinite. It is believed quite popularly that when these conserved integrals are in involution, there will be enough functionally independent ones among them to guarantee the Liouville integrability of the Hamiltonian systems. Indeed, this is the case for most known physically interesting systems, such as the equations in the AKNS system, Kaup–Newell system and many other examples including those derived from $2+1$ dimensional integrable systems.[3,4,7,9–11]

However, we will give a counterexample in this article to show that this is not always true.

This counterexample is based on a well-known physical model–the $U(n)$ principal chiral field [or, mathematically, the harmonic map from $\mathbf{R}^{1,1}$ to $U(n)$].[12–16] In this article, the equation of $U(n)$ principal chiral field can be first reduced to a set of Hamiltonian systems by the standard procedure of the nonlinearization method. This will be done in Secs. II and III. Then, in Sec. IV, we show that there are not enough conserved integrals in those given by $\mathrm{tr}(L^k(\lambda))$'s to guarantee the Liouville integrability of the systems. In Sec. V, it is proved that these Hamiltonian systems are actually Liouville integrable for $n=2$. That is, they still have a full set of involutive and independent conserved integrals. These conserved integrals are obtained from $\mathrm{tr}L^k(\lambda)$ and other obvious

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conserved integrals. When $n>2$, it is still open whether one can find enough involutive and independent conserved integrals by adding some obvious ones to $\mathrm{tr}L^k(\lambda)$'s. Therefore, at least for $n=2$, the Hamiltonian systems derived from the $U(2)$ principal chiral field are Liouville integrable, but their conserved integrals for Liouville integrability can not be fully obtained from $\mathrm{tr}L^k(\lambda)$.

II. HAMILTONIAN SYSTEMS DERIVED FROM $U(n)$ PRINCIPAL CHIRAL FIELD

The equation for the $U(n)$ principal chiral field is

$$(g_x g^{-1})_t + (g_t g^{-1})_x = 0, \tag{2.1}$$

where the field $g(x,t) \in U(n)$. Write

$$P = g_x g^{-1}, \quad Q = g_t g^{-1}. \tag{2.2}$$

Then $P,Q \in u(n)$ (i.e., $P^*+P=0$, $Q^*+Q=0$) and (2.1) becomes

$$P_t + Q_x = 0, \quad P_t - Q_x + [P,Q] = 0. \tag{2.3}$$

Here the second equation is the integrability condition of (2.2).

It is known that (2.3) has a Lax pair

$$\Phi_x = \frac{1}{1-\lambda} P\Phi, \quad \Phi_t = \frac{1}{1+\lambda} Q\Phi, \tag{2.4}$$

where λ is a complex spectral parameter.

Now we write down the corresponding finite dimensional Hamiltonian systems and their Lax operators.

Let $\lambda_1,\ldots,\lambda_N$ be N distinct real constants with $\lambda_j \neq \pm 1$ $(j=1,\ldots,N)$, and let $(\phi_{1\alpha},\ldots,\phi_{n\alpha})^T$ be an arbitrary solution of the Lax pair (2.4) with $\lambda=\lambda_\alpha$, $\Lambda=\mathrm{diag}(\lambda_1,...,\lambda_N)$, $\Phi_j=(\phi_{j1},\ldots,\phi_{jN})^T$. Let

$$L = \sum_{\alpha=1}^{N} \frac{1}{\lambda-\lambda_\alpha} \begin{pmatrix} \bar{\phi}_{1\alpha}\phi_{1\alpha} & \bar{\phi}_{2\alpha}\phi_{1\alpha} & \cdots & \bar{\phi}_{n\alpha}\phi_{1\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\phi}_{1\alpha}\phi_{n\alpha} & \bar{\phi}_{2\alpha}\phi_{n\alpha} & \cdots & \bar{\phi}_{n\alpha}\phi_{n\alpha} \end{pmatrix}. \tag{2.5}$$

Expand L to power series of $1-\lambda$ and $1+\lambda$, respectively:

$$\begin{aligned} L = L^{(1)} &= \sum_{k=1}^{\infty} (1-\lambda)^{k-1} \begin{pmatrix} \langle \Phi_1,(1-\Lambda)^{-k}\Phi_1\rangle & \cdots & \langle \Phi_n,(1-\Lambda)^{-k}\Phi_1\rangle \\ \vdots & \ddots & \vdots \\ \langle \Phi_1,(1-\Lambda)^{-k}\Phi_n\rangle & \cdots & \langle \Phi_n,(1-\Lambda)^{-k}\Phi_n\rangle \end{pmatrix}, \\ L = L^{(2)} &= -\sum_{k=1}^{\infty} (1+\lambda)^{k-1} \begin{pmatrix} \langle \Phi_1,(1+\Lambda)^{-k}\Phi_1\rangle & \cdots & \langle \Phi_n,(1+\Lambda)^{-k}\Phi_1\rangle \\ \vdots & \ddots & \vdots \\ \langle \Phi_1,(1+\Lambda)^{-k}\Phi_n\rangle & \cdots & \langle \Phi_n,(1+\Lambda)^{-k}\Phi_n\rangle \end{pmatrix}, \end{aligned} \tag{2.6}$$

where the inner product $\langle V_1,V_2\rangle$ of two vectors is defined as $V_1^* V_2$. The first series converges when $|\lambda-1| < \min\limits_{1\leqslant\alpha\leqslant N} |\lambda_\alpha-1|$ and the second one converges when $|\lambda+1| < \min\limits_{1\leqslant\alpha\leqslant N} |\lambda_\alpha+1|$.

