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Finite Dimensional Hamiltonians and Almost-Periodic Solutions for 2+1 Dimensional Three-Wave Equations

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For the 2+1 dimensional three-wave equation, by using the known nonlinear constraints from 2+1 dimensions to 1+1 dimensions, we reduce it further to 0+1 dimensional (finite dimensional) Hamiltonian systems with constraints of Neumann type. These Hamiltonian systems are proved to be Liouville integrable by finding a full set of involutive conserved integrals and proving their functional independence. Moreover, almost-periodic solutions of the 2+1 dimensional three-wave equation are obtained by solving these Hamiltonian systems explicitly.

KEYWORDS: 2+1 dimensional three-wave equation, finite dimensional Hamiltonian system, almost-periodic solution
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1. Introduction

The 2+1 dimensional three-wave equation is one of the most important integrable equations in 2+1 dimensions.[1,2] This system of equations is

$$\begin{aligned} w_{1,t} &= \alpha_1 w_{1,y} + \beta_1 w_{1,x} + (\alpha_3 - \alpha_2)\bar{w}_2\bar{w}_3, \\ w_{2,t} &= \alpha_2 w_{2,y} + \beta_2 w_{2,x} + (\alpha_1 - \alpha_3)\bar{w}_1\bar{w}_3, \\ w_{3,t} &= \alpha_3 w_{3,y} + \beta_3 w_{3,x} + (\alpha_2 - \alpha_1)\bar{w}_1\bar{w}_2 \end{aligned} \tag{1}$$

where

$$\begin{gathered} \alpha_1 = \frac{b_1 - b_2}{a_1 - a_2}, \qquad \alpha_2 = \frac{b_2 - b_3}{a_2 - a_3}, \qquad \alpha_3 = \frac{b_3 - b_1}{a_3 - a_1}, \\ \beta_1 = b_1 - \alpha_1 a_1, \quad \beta_2 = b_2 - \alpha_2 a_2, \quad \beta_3 = b_3 - \alpha_3 a_3, \end{gathered} \tag{2}$$

a_i's, b_i's are real constants with $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$.

Let

$$U = (u_{ij}) = \begin{pmatrix} 0 & w_1 & -\bar{w}_3 \\ -\bar{w}_1 & 0 & w_2 \\ w_3 & -\bar{w}_2 & 0 \end{pmatrix}, \qquad V = (v_{ij}) \tag{3}$$

with

$$v_{ij} = \frac{b_i - b_j}{a_i - a_j} u_{ij} \quad (i \neq j) \quad \text{and} \quad v_{ii} = 0. \tag{4}$$

Then (1) becomes the matrix equations

$$\begin{aligned} &[A, V] = [B, U], \\ &U_t - V_y + [U, V] + AV_x - BU_x = 0, \end{aligned} \tag{5}$$

where $A = \mathrm{diag}(a_1, \cdots, a_n)$, $B = \mathrm{diag}(b_1, \cdots, b_n)$.

After its integrability was known,[3] the 2+1 dimensional three-wave equation has been studied by various ways such as inverse scattering,[4–7] Bäcklund transformation,[8,9] Darboux transformation and binary Darboux transformation,[10,11] nonlinearization to 1+1 dimensional problems[12,13] etc. In general, higher dimensional three-wave equation is also integrable. However, in order to use the nonlinear constraints, we restrict our discussion here to the 2+1 dimensional three-wave equation.

The method of nonlinear constraints is an effective way to transform 1+1 dimensional integrable systems to finite dimensional integrable systems.[14] On the other hand, many 2+1 dimensional integrable systems can also be transformed to 1+1 dimensional integrable systems via nonlinear constraints.[15] Combining these two procedures, many works have been done to reduce 2+1 dimensional integrable systems to finite dimensional integrable systems.[16–20] In ref. 18, the Bargmann constraints for the 2+1 dimensional N-wave equation were discussed. In refs. 19 and 20 both Bargmann and Neumann constraints could be applied to the Davey-Stewartson I equation which also belongs to the 2+1 dimensional AKNS system as the 2+1 dimensional three-wave equation does. For the Neumann constraints, periodic solutions of Davey–Stewartson equation could be obtained.[19]

In the present paper, we use the Neumann constraints to get Liouville integrable Hamiltonian systems related with the 2+1 dimensional three-wave equation. Using the finite dimensional Hamiltonian systems, we can get explicit almost-periodic solutions of the 2+1 dimensional three-wave equation by solving the ODEs directly.

In §2, the 2+1 dimensional three-wave equation and its Lax set are presented. They are nonlinearized to finite dimensional Hamiltonian systems. In §3, it is shown that these Hamiltonian systems are Liouville integrable by proving that there are a full set of conserved integrals which are involutive and functionally independent in a dense open subset of a symplectic manifold. §4 gives an example of almost-periodic solutions of the 2+1 dimensional three-wave equation solved from these finite dimensional Hamiltonian systems directly.

2. Nonlinear Constraints and Finite Dimensional Hamiltonian Systems

We consider the following Lax set (generalized Lax pair)[12,13]

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$$\Phi_x = U^x\Phi \equiv \begin{pmatrix} \mathrm{i}\lambda & 0 & 0 & \mathrm{i}f_1 \\ 0 & \mathrm{i}\lambda & 0 & \mathrm{i}f_2 \\ 0 & 0 & \mathrm{i}\lambda & \mathrm{i}f_3 \\ \mathrm{i}\bar{f}_1 & \mathrm{i}\bar{f}_2 & \mathrm{i}\bar{f}_3 & 0 \end{pmatrix}\Phi,$$

$$\Phi_y = U^y\Phi \equiv \begin{pmatrix} \mathrm{i}a_1\lambda & u_{12} & u_{13} & \mathrm{i}a_1f_1 \\ u_{21} & \mathrm{i}a_2\lambda & u_{23} & \mathrm{i}a_2f_2 \\ u_{31} & u_{32} & \mathrm{i}a_3\lambda & \mathrm{i}a_3f_3 \\ \mathrm{i}a_1\bar{f}_1 & \mathrm{i}a_2\bar{f}_2 & \mathrm{i}a_3\bar{f}_3 & 0 \end{pmatrix}\Phi, \qquad (6)$$

$$\Phi_t = U^t\Phi \equiv \begin{pmatrix} \mathrm{i}b_1\lambda & v_{12} & v_{13} & \mathrm{i}b_1f_1 \\ v_{21} & \mathrm{i}b_2\lambda & v_{23} & \mathrm{i}b_2f_2 \\ v_{31} & v_{32} & \mathrm{i}b_3\lambda & \mathrm{i}b_3f_3 \\ \mathrm{i}b_1\bar{f}_1 & \mathrm{i}b_2\bar{f}_2 & \mathrm{i}b_3\bar{f}_3 & 0 \end{pmatrix}\Phi$$

where

$$\bar{u}_{kj} = -u_{jk}, \quad v_{jk} = \frac{b_j - b_k}{a_j - a_k}u_{jk} \quad (j,k = 1,2,3;\ j \neq k), \qquad (7)$$

$a_1, a_2, a_3, b_1, b_2, b_3$ are real constants such that $a_j \neq a_k$, $b_j \neq b_k$ for $j \neq k$ and $a_j \neq 0$, $b_j \neq 0$.