Lemma 1: If

$$P_{jk} = \mathrm{i}\langle \Phi_k,(1-\Lambda)^{-1}\Phi_j\rangle, \quad Q_{jk} = \mathrm{i}\langle \Phi_k,(1+\Lambda)^{-1}\Phi_j\rangle, \tag{2.7}$$

then

$$L_x=\frac{1}{1-\lambda}[P,L], \quad L_t=\frac{1}{1+\lambda}[Q,L], \tag{2.8}$$

and (P,Q) *gives a solution of (2.3).*

Proof: Let $\phi_\alpha=(\phi_{1\alpha},\ldots,\phi_{n\alpha})^T$. Then

$$L=\sum_{\alpha=1}^{N}\frac{1}{\lambda-\lambda_\alpha}\phi_\alpha\phi_\alpha^*. \tag{2.9}$$

Since $P^*=-P$ and λ_α's are real,

$$\begin{aligned}L_x&=\sum_{\alpha=1}^{N}\frac{1}{\lambda-\lambda_\alpha}\left(\phi_\alpha\frac{1}{1-\lambda_\alpha}\phi_\alpha^*P^*+\frac{1}{1-\lambda_\alpha}P\phi_\alpha\phi_\alpha^*\right)\\&=\sum_{\alpha=1}^{N}\frac{1}{\lambda-\lambda_\alpha}\frac{1}{1-\lambda_\alpha}[P,\phi_\alpha\phi_\alpha^*]\\&=\sum_{\alpha=1}^{N}\left(\frac{1}{\lambda-\lambda_\alpha}\frac{1}{1-\lambda}-\frac{1}{1-\lambda_\alpha}\frac{1}{1-\lambda}\right)[P,\phi_\alpha\phi_\alpha^*]=\frac{1}{1-\lambda}[P,L].\end{aligned} \tag{2.10}$$

The last equality holds due to (2.7). The equation for L_t in (2.8) is derived similarly. Finally, by computing the integrability condition $L_{xt}=L_{tx}$ from (2.8) or substituting (2.7) into (2.3) directly, we know that (P,Q) satisfies (2.3). The lemma is proved.

Now we always suppose (2.7) holds for the $U(n)$ principal chiral field, which gives the nonlinear constraints. Substituting (2.7) into (2.4), we get a system of partial differential equations

$$\begin{aligned}\Phi_{j,x}&=\mathrm{i}(1-\Lambda)^{-1}\sum_{k=1}^{n}\langle\Phi_k,(1-\Lambda)^{-1}\Phi_j\rangle\Phi_k,\\\Phi_{j,t}&=\mathrm{i}(1+\Lambda)^{-1}\sum_{k=1}^{n}\langle\Phi_k,(1+\Lambda)^{-1}\Phi_j\rangle\Phi_k,\end{aligned} \tag{2.11}$$

which can be studied as two systems of ordinary differential equations when t and x are considered as constants, respectively.

Now $\phi_{11},\phi_{12},\ldots,\phi_{1N},\ldots,\phi_{n1},\phi_{n2},\ldots,\phi_{nN}$ and their complex conjugations form the complex coordinates of $\mathbf{R}^{2nN}$. In this $\mathbf{R}^{2nN}$, let ω be the standard symplectic form

$$\omega=2\sum_{j=1}^{n}\sum_{\alpha=1}^{N}\mathrm{d}\,\mathrm{Im}(\phi_{j\alpha})\wedge\mathrm{d}\,\mathrm{Re}(\phi_{j\alpha})=\mathrm{i}\sum_{j=1}^{n}\sum_{\alpha=1}^{N}\mathrm{d}\overline{\phi}_{j\alpha}\wedge\mathrm{d}\phi_{j\alpha}. \tag{2.12}$$

Then the corresponding Poisson bracket for two functions f and g is

$$\{f,g\}=\frac{1}{\mathrm{i}}\sum_{j=1}^{n}\sum_{\alpha=1}^{N}\left(\frac{\partial f}{\partial\phi_{j\alpha}}\frac{\partial g}{\partial\overline{\phi}_{j\alpha}}-\frac{\partial g}{\partial\phi_{j\alpha}}\frac{\partial f}{\partial\overline{\phi}_{j\alpha}}\right). \tag{2.13}$$

From (2.8), the coefficients of $(1-\lambda)^j$ $(j=0,1,2,\ldots)$ in $\mathrm{tr}(L^{(1)})^k$ $(k=1,2,\ldots)$ and the coefficients of $(1+\lambda)^j$ $(j=0,1,2,\ldots)$ in $\mathrm{tr}(L^{(2)})^k$ $(k=1,2,\ldots)$ are all conserved. Suppose

$$\mathrm{tr}(L^{(1)})^m=\sum_{k=1}^{\infty}(1-\lambda)^{k-1}\mathcal{E}_{mk}^{(1)}, \quad \mathrm{tr}(L^{(2)})^m=(-1)^m\sum_{k=1}^{\infty}(1+\lambda)^{k-1}\mathcal{E}_{mk}^{(2)}. \tag{2.14}$$

Since $\mathrm{tr}P=\mathrm{i}\mathcal{E}_{11}^{(1)}$ and $\mathrm{tr}Q=\mathrm{i}\mathcal{E}_{11}^{(2)}$, both $\mathrm{tr}P$ and $\mathrm{tr}Q$ are conserved. On the other hand, the Hamiltonians for Eqs. (2.11) are given by $\mathcal{E}_{21}^{(1)}$ and $\mathcal{E}_{21}^{(2)}$ according to the following lemma.