Denote $u_{jj} = v_{jj} = 0$ $(j = 1,2,3)$,

$$\begin{aligned} &A = \mathrm{diag}(a_1, a_2, a_3), \quad B = \mathrm{diag}(b_1, b_2, b_3), \\ &U = (u_{jk})_{1\le j,k\le 3}, \quad V = (v_{jk})_{1\le j,k\le 3}, \\ &F = (f_1, f_2, f_3)^T. \end{aligned} \qquad (8)$$

The integrability conditions of (6) are

$$\begin{aligned} &[A, V] = [B, U], \\ &U_t - V_y + [U, V] + AV_x - BU_x = 0, \end{aligned} \qquad (9)$$

$$F_y = AF_x + UF, \quad F_t = BF_x + VF, \qquad (10)$$

$$U_x = [A, FF^*], \quad V_x = [B, FF^*]. \qquad (11)$$

Here (9) is the 2+1 dimensional three-wave equation (5), (10) is its standard Lax pair, and (11) is an extra nonlinear constraint between U and F. Therefore, any solution of (9)–(11) is a solution of the 2+1 dimensional three-wave equation.

To simplify the matrices in (6), denote

$$\begin{aligned} &u_{j4} = \mathrm{i}a_jf_j, \quad v_{j4} = \mathrm{i}b_jf_j, \quad u_{4j} = -\bar{u}_{j4}, \\ &v_{4j} = -\bar{v}_{j4} \quad (j = 1,2,3) \end{aligned} \qquad (12)$$

and $a_4 = b_4 = 0$, then

$$\begin{aligned} &\bar{u}_{kj} = -u_{jk}, \quad v_{jk} = \frac{b_j - b_k}{a_j - a_k}u_{jk} \\ &(j,k = 1,2,3,4;\ j \neq k). \end{aligned} \qquad (13)$$

Now we nonlinearize (6) further to get finite dimensional Hamiltonian systems.

Take N distinct real numbers $\lambda_1, \cdots, \lambda_N$. Let $\phi_\alpha = (\phi_{1\alpha}, \phi_{2\alpha}, \phi_{3\alpha}, \phi_{4\alpha})^T$ be a column solution of (6) with $\lambda = \lambda_\alpha$ $(\alpha = 1, 2, \cdots, N)$, $\Phi_j = (\phi_{j1}, \cdots, \phi_{jN})^T$ $(j = 1,2,3,4)$. For any two vectors V_1 and V_2, define $\langle V_1, V_2\rangle = V_1^*V_2$.

Lemma 1. *Let* $\Gamma = \mathrm{diag}(1,1,0,0)$,

$$L = \Gamma + \sum_{\alpha=1}^{N} \frac{\phi_\alpha\phi_\alpha^*}{\lambda - \lambda_\alpha}, \qquad (14)$$

then L satisfies

$$L_x = [U^x, L], \quad L_y = [U^y, L], \quad L_t = [U^t, L] \qquad (15)$$

if and only if

$$\begin{aligned} &\langle\Phi_1, \Phi_2\rangle = 0, \quad \langle\Phi_3, \Phi_4\rangle = 0, \\ &u_{13} = \hat{u}_{13} := \mathrm{i}(a_1 - a_3)\langle\Phi_3, \Phi_1\rangle, \\ &u_{14} = \hat{u}_{14} := \mathrm{i}a_1\langle\Phi_4, \Phi_1\rangle, \\ &u_{23} = \hat{u}_{23} := \mathrm{i}(a_2 - a_3)\langle\Phi_3, \Phi_2\rangle, \\ &u_{24} = \hat{u}_{24} := \mathrm{i}a_2\langle\Phi_4, \Phi_2\rangle. \end{aligned} \qquad (16)$$

Proof. We first prove the result for the second equation of (15). Denote $J = \mathrm{diag}(a_1, a_2, a_3, a_4)$, then by (14),

$$\begin{aligned} L_y &= \sum_{\alpha=1}^{N} \frac{1}{\lambda - \lambda_\alpha}\left(U^y(\lambda_\alpha)\phi_\alpha\phi_\alpha^* + \phi_\alpha\phi_\alpha^*U^y(\lambda_\alpha)^*\right) \\ &= \sum_{\alpha=1}^{N} \frac{1}{\lambda - \lambda_\alpha}\left[U^y(\lambda_\alpha), \phi_\alpha\phi_\alpha^*\right] \\ &= \sum_{\alpha=1}^{N} \frac{1}{\lambda - \lambda_\alpha}\left[U^y(\lambda) - \mathrm{i}J(\lambda - \lambda_\alpha), \phi_\alpha\phi_\alpha^*\right] \\ &= [U^y(\lambda), L(\lambda)] - \mathrm{i}\left[J, \sum_{\alpha=1}^{N}\phi_\alpha\phi_\alpha^*\right] - [U^y(\lambda), \Gamma]. \end{aligned}$$

Hence $L_y(\lambda) = [U^y, L(\lambda)]$ if and only if $\mathrm{i}[J, \sum_{\alpha=1}^{N}\phi_\alpha\phi_\alpha^*] = [\Gamma, U^y(\lambda)]$. Written in components, it is just (16).

When (16) holds, it is easy to check that $L_x = [U^x, L]$ and $L_t = [U^t, L]$ hold. The lemma is proved.

By computing the derivatives of the first two equations of (16) with respect to x, we get

$$\begin{aligned} &u_{12} = \hat{u}_{12} := \frac{\mathrm{i}(a_1 - a_2)(\langle\Phi_2, \Lambda\Phi_1\rangle + \langle\Phi_2, \Phi_3\rangle\langle\Phi_3, \Phi_1\rangle + \langle\Phi_2, \Phi_4\rangle\langle\Phi_4, \Phi_1\rangle)}{\langle\Phi_1, \Phi_1\rangle - \langle\Phi_2, \Phi_2\rangle}, \\ &u_{34} = \hat{u}_{34} := \frac{\mathrm{i}a_3(\langle\Phi_4, \Lambda\Phi_3\rangle - \langle\Phi_4, \Phi_1\rangle\langle\Phi_1, \Phi_3\rangle - \langle\Phi_4, \Phi_2\rangle\langle\Phi_2, \Phi_3\rangle)}{\langle\Phi_3, \Phi_3\rangle - \langle\Phi_4, \Phi_4\rangle}. \end{aligned} \qquad (17)$$

By (12), (13), (16) and (17), define

$$\begin{aligned} &\hat{v}_{jk} = \frac{b_j - b_k}{a_j - a_k}\hat{u}_{jk} \quad (j,k = 1,2,3,4;\ j \neq k) \\ &\hat{f}_j = \frac{1}{\mathrm{i}a_j}\hat{u}_{j4} \quad (j = 1,2,3). \end{aligned} \qquad (18)$$

$\mathrm{Re}\,\phi_{j\alpha}$ and $\mathrm{Im}\,\phi_{j\alpha}$ $(j = 1,2,3,4;\ \alpha = 1,2,\cdots,N)$ form the coordinates of a Euclidean space $\boldsymbol{R}^{8N}$. This $\boldsymbol{R}^{8N}$ has the standard symplectic form

$$\omega = 2\sum_{j=1}^{4}\sum_{\alpha=1}^{N} \mathrm{d}\,\mathrm{Im}\,\phi_{j\alpha} \wedge \mathrm{d}\,\mathrm{Re}\,\phi_{j\alpha} = \mathrm{i}\sum_{j=1}^{4}\sum_{\alpha=1}^{N} \mathrm{d}\bar{\phi}_{j\alpha} \wedge \mathrm{d}\phi_{j\alpha}. \qquad (19)$$

With this symplectic form, the corresponding Poisson bracket of two smooth functions ξ, η of $\phi_{j\alpha}$'s and $\bar{\phi}_{j\alpha}$'s is

$$\{\xi, \eta\} = \frac{1}{\mathrm{i}}\sum_{j=1}^{4}\sum_{\alpha=1}^{N}\left(\frac{\partial\xi}{\partial\phi_{j\alpha}}\frac{\partial\eta}{\partial\bar{\phi}_{j\alpha}} - \frac{\partial\xi}{\partial\bar{\phi}_{j\alpha}}\frac{\partial\eta}{\partial\phi_{j\alpha}}\right). \qquad (20)$$

Let

$$S = \Big\{ (\phi_{j\alpha}, \bar{\phi}_{j\alpha}) \in \boldsymbol{R}^{8N} (j = 1, 2, 3, 4;\ \alpha = 1, \cdots, N) \mid \langle \Phi_1, \Phi_2 \rangle = \langle \Phi_3, \Phi_4 \rangle = 0, \langle \Phi_1, \Phi_1 \rangle \neq \langle \Phi_2, \Phi_2 \rangle, \langle \Phi_3, \Phi_3 \rangle \neq \langle \Phi_4, \Phi_4 \rangle \Big\}. \tag{21}$$

Denote $h_1 = \mathrm{Re}\langle \Phi_1, \Phi_2 \rangle$, $h_2 = \mathrm{Im}\langle \Phi_1, \Phi_2 \rangle$, $h_3 = \mathrm{Re}\langle \Phi_3, \Phi_4 \rangle$, $h_4 = \mathrm{Im}\langle \Phi_3, \Phi_4 \rangle$. Then

$$\det(\{h_j, h_k\})_{1 \le j,k \le 4} = \frac{1}{16} (\langle \Phi_1, \Phi_1 \rangle - \langle \Phi_2, \Phi_2 \rangle)^2 (\langle \Phi_3, \Phi_3 \rangle - \langle \Phi_4, \Phi_4 \rangle)^2 \neq 0 \tag{22}$$

on S. Hence S is a symplectic manifold with the symplectic form induced from (19). For any two functions ξ and η satisfying $\{\xi, \langle \Phi_1, \Phi_2 \rangle\} = \{\xi, \langle \Phi_3, \Phi_4 \rangle\} = 0$, $\{\eta, \langle \Phi_1, \Phi_2 \rangle\} = \{\eta, \langle \Phi_3, \Phi_4 \rangle\} = 0$, the Poisson bracket of ξ and η is still given by (20).

With the constraints (16) and (17), the system (6) becomes a system of ODEs of $\phi_{j\alpha}$'s:

$$\begin{aligned} &\phi_{j\alpha,x} = \mathrm{i}\lambda_\alpha \phi_{j\alpha} + \mathrm{i}\hat{f}_j \phi_{4\alpha} \ (1 \le j \le 3), \\ &\phi_{4\alpha,x} = \mathrm{i} \sum_{k=1}^{3} \bar{\hat{f}}_k \phi_{k\alpha}, \\ &\phi_{j\alpha,y} = \mathrm{i} a_j \lambda_\alpha \phi_{j\alpha} + \sum_{\substack{k=1 \\ k \neq j}}^{4} \hat{u}_{jk} \phi_{k\alpha} \ (1 \le j \le 4), \\ &\phi_{j\alpha,t} = \mathrm{i} b_j \lambda_\alpha \phi_{j\alpha} + \sum_{\substack{k=1 \\ k \neq j}}^{4} \hat{v}_{jk} \phi_{k\alpha} \ (1 \le j \le 4). \end{aligned} \tag{23}$$

Let

$$\begin{aligned} K_1 &= \langle \Phi_1, \Lambda \Phi_1 \rangle + |\langle \Phi_3, \Phi_1 \rangle|^2 + |\langle \Phi_4, \Phi_1 \rangle|^2 \\ &\quad + \tfrac{1}{\Omega_1 - \Omega_2} (q_{12} \langle \Phi_1, \Phi_2 \rangle + \bar{q}_{12} \langle \Phi_2, \Phi_1 \rangle), \\ K_2 &= \langle \Phi_2, \Lambda \Phi_2 \rangle + |\langle \Phi_3, \Phi_2 \rangle|^2 + |\langle \Phi_4, \Phi_2 \rangle|^2 \\ &\quad - \tfrac{1}{\Omega_1 - \Omega_2} (q_{12} \langle \Phi_1, \Phi_2 \rangle + \bar{q}_{12} \langle \Phi_2, \Phi_1 \rangle), \\ K_3 &= \langle \Phi_3, \Lambda \Phi_3 \rangle - |\langle \Phi_3, \Phi_1 \rangle|^2 - |\langle \Phi_3, \Phi_2 \rangle|^2 \\ &\quad + \tfrac{1}{\Omega_3 - \Omega_4} (q_{34} \langle \Phi_3, \Phi_4 \rangle + \bar{q}_{34} \langle \Phi_3, \Phi_4 \rangle) \end{aligned} \tag{24}$$