Moreover, direct computation shows that they commute with each other under the Poisson bracket (2.13) (this can also be derived directly from Lemma 3 in Sec. III).

Lemma 2: The Hamiltonians for the x-equation and the t-equation of (2.11) are given by

$$\begin{aligned} H^x &= -\tfrac{1}{2}\mathcal{E}_{21}^{(1)} = -\tfrac{1}{2}\sum_{j,k=1}^{n} \langle \Phi_k, (1-\Lambda)^{-1}\Phi_j\rangle\langle \Phi_j, (1-\Lambda)^{-1}\Phi_k\rangle, \\ H^t &= -\tfrac{1}{2}\mathcal{E}_{21}^{(2)} = -\tfrac{1}{2}\sum_{j,k=1}^{n} \langle \Phi_k, (1+\Lambda)^{-1}\Phi_j\rangle\langle \Phi_j, (1+\Lambda)^{-1}\Phi_k\rangle, \end{aligned} \tag{2.15}$$

respectively. That is, (2.11) is equivalent to the Hamiltonian equations

$$\mathrm{i}\phi_{j\alpha,x} = \frac{\partial H^x}{\partial \overline{\phi}_{j\alpha}}, \quad -\mathrm{i}\overline{\phi}_{j\alpha,x} = \frac{\partial H^x}{\partial \phi_{j\alpha}}, \quad \mathrm{i}\phi_{j\alpha,t} = \frac{\partial H^t}{\partial \overline{\phi}_{j\alpha}}, \quad -\mathrm{i}\overline{\phi}_{j\alpha,t} = \frac{\partial H^t}{\partial \phi_{j\alpha}}. \tag{2.16}$$

Moreover, $\{H^x, H^t\} = 0$.

Remark 1: The above procedure can also be used for the harmonic map from $\mathbf{R}^2$ *to* $U(n)$. *In this case, the equation is*

$$(g_z g^{-1})_{\overline{z}} + (g_{\overline{z}} g^{-1})_z = 0, \tag{2.17}$$

where z *is the complex coordinate of* $\mathbf{R}^2$, $g(z,\overline{z}) \in U(n)$. *The Lax pair is*

$$\Phi_z = \frac{1}{1-\mathrm{i}\lambda} g_z g^{-1}\Phi, \quad \Phi_{\overline{z}} = \frac{1}{1+\mathrm{i}\lambda} g_{\overline{z}} g^{-1}\Phi, \tag{2.18}$$

where λ *is a complex spectral parameter. Using the same method in this section, we can also get finite dimensional Hamiltonian systems whose Lax operator is completely the same as (2.5).*

III. CONSERVED INTEGRALS

Lemma 3: With the Poisson bracket (2.13), the following two conclusions hold.

(1) For any two complex numbers λ, μ *and two positive integers* k, l,

$$\{\mathrm{tr}L^k(\lambda), \mathrm{tr}L^l(\mu)\} = 0. \tag{3.1}$$

(2) For any complex number λ *and integers* j, k, l *with* $1 \leqslant j,k \leqslant n$,

$$\{\langle \Phi_j, \Phi_k\rangle, \mathrm{tr}L^l(\lambda)\} = 0. \tag{3.2}$$

This can be verified by direct computation of the Poisson brackets and was given in Ref. 11. Suppose the eigenvalues of $L(\lambda)$ are $\nu_1(\lambda)$, $\nu_2(\lambda), \ldots, \nu_n(\lambda)$. Then

$$\begin{aligned} &\mathrm{tr}L^k(\lambda) = \nu_1^k(\lambda) + \cdots + \nu_n^k(\lambda), \quad (k=1,2,\ldots), \\ &\det(\mu - L(\lambda)) = \mu^n - p_1(\lambda)\mu^{n-1} + \cdots + (-1)^n p_n(\lambda), \end{aligned} \tag{3.3}$$

for any complex number μ where

$$p_k(\lambda) = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \nu_{j_1}(\lambda) \cdots \nu_{j_k}(\lambda) \tag{3.4}$$

is the sum of all the determinants of the principal submatrices of $L(\lambda)$ of order k. Hence $\mathrm{tr}L^k(\lambda)$ $(k=1,2,\ldots)$ are uniquely determined by $p_k(\lambda)$ $(k=1,2,\ldots,n)$ and vise versa. Moreover, $\mathrm{tr}(L^{(1)})^k$ and $\mathrm{tr}(L^{(2)})^k$ in (2.14) can all be uniquely determined by $p_k(\lambda)$ $(k=1,2,\ldots,n)$.