where

$$\begin{aligned} &\Omega_k = \langle \Phi_k, \Phi_k \rangle \quad (k = 1, 2, 3, 4), \\ &q_{12} = \langle \Phi_2, \Lambda \Phi_1 \rangle + \langle \Phi_2, \Phi_3 \rangle \langle \Phi_3, \Phi_1 \rangle + \langle \Phi_2, \Phi_4 \rangle \langle \Phi_4, \Phi_1 \rangle, \\ &q_{34} = \langle \Phi_4, \Lambda \Phi_3 \rangle - \langle \Phi_4, \Phi_1 \rangle \langle \Phi_1, \Phi_3 \rangle - \langle \Phi_4, \Phi_2 \rangle \langle \Phi_2, \Phi_3 \rangle, \end{aligned} \tag{25}$$

then with direct computation we have the following lemma.

Lemma 2. *The equations in (23) are three Hamiltonian systems on S with the Hamiltonians*

$$\begin{aligned} H^x &= -(K_1 + K_2 + K_3), \\ H^y &= -(a_1 K_1 + a_2 K_2 + a_3 K_3), \\ H^t &= -(b_1 K_1 + b_2 K_2 + b_3 K_3) \end{aligned} \tag{26}$$

respectively. That is, the equations in (23) are equivalent to the Hamiltonian equations

$$\begin{aligned} &\mathrm{i} \frac{\partial \phi_{j\alpha}}{\partial x} = \frac{\partial H^x}{\partial \bar{\phi}_{j\alpha}}, \quad \mathrm{i} \frac{\partial \phi_{j\alpha}}{\partial y} = \frac{\partial H^y}{\partial \bar{\phi}_{j\alpha}}, \quad \mathrm{i} \frac{\partial \phi_{j\alpha}}{\partial t} = \frac{\partial H^t}{\partial \bar{\phi}_{j\alpha}}, \\ &\frac{1}{\mathrm{i}} \frac{\partial \bar{\phi}_{j\alpha}}{\partial x} = \frac{\partial H^x}{\partial \phi_{j\alpha}}, \quad \frac{1}{\mathrm{i}} \frac{\partial \bar{\phi}_{j\alpha}}{\partial y} = \frac{\partial H^y}{\partial \phi_{j\alpha}}, \quad \frac{1}{\mathrm{i}} \frac{\partial \bar{\phi}_{j\alpha}}{\partial t} = \frac{\partial H^t}{\partial \phi_{j\alpha}}. \end{aligned} \tag{27}$$

3. Integrability of Hamiltonian Systems

In order to get the integrability of the Hamiltonian systems on the symplectic manifold S, we need to prove the following facts.

(i) There are $\{E_{j\alpha}\}$ $(j = 1, 2, \cdots;\ \alpha = 1, 2, \cdots)$ such that

$$\{E_{j\alpha}, E_{k\beta}\} = 0, \quad \{E_{j\alpha}, \langle \Phi_1, \Phi_2 \rangle\} = 0, \quad \{E_{j\alpha}, \langle \Phi_3, \Phi_4 \rangle\} = 0 \tag{28}$$

on whole $\boldsymbol{R}^{8N}$.

(ii) H^x, H^y, H^t commute with each other on S.

(iii) H^x, H^y, H^t commute with all $\{E_{j\alpha}\}$, $\langle \Phi_1, \Phi_2 \rangle$ and $\langle \Phi_3, \Phi_4 \rangle$ on S.

(iv) There are $4N - 2$ functions in $\{E_{j\alpha}\}$ which are functionally independent in a dense open subset of S.

We first give the following lemma.

Lemma 3. *For any complex numbers λ, μ and positive integers k, l,*

$$\{\mathrm{tr}\, L^k(\lambda), \mathrm{tr}\, L^l(\mu)\} = 0, \quad \{\mathrm{tr}\, L^k(\lambda), \langle \Phi_1, \Phi_2 \rangle\} = 0, \quad \{\mathrm{tr}\, L^k(\lambda), \langle \Phi_3, \Phi_4 \rangle\} = 0. \tag{29}$$

This is derived from Lemma 1 of ref. 19, or proved directly by computing the Poisson brackets.

Suppose $\nu_1(\lambda)$, $\nu_2(\lambda)$, $\nu_3(\lambda)$, $\nu_4(\lambda)$ are eigenvalues of $L(\lambda)$, then

$$\mathrm{tr}\, L^k(\lambda) = \nu_1^k(\lambda) + \nu_2^k(\lambda) + \nu_3^k(\lambda) + \nu_4^k(\lambda). \tag{30}$$

On the other hand, for a complex number μ,

$$\det(\mu I - L(\lambda)) = (\mu - \nu_1)(\mu - \nu_2)(\mu - \nu_3)(\mu - \nu_4). \tag{31}$$

Denote

$$Q_k(\lambda) = \sum_{1 \le j_1 < \cdots < j_k \le 4} \nu_{j_1} \cdots \nu_{j_k} \quad (k = 1, 2, 3, 4), \tag{32}$$

then $Q_k(\lambda)$ is the sum of all the determinants of $k \times k$ principal submatrices of $L(\lambda)$. Therefore, Lemma 3 implies that

$$\{Q_k(\lambda), Q_l(\mu)\} = 0, \quad \{Q_k(\lambda), \langle \Phi_1, \Phi_2 \rangle\} = 0, \quad \{Q_k(\lambda), \langle \Phi_3, \Phi_4 \rangle\} = 0 \tag{33}$$

for any complex numbers λ, μ and positive integers k, l.

For integers j, k, $p_1, \cdots, p_j$, $r_1, \cdots, r_j$ with $j \ge 1$, $1 \le p_l \le 4$, $r_l \ge 0$ $(l = 1, \cdots, j)$, define

$$D_{p_1 \cdots p_j}^{r_1 \cdots r_j} = \begin{vmatrix} \langle \Phi_{p1}, \Lambda^{r_1} \Phi_{p1} \rangle & \cdots & \langle \Phi_{pj}, \Lambda^{r_j} \Phi_{p1} \rangle \\ \vdots & \ddots & \vdots \\ \langle \Phi_{p1}, \Lambda^{r_1} \Phi_{pj} \rangle & \cdots & \langle \Phi_{pj}, \Lambda^{r_j} \Phi_{pj} \rangle \end{vmatrix}, \tag{34}$$

$$W_{p_1 \cdots p_j}^{(k)} = \sum_{\substack{r_1 + \cdots + r_j = k \\ r_1, \cdots, r_j \ge 0}} D_{p_1 \cdots p_j}^{r_1 \cdots r_j} \tag{35}$$

where the result is zero for empty summation.

Expand $Q_k(\lambda)$ as a Laurent series of λ as

$$Q_k(\lambda) = \sum_{j=-1}^{\infty} F_{kj}\lambda^{-j-1} \quad (k = 1, 2, 3, 4), \tag{36}$$

then using the expression (14), we have, for $k \geq 0$,

$$\begin{aligned} F_{1k} &= \sum_{j=1}^{4} W_j^{(k)}, \\ F_{2k} &= F_{1k} + W_3^{(k)} + W_4^{(k)} + \sum_{1\leq p_1<p_2\leq 4} W_{p_1p_2}^{(k-1)}, \\ F_{3k} &= F_{2k} - F_{1k} + W_{34}^{(k-1)} - W_{12}^{(k-1)} + \sum_{1\leq p_1<p_2<p_3\leq 4} W_{p_1p_2p_3}^{(k-2)}, \\ F_{4k} &= W_{34}^{(k-1)} + W_{134}^{(k-2)} + W_{234}^{k-2} + W_{1234}^{(k-3)}. \end{aligned} \tag{37}$$

Define

$$\begin{aligned} E_{1k} &= 2F_{1k} - F_{2k} = W_1^{(k)} + W_2^{(k)} - \sum_{1\leq p\leq q\leq 4} W_{pq}^{(k-1)}, \\ E_{2k} &= F_{4,k+1} - F_{3,k+1} + F_{2,k+1} - F_{1,k+1} \\ &= W_{12}^{(k)} - W_{123}^{(k-1)} - W_{124}^{(k-1)} + W_{1234}^{(k-2)}, \\ E_{3k} &= F_{2k} - F_{1k} = W_3^{(k)} + W_4^{(k)} + \sum_{1\leq p\leq q\leq 4} W_{pq}^{(k-1)}, \\ E_{4k} &= F_{4,k+1} = W_{34}^{(k)} + W_{134}^{(k-1)} + W_{234}^{(k-1)} + W_{1234}^{(k-2)}, \end{aligned} \tag{38}$$

then

$$\begin{aligned} E_{10} &= \langle \Phi_1, \Phi_1 \rangle + \langle \Phi_2, \Phi_2 \rangle, \\ E_{20} &= \langle \Phi_1, \Phi_1 \rangle \langle \Phi_2, \Phi_2 \rangle - |\langle \Phi_1, \Phi_2 \rangle|^2, \\ E_{30} &= \langle \Phi_3, \Phi_3 \rangle + \langle \Phi_4, \Phi_4 \rangle, \\ E_{40} &= \langle \Phi_3, \Phi_3 \rangle \langle \Phi_4, \Phi_4 \rangle - |\langle \Phi_3, \Phi_4 \rangle|^2. \end{aligned} \tag{39}$$

Since $\langle \Phi_1, \Phi_1 \rangle \neq \langle \Phi_2, \Phi_2 \rangle$ and $\langle \Phi_3, \Phi_3 \rangle \neq \langle \Phi_4, \Phi_4 \rangle$ on S, we should have

$$E_{10}^2 > 4E_{20}, \quad E_{30}^2 > 4E_{40}. \tag{40}$$

According to (33), we have

Lemma 4. *The identities*

$$\begin{aligned} &\{E_{j\alpha}, E_{k\beta}\} = 0, \quad \{E_{j\alpha}, \langle \Phi_1, \Phi_2 \rangle\} = 0, \\ &\{E_{j\alpha}, \langle \Phi_3, \Phi_4 \rangle\} = 0 \end{aligned} \tag{41}$$

hold for all $j, k = 1, 2, 3, 4$; $\alpha, \beta = 0, 1, 2, \cdots$.

Now we consider the independence of $E_{j\alpha}$'s.

Lemma 5. $E_{j\alpha}$ ($j = 1, 2, 3, 4$; $0 \leq k \leq N-1$ *for* $j = 1, 3$, $0 \leq k \leq N-2$ *for* $j = 2, 4$) *are functionally independent in a dense open subset of S.*

Proof. Near the point of S with $\phi_{1N} \neq 0$ and $\phi_{3N} \neq 0$, the coordinates can be chosen as ($z_{11} = \phi_{11}, \cdots, z_{1N} = \phi_{1N}$, $z_{21} = \phi_{21}, \cdots, z_{2,N-1} = \phi_{2,N-1}$, $z_{31} = \phi_{31}, \cdots, z_{3N} = \phi_{3N}$, $z_{41} = \phi_{41}, \cdots, z_{4,N-1} = \phi_{4,N-1}$) and their complex conjugates. In this case, ϕ_{2N} and ϕ_{4N} are given by

$$\begin{aligned} \bar{\phi}_{2N} &= -\frac{1}{z_{1N}}(z_{11}\bar{z}_{21} + \cdots + z_{1,N-1}\bar{z}_{2,N-1}), \\ \bar{\phi}_{4N} &= -\frac{1}{z_{3N}}(z_{31}\bar{z}_{41} + \cdots + z_{3,N-1}\bar{z}_{4,N-1}). \end{aligned} \tag{42}$$

On S,

$$\begin{aligned} \frac{\partial}{\partial \bar{z}_{1\alpha}} &= \frac{\partial}{\partial \bar{\phi}_{1\alpha}} - \frac{\phi_{2\alpha}}{\bar{\phi}_{1N}} \frac{\partial}{\partial \phi_{2N}} \quad (\alpha \leq N), \\ \frac{\partial}{\partial \bar{z}_{2\alpha}} &= \frac{\partial}{\partial \bar{\phi}_{2\alpha}} - \frac{\phi_{1\alpha}}{\phi_{1N}} \frac{\partial}{\partial \bar{\phi}_{2N}} \quad (\alpha \leq N-1), \\ \frac{\partial}{\partial \bar{z}_{3\alpha}} &= \frac{\partial}{\partial \bar{\phi}_{3\alpha}} - \frac{\phi_{4\alpha}}{\bar{\phi}_{3N}} \frac{\partial}{\partial \phi_{4N}} \quad (\alpha \leq N), \\ \frac{\partial}{\partial \bar{z}_{4\alpha}} &= \frac{\partial}{\partial \bar{\phi}_{4\alpha}} - \frac{\phi_{3\alpha}}{\phi_{3N}} \frac{\partial}{\partial \bar{\phi}_{4N}} \quad (\alpha \leq N-1). \end{aligned}$$