Each $p_k(\lambda)$ is a holomorphic function of λ near $\lambda=\infty$. Let

$$p_m(\lambda)=\sum_{k=0}^{\infty} E_k^{(m)}\lambda^{-k-m}. \tag{3.5}$$

Then

$$\begin{aligned}E_k^{(m)}&=\sum_{1\leqslant i_1<\cdots<i_m\leqslant n}\ \sum_{\substack{r_1+\cdots+r_m=k\\ r_1,\cdots,r_m\geqslant 0}}\begin{vmatrix}\langle\Phi_{i_1},\Lambda^{r_1}\Phi_{i_1}\rangle & \langle\Phi_{i_2},\Lambda^{r_2}\Phi_{i_1}\rangle & \cdots & \langle\Phi_{i_m},\Lambda^{r_m}\Phi_{i_1}\rangle\\ \langle\Phi_{i_1},\Lambda^{r_1}\Phi_{i_2}\rangle & \langle\Phi_{i_2},\Lambda^{r_2}\Phi_{i_2}\rangle & \cdots & \langle\Phi_{i_m},\Lambda^{r_m}\Phi_{i_2}\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\Phi_{i_1},\Lambda^{r_1}\Phi_{i_m}\rangle & \langle\Phi_{i_2},\Lambda^{r_2}\Phi_{i_m}\rangle & \cdots & \langle\Phi_{i_m},\Lambda^{r_m}\Phi_{i_m}\rangle\end{vmatrix}\\ &=\sum_{1\leqslant i_1<\cdots<i_m\leqslant n}\ \sum_{\substack{r_1+\cdots+r_m=k\\ r_1,\cdots,r_m\geqslant 0}}\ \sum_{\substack{\alpha_1,\cdots,\alpha_m=1\\ \alpha_a\neq\alpha_b\ \text{for}\ a\neq b}}^{N}\lambda_{\alpha_1}^{r_1}\cdots\lambda_{\alpha_m}^{r_m}\\ &\quad\times\begin{vmatrix}\bar{\phi}_{i_1\alpha_1}\phi_{i_1\alpha_1} & \bar{\phi}_{i_2\alpha_2}\phi_{i_1\alpha_2} & \cdots & \bar{\phi}_{i_m\alpha_m}\phi_{i_1\alpha_m}\\ \bar{\phi}_{i_1\alpha_1}\phi_{i_2\alpha_1} & \bar{\phi}_{i_2\alpha_2}\phi_{i_2\alpha_2} & \cdots & \bar{\phi}_{i_m\alpha_m}\phi_{i_2\alpha_m}\\ \vdots & \vdots & \ddots & \vdots\\ \bar{\phi}_{i_1\alpha_1}\phi_{i_m\alpha_1} & \bar{\phi}_{i_2\alpha_2}\phi_{i_m\alpha_2} & \cdots & \bar{\phi}_{i_m\alpha_m}\phi_{i_m\alpha_m}\end{vmatrix}.\end{aligned} \tag{3.6}$$

In the last summation, the condition "$\alpha_a\neq\alpha_b$ for $a\neq b$" is added since the determinants with $\alpha_a=\alpha_b$ $(a\neq b)$ are all zero.

Remark 2: When $m\geqslant N+1$, the last summation in (3.6) for "$1\leqslant\alpha_1,\dots,\alpha_m\leqslant N$ with $\alpha_a\neq\alpha_b$ for $a\neq b$" is empty. This means that $E_k^{(m)}\equiv 0$ for $m\geqslant N+1$.

According to (2.8), all $E_k^{(m)}$'s are conserved.

From the first part of Lemma 3, all $E_k^{(m)}$'s are in involution. The second part of Lemma 3 implies that all $\langle\Phi_j,\Phi_k\rangle$'s commute with $E_k^{(m)}$'s. However, these $\langle\Phi_j,\Phi_k\rangle$'s may not commute with each other.

Remark 3: For the Heisenberg ferromagnetic equation, the x-equation of its Lax pair is similar to that of the $U(2)$ principal chiral field. The nonlinearization for this equation was dealt with in Ref. 2 and a set of involutive conserved integrals was obtained there.

Remark 4: Since tr$P$, tr$Q$ *are conserved, if (P,Q) is a solution of (2.3) in $u(n)$, then*

$$P'=P-\frac{1}{n}\mathrm{tr}P,\quad Q'=Q-\frac{1}{n}\mathrm{tr}Q \tag{3.7}$$

gives a solution of the same equation (2.3) in $su(n)$.

IV. DEPENDENCE OF CONSERVED INTEGRALS

In order to consider the integrability of the Hamiltonian systems, we should find a full set of involutive and independent conserved integrals. Unlike many other cases, here we cannot get a full set of independent conserved integrals simply from tr$L^k(\lambda)$'s.

For further discussion, we need the following lemma.

Lemma 4: Suppose k, m are two integers with $k\geqslant 0$ and $m\geqslant 2$, and $\mu_1,\dots,\mu_m$ are distinct complex numbers. Then

$$\sum_{j=1}^{m}\mu_j^k\prod_{\substack{r=1\\ r\neq j}}^{m}(\mu_j-\mu_r)^{-1}=\begin{cases}0 & \text{if } k<m-1,\\ \displaystyle\sum_{\substack{p_1+\cdots+p_m=k-m+1\\ p_1,\cdots,p_m\geqslant 0}}\mu_1^{p_1}\cdots\mu_m^{p_m} & \text{if } k\geqslant m-1.\end{cases}\tag{4.1}$$

Proof: Let

$$f(\zeta)=\zeta^k\prod_{r=1}^{m}(\zeta-\mu_r)^{-1}.\tag{4.2}$$

Then $f(\zeta)$ is a meromorphic function of ζ with poles $\mu_1,\dots,\mu_m$. Let C_R be a circle with radius R, center 0 and positive orientation. Then, when $R>\max_{1\leqslant j\leqslant m}|\mu_j|$,

$$\frac{1}{2\pi\mathrm{i}}\int_{C_R}f(\zeta)\mathrm{d}\zeta=\sum_{j=1}^{m}\operatorname*{Res}_{\zeta=\mu_j}f(\zeta)=\sum_{j=1}^{m}\mu_j^k\prod_{\substack{r=1\\ r\neq j}}^{m}(\mu_j-\mu_r)^{-1}.\tag{4.3}$$

On the other hand, let $\xi=\zeta^{-1}$. Then

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_{C_R}f(\zeta)\mathrm{d}\zeta&=\lim_{R\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{C_{1/R}}\xi^{m-k-2}\prod_{r=1}^{m}(1-\mu_r\xi)^{-1}\mathrm{d}\xi\\ &=\begin{cases}0 & \text{if } k-m+1<0,\\ \displaystyle\sum_{\substack{p_1+\cdots+p_m=k-m+1\\ p_1,\cdots,p_m\geqslant 0}}\mu_1^{p_1}\cdots\mu_m^{p_m} & \text{if } k-m+1\geqslant 0\end{cases}\end{aligned}\tag{4.4}$$

by expanding all the terms $(1-\mu_j\xi)^{-1}$ at $\xi=0$. The lemma is proved.

Theorem 1: *For $1\leqslant m\leqslant n$, there are at most $\max(0,N-m+1)$ linearly independent functions in $E_k^{(m)}$ $(k=0,1,2,\dots)$. Therefore, the number of linearly independent functions in $E_k^{(m)}$ $(m=1,2,\dots,n;\ k=0,1,2,\dots)$ cannot exceed $nN-\frac{1}{2}n(n-1)$ if $N\geqslant n$ or $\frac{1}{2}N(N+1)$ if $N<n$.*

Proof: According to Remark 2, $E_k^{(m)}\equiv 0$ for $m\geqslant N+1$. Hence we always suppose $m\leqslant N$.

By definition, $\Lambda=\mathrm{diag}(\lambda_1,\dots,\lambda_N)$, $\lambda_j\neq\lambda_k$ $(j\neq k)$. Clearly, for any non-negative integers $(k_1,\dots,k_l)$ with $k_i\neq k_j$ $(i\neq j)$ and $l\geqslant N+1$, $E_{k_1}^{(1)},\dots,E_{k_l}^{(1)}$ are linearly dependent. On the other hand, since the Van de Monde determinant of $\lambda_1,\dots,\lambda_N$ is not zero, there are exactly N independent functions in $E_k^{(1)}$ $(k=0,1,2,\dots)$.

For $m\geqslant 2$, we show that there are at most $N-m+1$ independent functions in $E_k^{(m)}$ $(k=0,1,2,\dots)$ for fixed m.

Let $k_1,\dots,k_N$ be N arbitrary distinct non-negative integers. For fixed s with $0\leqslant s\leqslant m-2$, let $(\gamma_1^{(s)},\dots,\gamma_N^{(s)})$ be a solution of the linear algebraic system

$$\sum_{j=1}^{N}\lambda_\alpha^{k_j+m-1}\gamma_j^{(s)}=\lambda_\alpha^s\quad(\alpha=1,2,\dots,N).\tag{4.5}$$

Since the coefficient matrix in (4.5) is $(\lambda_\alpha^{k_j+m+1})_{\alpha,j=1,\dots,N}$ which is invertible, $(\gamma_1^{(s)},\dots,\gamma_N^{(s)})$ exists uniquely.

Let

$$F_s^{(m)}=\sum_{j=1}^{N}\gamma_j^{(s)}E_{k_j}^{(m)}\quad(s=0,1,\dots,m-2).\tag{4.6}$$

Then

$$F_s^{(m)}=\sum_{j=1}^{N}\sum_{1\leqslant i_1<\cdots<i_m\leqslant n}\sum_{\substack{r_1+\cdots+r_m=k_j\\ r_1,\cdots,r_m\geqslant 0}}\sum_{\substack{\alpha_1,\cdots,\alpha_m=1\\ \alpha_a\neq\alpha_b\ \text{for}\ a\neq b}}^{N}\gamma_j^{(s)}\lambda_{\alpha_1}^{r_1}\cdots\lambda_{\alpha_m}^{r_m}$$
$$\times\begin{vmatrix}\bar{\phi}_{i_1\alpha_1}\phi_{i_1\alpha_1} & \bar{\phi}_{i_2\alpha_2}\phi_{i_1\alpha_2} & \cdots & \bar{\phi}_{i_m\alpha_m}\phi_{i_1\alpha_m}\\ \bar{\phi}_{i_1\alpha_1}\phi_{i_2\alpha_1} & \bar{\phi}_{i_2\alpha_2}\phi_{i_2\alpha_2} & \cdots & \bar{\phi}_{i_m\alpha_m}\phi_{i_2\alpha_m}\\ \vdots & \vdots & \ddots & \vdots\\ \bar{\phi}_{i_1\alpha_1}\phi_{i_m\alpha_1} & \bar{\phi}_{i_2\alpha_2}\phi_{i_m\alpha_2} & \cdots & \bar{\phi}_{i_m\alpha_m}\phi_{i_m\alpha_m}\end{vmatrix}. \tag{4.7}$$