Let $P_0 \in S$ be given by $\Phi_1 = \Phi_3 = (1, 1, \cdots, 1)$, $\Phi_2 = \Phi_4 = \epsilon(1, 1, \cdots, 1, -N+1)$, where ϵ is a small constant. Then, at P_0,

$$\begin{aligned} &\frac{\partial E_{1k}}{\partial \bar{z}_{1\alpha}} = \lambda_\alpha^k + O(\epsilon^2), \quad \frac{\partial E_{1k}}{\partial \bar{z}_{2\alpha}} = O(\epsilon), \\ &\frac{\partial E_{1k}}{\partial \bar{z}_{3\alpha}} = O(\epsilon^2), \quad \frac{\partial E_{1k}}{\partial \bar{z}_{4\alpha}} = O(\epsilon), \\ &\frac{\partial E_{2k}}{\partial \bar{z}_{1\alpha}} = O(\epsilon^2), \\ &\frac{\partial E_{2k}}{\partial \bar{z}_{2\alpha}} = \epsilon N \sum_{r_1+r_2=k} \left(\sum_{\beta=1}^{N} \lambda_\beta^{r_1}\lambda_N^{r_2} + \lambda_\alpha^{r_1}\lambda_N^{r_2} - \lambda_N^k \right) + O(\epsilon^3), \\ &\frac{\partial E_{2k}}{\partial \bar{z}_{3\alpha}} = O(\epsilon^2), \quad \frac{\partial E_{2k}}{\partial \bar{z}_{4\alpha}} = O(\epsilon^3), \\ &\frac{\partial E_{3k}}{\partial \bar{z}_{1\alpha}} = O(\epsilon^2), \quad \frac{\partial E_{3k}}{\partial \bar{z}_{2\alpha}} = O(\epsilon), \\ &\frac{\partial E_{3k}}{\partial \bar{z}_{3\alpha}} = \lambda_\alpha^k + O(\epsilon^2), \quad \frac{\partial E_{3k}}{\partial \bar{z}_{4\alpha}} = O(\epsilon), \\ &\frac{\partial E_{4k}}{\partial \bar{z}_{1\alpha}} = O(\epsilon^2), \quad \frac{\partial E_{4k}}{\partial \bar{z}_{2\alpha}} = O(\epsilon^3), \quad \frac{\partial E_{4k}}{\partial \bar{z}_{3\alpha}} = O(\epsilon^2), \\ &\frac{\partial E_{4k}}{\partial \bar{z}_{4\alpha}} = \epsilon N \sum_{r_1+r_2=k} \left(\sum_{\beta=1}^{N} \lambda_\beta^{r_1}\lambda_N^{r_2} + \lambda_\alpha^{r_1}\lambda_N^{r_2} - \lambda_N^k \right) + O(\epsilon^3). \end{aligned}$$

The Jacobian determinant of $4N-2$ real functions ($E_{10}, \cdots, E_{1,N-1}, E_{20}, \cdots, E_{2,N-2}, E_{30}, \cdots, E_{3,N-1}, E_{40}, \cdots, E_{4,N-2}$) to the variables ($\bar{z}_{11}, \cdots, \bar{z}_{1N}, \bar{z}_{21}, \cdots, \bar{z}_{2,N-1}, \bar{z}_{31}, \cdots, \bar{z}_{3N}, \bar{z}_{41}, \cdots, \bar{z}_{4,N-1}$) is

$$\left(\epsilon^{N-1} N^N \prod_{1\leq j<k\leq N} (\lambda_j - \lambda_k) \prod_{1\leq j<k\leq N-1} (\lambda_j - \lambda_k) \right)^2 + O(\epsilon^{2N-3}) \tag{43}$$

which is not zero when ϵ is small enough. Since $4N-2$ functions $E_{10}, \cdots, E_{1,N-1}, E_{20}, \cdots, E_{2,N-2}, E_{30}, \cdots, E_{3,N-1}, E_{40}, \cdots, E_{4,N-2}$ are real analytic functions on S, they are functionally independent in a dense open subset of S for $\langle \Phi_1, \Phi_1 \rangle > \langle \Phi_2, \Phi_2 \rangle$ and $\langle \Phi_3, \Phi_3 \rangle > \langle \Phi_4, \Phi_4 \rangle$. Similarly, this is true for the other three connected components of S. The lemma is proved.

It can be checked directly that K_1, K_2, K_3 in (24) are given by

$$
\begin{aligned}
(\Omega_1-\Omega_2)K_1 &= \Omega_1 E_{11} - E_{21} \\
&\quad + \Omega_1^2(\Omega_2+\Omega_3+\Omega_4) + \Omega_1\Omega_3\Omega_4 \\
&\quad - (\Omega_1-\Omega_3-\Omega_4)|\langle\Phi_2,\Phi_1\rangle|^2 - \Omega_1|\langle\Phi_4,\Phi_3\rangle|^2, \\
(\Omega_1-\Omega_2)K_2 &= E_{21} - \Omega_2 E_{11} \\
&\quad - \Omega_2^2(\Omega_1+\Omega_3+\Omega_4) - \Omega_2\Omega_3\Omega_4 \\
&\quad + (\Omega_2-\Omega_3-\Omega_4)|\langle\Phi_2,\Phi_1\rangle|^2 + \Omega_2|\langle\Phi_4,\Phi_3\rangle|^2, \\
(\Omega_3-\Omega_4)K_3 &= \Omega_3 E_{31} - E_{41} \\
&\quad - \Omega_3^2(\Omega_1+\Omega_2+\Omega_4) - \Omega_1\Omega_2\Omega_3 \\
&\quad + \Omega_3|\langle\Phi_2,\Phi_1\rangle|^2 + (\Omega_3-\Omega_1-\Omega_2)|\langle\Phi_4,\Phi_3\rangle|^2.
\end{aligned} \tag{44}
$$

With the conditions (40), (39) implies that Ω_j's are smooth functions of E_{jk}'s and $|\langle\Phi_2,\Phi_1\rangle|^2$, $|\langle\Phi_4,\Phi_3\rangle|^2$ near S. Hence K_1, K_2 and K_3 are smooth functions of E_{jk}'s and $|\langle\Phi_2,\Phi_1\rangle|^2$, $|\langle\Phi_4,\Phi_3\rangle|^2$ near S. On the other hand, the terms $|\langle\Phi_2,\Phi_1\rangle|^2$ and $|\langle\Phi_4,\Phi_3\rangle|^2$ do not take effect in computing Poisson brackets on S. Therefore, we have the following conclusions.