For fixed $i_1,\ldots,i_m$, $\alpha_1,\ldots,\alpha_m$ and s with $0\leqslant s\leqslant m-2$, let

$$\Delta=\sum_{j=1}^{N}\sum_{\substack{r_1+\ldots+r_m=k_j\\ r_1,\ldots,r_m\geqslant 0}}\gamma_j^{(s)}\lambda_{\alpha_1}^{r_1}\cdots\lambda_{\alpha_m}^{r_m}. \tag{4.8}$$

Then

$$\Delta=\sum_{j=1}^{N}\gamma_j^{(s)}\sum_{a=1}^{m}\lambda_{\alpha_a}^{k_j+m-1}\prod_{\substack{r=1\\ r\neq a}}^{m}(\lambda_{\alpha_a}-\lambda_{\alpha_r})^{-1} \tag{4.9}$$

by Lemma 4. The relations (4.5) imply

$$\Delta=\sum_{a=1}^{m}\lambda_{\alpha_a}^{s}\prod_{\substack{r=1\\ r\neq a}}^{m}(\lambda_{\alpha_a}-\lambda_{\alpha_r})^{-1}. \tag{4.10}$$

Using Lemma 4 again, we get $\Delta=0$ for $s=0,1,2,\ldots,m-2$. Hence

$$F_0^{(m)}=F_1^{(m)}=\cdots=F_{m-2}^{(m)}=0. \tag{4.11}$$

By (4.5), the matrix $(\gamma_j^{(s)})_{1\leqslant j\leqslant N;\ 0\leqslant s\leqslant m-2}$ has rank $m-1$. Hence $E_{k_j}^{(m)}$ $(j=1,2,\ldots,N)$ satisfy $m-1$ independent linear relations for fixed m. This means that there are at most $N-m+1$ independent functions in N functions $E_{k_j}^{(m)}$ for fixed m.

Since $k_1,\ldots,k_N$ are arbitrary, there are at most $N-m+1$ independent functions in $E_k^{(m)}$ $(k=0,1,2,\ldots)$.

The total number of possible linearly independent functions in $E_k^{(m)}$ $(m=1,2,\ldots,n;\ k=0,1,2,\ldots)$ is

$$\sum_{m=1}^{n}(N-m+1)=nN-\tfrac{1}{2}n(n-1) \tag{4.12}$$

for $N\geqslant n$ and

$$\sum_{m=1}^{N}(N-m+1)=\tfrac{1}{2}N(N+1) \tag{4.13}$$

for $N<n$. The theorem is proved.

A completely integrable Hamiltonian system in $\mathbf{R}^{2nN}$ needs nN independent involutive conserved integrals. Hence the above theorem shows that it is not possible to find enough conserved integrals only from $E_k^{(m)}$'s for Liouville integrability.

V. LIOUVILLE INTEGRABILITY OF THE HAMILTONIAN SYSTEMS

In general, we have not been able to determine whether the Hamiltonian systems for the $U(n)$ principal chiral field are Liouville integrable or not. However, when $n=2$, the answer is positive.

Hereafter, we suppose $n=2$. Therefore, we want to find $2N$ independent conserved integrals for the Hamiltonian systems in $\mathbf{R}^{4N}$.

If $N=1$, let

$$\begin{aligned}\widetilde{E}_0^{(1)}&=E_0^{(1)}=\langle\Phi_1,\Phi_1\rangle+\langle\Phi_2,\Phi_2\rangle,\\ \widetilde{E}_0^{(2)}&=\langle\Phi_1,\Phi_2\rangle+\langle\Phi_2,\Phi_1\rangle.\end{aligned}\tag{5.1}$$

If $N\geqslant 2$, let

$$\begin{aligned}\widetilde{E}_k^{(1)}&=E_k^{(1)}=\langle\Phi_1,\Lambda^k\Phi_1\rangle+\langle\Phi_2,\Lambda^k\Phi_2\rangle\quad(k=0,1,\ldots,N-1),\\ \widetilde{E}_k^{(2)}&=E_k^{(2)}=\sum_{j=0}^{k}\begin{vmatrix}\langle\Phi_1,\Lambda^j\Phi_1\rangle & \langle\Phi_2,\Lambda^{k-j}\Phi_1\rangle\\ \langle\Phi_1,\Lambda^j\Phi_2\rangle & \langle\Phi_2,\Lambda^{k-j}\Phi_2\rangle\end{vmatrix}\quad(k=0,1,\ldots,N-2),\\ \widetilde{E}_{N-1}^{(2)}&=\langle\Phi_1,\Phi_2\rangle+\langle\Phi_2,\Phi_1\rangle.\end{aligned}\tag{5.2}$$

Here the last one is chosen to be $\langle\Phi_1,\Phi_2\rangle+\langle\Phi_2,\Phi_1\rangle$ because all the conserved integrals should take real value.

Theorem 2: *When $n=2$, $\widetilde{E}_k^{(m)}$ $(m=1,2;\ k=0,1,\ldots,N-1)$ are in involution and are functionally independent in a dense open subset of $\mathbf{R}^{4N}$.*

Proof: By Lemma 3, $\widetilde{E}_k^{(m)}$ $(m=1,2;\ k=0,1,\ldots,N-1)$ are in involution.