Theorem 1. *(1) The Hamiltonians H^x, H^y and H^t defined by (26) are in involution:*

$$\{H^x,H^y\} = \{H^x,H^t\} = \{H^y,H^t\} = 0. \tag{45}$$

They also commute with the functions defining S:

$$
\begin{aligned}
\{H^x,\langle\Phi_1,\Phi_2\rangle\} &= \{H^x,\langle\Phi_3,\Phi_4\rangle\} = 0, \\
\{H^y,\langle\Phi_1,\Phi_2\rangle\} &= \{H^y,\langle\Phi_3,\Phi_4\rangle\} = 0, \\
\{H^t,\langle\Phi_1,\Phi_2\rangle\} &= \{H^t,\langle\Phi_3,\Phi_4\rangle\} = 0.
\end{aligned} \tag{46}
$$

Hence they are involutive Hamiltonians on S.

(2) $E_{j\alpha}$'s are conserved integrals of the Hamiltonian flows on S given by H^x, H^y and H^t respectively. $4N-2$ of them: $E_{10}, \cdots, E_{1,N-1}$, $E_{20}, \cdots, E_{2,N-2}$, $E_{30}, \cdots, E_{3,N-1}$, $E_{40}, \cdots, E_{4,N-2}$ are functionally independent in a dense open subset of $8N-4$ dimensional symplectic manifold S.

Therefore, H^x, H^y and H^t give three involutive Liouville integrable Hamiltonian systems. Moreover, each solution of these Hamiltonian systems gives a solution of the 2+1 dimensional three-wave equation.

Remark 1. *If Γ in Lemma 1 is chosen as other constant matrices, we can get other finite dimensional Hamiltonian systems. For example, when all the eigenvalues of Γ are different, this was discussed in ref. 18. The choice of Γ in this paper guarantees that we can solve the finite dimensional Hamiltonian systems and get explicit almost-periodic solutions.*

Remark 2. *The finite dimensional Hamiltonian systems related with the 2+1 dimensional n-wave equation can also be obtained in this way. However, these Hamiltonian systems with Neumann constraints can only be solved explicitly for $n=3$. It is still interesting to find appropriate Lax operator to have the Hamiltonian systems solved explicitly.*

4. Example

Here we consider the example for $N=2$, $\lambda_1=\lambda$, $\lambda_2=\mu$. In this case, the symplectic manifold S has real dimension $2\times(4\times 2-2)=12$. Hence we need 6 independent involutive conserved integrals.

Let

$$
\begin{aligned}
R_j^2 &= |\phi_{j1}|^2 + |\phi_{j2}|^2 \quad (j=1,2,3,4), \\
G &= |\phi_{11}|^2 + |\phi_{21}|^2 + |\phi_{31}|^2 + |\phi_{41}|^2 - R_2^2 - R_3^2, \\
K &= \frac{1}{R_1^2}(\langle\Phi_1,\Lambda\Phi_1\rangle + \langle\Phi_1,\Phi_3\rangle\langle\Phi_3,\Phi_1\rangle \\
&\quad + \langle\Phi_1,\Phi_4\rangle\langle\Phi_4,\Phi_1\rangle)
\end{aligned} \tag{47}
$$

Moreover, suppose R_1, R_2, R_3, R_4 are positive and $R_1\neq R_2$, $R_3\neq R_4$. According to (24), (39) and (44),

$$R_j^2 = \Omega_j, \quad G = \frac{E_{11}-\mu E_{10}}{\lambda-\mu} - R_2^2 - R_3^2, \quad K = \frac{K_1}{R_1^2}. \tag{48}$$

They are all constants for the Hamiltonian flows on S given by H^x, H^y and H^t. Let

$$\phi_{j1} = R_j\cos\theta_j e^{i\alpha_j}, \quad \phi_{j2} = R_j\sin\theta_j e^{i\beta_j} \tag{49}$$

where θ_j's, α_j's and β_j's are real unknown functions.

The constraint $\langle\Phi_1,\Phi_2\rangle = 0$ gives

$$\alpha_1-\alpha_2-\beta_1+\beta_2 = 2k\pi + l\pi, \quad \theta_2 = (-1)^l\theta_1 + m\pi + \pi/2$$

where k, l and m are integers. Let $k=l=m=0$, we have $\theta_2=\theta_1+\pi/2$ and $\beta_2-\alpha_2=\beta_1-\alpha_1$. Likewise, we can also want $\theta_4=\theta_3+\pi/2$ and $\beta_3-\alpha_3=\beta_4-\alpha_4$.

Let $\delta=\beta_1-\beta_3-\alpha_1+\alpha_3$, then $\beta_2-\beta_4-\alpha_2+\alpha_4=\delta$. By (47),

$$
\begin{aligned}
G &= (R_1^2-R_2^2)\cos^2\theta_1 - (R_3^2-R_4^2)\cos^2\theta_4, \\
K &= (\lambda + R_3^2\sin^2\theta_4 + R_4^2\cos^2\theta_4)\cos^2\theta_1 \\
&\quad + (\mu + R_3^2\cos^2\theta_4 + R_4^2\sin^2\theta_4)\sin^2\theta_1 \\
&\quad - 2(R_3^2-R_4^2)\sin\theta_1\cos\theta_1\sin\theta_4\cos\theta_4\cos\delta.
\end{aligned} \tag{50}
$$