It is obvious that they are independent for $N=1$. Hence we suppose $N\geqslant 2$. Let $a_{1\alpha}$ $(\alpha=1,2,\ldots,N)$ be N nonzero real numbers,

$$a_{2\alpha}=a_{1\alpha}^{-1}\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{N}(\lambda_\alpha-\lambda_\beta)^{-1}\quad(\alpha=1,2,\ldots,N).\tag{5.3}$$

Then Lemma 4 implies

$$\begin{aligned}\sum_{\beta=1}^{N}\lambda_\beta^k\overline{a_{2\beta}}a_{1\beta}&=0\quad(k=0,1,\ldots,N-2),\\ \sum_{\beta=1}^{N}\lambda_\beta^{N-1}\overline{a_{2\beta}}a_{1\beta}&=1.\end{aligned}\tag{5.4}$$

Let $P_0\in\mathbf{R}^{4N}$ be given by $\phi_{1\beta}=a_{1\beta}$, $\phi_{2\beta}=\epsilon a_{2\beta}$ $(\beta=1,2,\ldots,N)$. Here ϵ is a nonzero small real constant to be determined. Then, at P_0,

$$\begin{aligned}&\frac{\partial\widetilde{E}_k^{(1)}}{\partial\overline{\phi}_{1\alpha}}=\lambda_\alpha^k\phi_{1\alpha},\quad\frac{\partial\widetilde{E}_k^{(1)}}{\partial\overline{\phi}_{2\alpha}}=\lambda_\alpha^k\phi_{2\alpha}\quad(k=0,1,\ldots,N-1),\\ &\frac{\partial\widetilde{E}_k^{(2)}}{\partial\overline{\phi}_{1\alpha}}=\sum_{j=0}^{k}\begin{vmatrix}\lambda_\alpha^j\phi_{1\alpha} & \sum_{\beta=1}^{N}\lambda_\beta^{k-j}\overline{\phi_{2\beta}}\phi_{1\beta}\\ \lambda_\alpha^j\phi_{2\alpha} & \sum_{\beta=1}^{N}\lambda_\beta^{k-j}|\phi_{2\beta}|^2\end{vmatrix}=\sum_{j=0}^{k}r_{2,k-j}\lambda_\alpha^j\phi_{1\alpha},\end{aligned}\tag{5.5}$$

$$\frac{\partial \widetilde{E}_k^{(2)}}{\partial \overline{\phi}_{2\alpha}} = \sum_{j=0}^{k} \begin{vmatrix} \sum_{\beta=1}^{N} \lambda_\beta^{k-j} |\phi_{1\beta}|^2 & \lambda_\alpha^j \phi_{1\alpha} \\ \sum_{\beta=1}^{N} \lambda_\beta^{k-j} \overline{\phi}_{1\beta}\phi_{2\beta} & \lambda_\alpha^j \phi_{2\alpha} \end{vmatrix} = \sum_{j=0}^{k} r_{1,k-j}\lambda_\alpha^j \phi_{2\alpha} \qquad (k=0,1,\ldots,N-2),$$

$$\frac{\partial \widetilde{E}_{N-1}^{(2)}}{\partial \overline{\phi}_{1\alpha}} = \phi_{2\alpha}, \quad \frac{\partial \widetilde{E}_{N-1}^{(2)}}{\partial \overline{\phi}_{2\alpha}} = \phi_{1\alpha}$$

by using (5.4) where

$$r_{jk} = \sum_{\beta=1}^{N} \lambda_\beta^k |\phi_{j\beta}|^2. \tag{5.6}$$

Let J be the Jacobian matrix

$$J = \left. \frac{\partial(\widetilde{E}_0^{(1)},\ldots,\widetilde{E}_{N-1}^{(1)},\widetilde{E}_0^{(2)},\ldots,\widetilde{E}_{N-1}^{(2)})}{\partial(\overline{\phi}_{11},\ldots,\overline{\phi}_{1N},\overline{\phi}_{21},\ldots,\overline{\phi}_{2N})} \right|_{P_0}. \tag{5.7}$$

Denote ROW_j to be the jth row of J. Take the elementary transformations for the rows of J as follows:

(1) k from 1 to $N-1$:

$$\mathrm{ROW}_{N+k} - \sum_{j=0}^{k-1} r_{2,k-1-j}\mathrm{ROW}_{j+1} \to \mathrm{ROW}_{N+k},$$

(2) k from 2 to $N-1$:

$$\mathrm{ROW}_{N+k} - \sum_{j=1}^{k-1} \frac{r_{1j}-r_{2j}}{r_{10}-r_{20}} \mathrm{ROW}_{N+k-j} \to \mathrm{ROW}_{N+k}, \tag{5.8}$$

(3) k from 1 to $N-1$:

$$(r_{10}-r_{20})^{-1}\mathrm{ROW}_{N+k} \to \mathrm{ROW}_{N+k},$$

(4) k from 1 to $N-1$:

$$\mathrm{ROW}_k - \mathrm{ROW}_{N+k} \to \mathrm{ROW}_k.$$

Then J is transformed to

$$\tilde{J}=\begin{pmatrix}
\phi_{11} & \phi_{12} & \cdots & \phi_{1N} & 0 & 0 & \cdots & 0\\
\lambda_1\phi_{11} & \lambda_2\phi_{12} & \cdots & \lambda_N\phi_{1N} & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\lambda_1^{N-2}\phi_{11} & \lambda_2^{N-2}\phi_{12} & \cdots & \lambda_N^{N-2}\phi_{1N} & 0 & 0 & \cdots & 0\\
\lambda_1^{N-1}\phi_{11} & \lambda_2^{N-1}\phi_{12} & \cdots & \lambda_N^{N-1}\phi_{1N} & \lambda_1^{N-1}\phi_{21} & \lambda_2^{N-1}\phi_{22} & \cdots & \lambda_N^{N-1}\phi_{2N}\\
0 & 0 & \cdots & 0 & \phi_{21} & \phi_{22} & \cdots & \phi_{2N}\\
0 & 0 & \cdots & 0 & \lambda_1\phi_{21} & \lambda_2\phi_{22} & \cdots & \lambda_N\phi_{2N}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & 0 & \lambda_1^{N-2}\phi_{21} & \lambda_2^{N-2}\phi_{22} & \cdots & \lambda_N^{N-2}\phi_{2N}\\
\phi_{21} & \phi_{22} & \cdots & \phi_{2N} & \phi_{11} & \phi_{12} & \cdots & \phi_{1N}
\end{pmatrix}. \tag{5.9}$$

Let

$$T=\begin{pmatrix} T_2 & \\ & T_1\end{pmatrix}, \tag{5.10}$$

where

$$T_j=\left.\begin{pmatrix}
\lambda_1^{N-1}\overline{\phi}_{j1} & \lambda_1^{N-2}\overline{\phi}_{j1} & \cdots & \overline{\phi}_{j1}\\
\lambda_2^{N-1}\overline{\phi}_{j2} & \lambda_2^{N-2}\overline{\phi}_{j2} & \cdots & \overline{\phi}_{j2}\\
\vdots & \vdots & \ddots & \vdots\\
\lambda_N^{N-1}\overline{\phi}_{jN} & \lambda_N^{N-2}\overline{\phi}_{jN} & \cdots & \overline{\phi}_{jN}
\end{pmatrix}\right|_{P_0} \qquad (j=1,2). \tag{5.11}$$

Then

$$\det T=\prod_{1\leqslant\alpha<\beta\leqslant N}(\lambda_\alpha-\lambda_\beta)^2\prod_{\gamma=1}^{N}\overline{\phi}_{1\gamma}\overline{\phi}_{2\gamma}\big|_{P_0}\neq 0. \tag{5.12}$$

Using the relations (5.4), we have, at P_0,

$$\tilde{J}T=\begin{pmatrix}
\rho & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
* & \rho & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
* & * & \cdots & \rho & 0 & 0 & 0 & \cdots & 0 & 0\\
* & * & \cdots & * & \rho & * & * & \cdots & * & \overline{\rho}\\
0 & 0 & \cdots & 0 & 0 & \rho & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & * & \rho & \cdots & 0 & 0\\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & * & * & \cdots & \rho & 0\\
* & * & \cdots & * & \sum_{j=1}^{N}|\phi_{2j}|^2 & * & * & \cdots & * & \sum_{j=1}^{N}|\phi_{1j}|^2
\end{pmatrix}, \tag{5.13}$$

where

$$\rho=\sum_{\beta=1}^{N}\lambda_{\beta}^{N-1}\overline{\phi_{2\beta}}\phi_{1\beta}\bigg|_{P_0}=\epsilon\neq 0 \tag{5.14}$$

and $*$ represents the entries which may not be zero.

Hence, at P_0,

$$\begin{aligned}(r_{10}-r_{20})^{-N+1}\det(JT)=\det(\tilde{J}T)&=\rho^{2N-2}\left(\rho\sum_{j=1}^{N}|\phi_{1j}|^2-\bar{\rho}\sum_{j=1}^{N}|\phi_{2j}|^2\right)\\ &=\epsilon^{2N-1}\left(\sum_{j=1}^{N}a_{1j}^2-\epsilon^2\sum_{j=1}^{N}a_{2j}^2\right).\end{aligned} \tag{5.15}$$

It is not zero when ϵ is small enough. Since $\det J$ is a real analytical function on $\mathbf{R}^{4N}$, $\det J$ is not zero in a dense open subset of $\mathbf{R}^{4N}$. The theorem is proved.

Remark 5: Although the constraint here is of Bargmann type, the proof of the independence of the conserved integrals is not so simple as in the AKNS system. In that case, P_0 *is simply chosen as a point near* 0. *However, here* $L(\lambda)$ *is homogeneous to all* Φ_j's *so the choice of* P_0 *near* 0 *does not have any effect on the simplification of the computation on* J.

The Liouville integrability of the $U(2)$ principal chiral field follows from Lemma 2 and Theorem 2. It is given by the following theorem.

Theorem 3: *When* $n=2$, *the Hamiltonian systems given by* (2.15) *are completely integrable in the Liouville sense. Each solution of the Hamiltonian systems* (2.15) *gives a solution* (P,Q) *of* (2.3), *the equation of the* $U(2)$ *principal chiral field, and* $(P-\frac{1}{2}\mathrm{tr}P, Q-\frac{1}{2}\mathrm{tr}Q)$ *is a solution of the* $SU(2)$ *principal chiral field.*

Remark 6: Theorem 1 implies that one needs at least $n(n+1)/2$ *extra conserved integrals together with* $E_k^{(m)}$*'s to form a full set of conserved integrals for the complete integrability of the Hamiltonian systems. According to Lemma 3, all* $\langle\Phi_k,\Phi_j\rangle$*'s commute with* $E_k^{(m)}$. *However, two elements in* $\{\langle\Phi_k,\Phi_j\rangle\}$ *may not commute with each other. Therefore, it is not obvious how to add at least* $n(n+1)/2$ *extra conserved integrals to* $E_k^{(m)}$ *in general.*

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