Substituting (49) into (23), we get

$$\theta_{1,x} = R_4^2\sin\theta_4\cos\theta_4\sin\delta, \tag{51}$$

$$
\begin{aligned}
\alpha_{1,x} &= \lambda + \frac{1}{2}R_4^2(1+\pi_1+\sigma_1), \\
\alpha_{2,x} &= \lambda + \frac{1}{2}R_4^2(1-\pi_1+\sigma_1), \\
\alpha_{3,x} &= \lambda + c_4\cos^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2), \\
\alpha_{4,x} &= \frac{1}{2}R_1^2(1+\pi_2+\sigma_2) + \frac{1}{2}R_2^2(1-\pi_2-\sigma_2) \\
&\quad + c_3\sin^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2),
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
\beta_{1,x} &= \mu + \frac{1}{2}R_4^2(1+\pi_1-\sigma_1), \\
\beta_{2,x} &= \mu + \frac{1}{2}R_4^2(1-\pi_1-\sigma_1), \\
\beta_{3,x} &= \mu - c_4\sin^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2), \\
\beta_{4,x} &= \frac{1}{2}R_1^2(1+\pi_2-\sigma_2) + \frac{1}{2}R_2^2(1-\pi_2+\sigma_2) \\
&\quad - c_3\cos^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2),
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
\theta_{1,y} &= (a_3R_4^2 - \gamma(\kappa_1,\kappa_2,\kappa_3)(R_3^2-R_4^2)) \\
&\quad \cdot \sin\theta_4\cos\theta_4\sin\delta,
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
\alpha_{1,y} &= \lambda a_1 + \frac{1}{2}((a_1-a_3)R_3^2 + a_1R_4^2) \\
&\quad - \frac{1}{2}((a_1-a_3)R_3^2 - a_1R_4^2)(\pi_1+\sigma_1) \\
&\quad + c_2(a_1-a_2)\sin^2\theta_1(\lambda-\mu-(R_3^2-R_4^2)\sigma_1), \\
\alpha_{2,y} &= \lambda a_2 + \frac{1}{2}((a_2-a_3)R_3^2 + a_2R_4^2) \\
&\quad + \frac{1}{2}((a_2-a_3)R_3^2 - a_2R_4^2)(\pi_1-\sigma_1) \\
&\quad + c_1(a_1-a_2)\cos^2\theta_1(\lambda-\mu-(R_3^2-R_4^2)\sigma_1), \\
\alpha_{3,y} &= \lambda a_3 + \frac{1}{2}((a_1-a_3)R_1^2 + (a_2-a_3)R_2^2) \\
&\quad - \frac{1}{2}((a_1-a_3)R_1^2 - (a_2-a_3)R_2^2)(\pi_2-\sigma_2) \\
&\quad + c_4a_3\cos^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2), \\
\alpha_{4,y} &= \frac{1}{2}(a_1R_1^2 + a_2R_2^2) + \frac{1}{2}(a_1R_1^2 - a_2R_2^2)(\pi_2+\sigma_2) \\
&\quad + c_3a_3\sin^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2),
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
\beta_{1,y} &= \mu a_1 + \frac{1}{2}((a_1-a_3)R_3^2 + a_1R_4^2) \\
&\quad - \frac{1}{2}((a_1-a_3)R_3^2 - a_1R_4^2)(\pi_1-\sigma_1) \\
&\quad - c_2(a_1-a_2)\cos^2\theta_1(\lambda-\mu-(R_3^2-R_4^2)\sigma_1), \\
\beta_{2,y} &= \mu a_2 + \frac{1}{2}((a_2-a_3)R_3^2 + a_2R_4^2) \\
&\quad + \frac{1}{2}((a_2-a_3)R_3^2 - a_2R_4^2)(\pi_1+\sigma_1) \\
&\quad - c_1(a_1-a_2)\sin^2\theta_1(\lambda-\mu-(R_3^2-R_4^2)\sigma_1), \\
\beta_{3,y} &= \mu a_3 + \frac{1}{2}((a_1-a_3)R_1^2 + (a_2-a_3)R_2^2) \\
&\quad - \frac{1}{2}((a_1-a_3)R_1^2 - (a_2-a_3)R_2^2)(\pi_2+\sigma_2) \\
&\quad - c_4a_3\sin^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2), \\
\beta_{4,y} &= \frac{1}{2}(a_1R_1^2 + a_2R_2^2) + \frac{1}{2}(a_1R_1^2 - a_2R_2^2)(\pi_2-\sigma_2) \\
&\quad - c_3a_3\cos^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2),
\end{aligned} \tag{56}
$$

where

$$
\begin{aligned}
\sigma_1 &= \cos 2\theta_4 - \sin 2\theta_4 \cot 2\theta_1 \cos\delta, \\
\sigma_2 &= \cos 2\theta_1 - \sin 2\theta_1 \cot 2\theta_4 \cos\delta, \\
\pi_1 &= \frac{\sin 2\theta_4}{\sin 2\theta_1}\cos\delta, \quad \pi_2 = \frac{\sin 2\theta_1}{\sin 2\theta_4}\cos\delta,
\end{aligned} \tag{57}
$$

and

$$
\begin{aligned}
c_1 &= \frac{R_1^2}{R_1^2-R_2^2}, \quad c_2 = \frac{R_2^2}{R_1^2-R_2^2}, \\
c_3 &= \frac{R_3^2}{R_3^2-R_4^2}, \quad c_4 = \frac{R_4^2}{R_3^2-R_4^2}, \\
\gamma(\kappa_1,\kappa_2,\kappa_3) &= c_1\kappa_1 - c_2\kappa_2 - \kappa_3.
\end{aligned} \tag{58}
$$

The equations similar to (54)–(56) hold for $\theta_{1,t}$, $\alpha_{j,t}$'s and $\beta_{j,t}$'s, provided that a_j's are changed to b_j's.

Comparing (51) with (54), let

$$
\xi = x - \gamma_1 y - \gamma_2 t, \quad \eta = y, \quad \tau = t, \tag{59}
$$

where

$$
\gamma_1 = \frac{\gamma(a_1,a_2,a_3) - c_4a_3}{c_4}, \quad \gamma_2 = \frac{\gamma(b_1,b_2,b_3) - c_4b_3}{c_4}. \tag{60}
$$

With (50), the equations in (ξ,η,τ) coordinate are

$$
\begin{aligned}
\theta_{1,\xi} &= R_4^2\sin\theta_4\cos\theta_4\sin\delta, \\
\alpha_{1,\xi} &= \lambda + \frac{1}{2}R_4^2(1+\pi_1+\sigma_1), \\
\alpha_{2,\xi} &= \lambda + \frac{1}{2}R_4^2(1-\pi_1+\sigma_1), \\
\alpha_{3,\xi} &= \lambda + c_4\cos^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2), \\
\alpha_{4,\xi} &= \frac{1}{2}R_1^2(1+\pi_2+\sigma_2) + \frac{1}{2}R_2^2(1-\pi_2-\sigma_2) \\
&\quad + c_3\sin^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2), \\
\beta_{1,\xi} &= \mu + \frac{1}{2}R_4^2(1+\pi_1-\sigma_1), \\
\beta_{2,\xi} &= \mu + \frac{1}{2}R_4^2(1-\pi_1-\sigma_1), \\
\beta_{3,\xi} &= \mu - c_4\sin^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2), \\
\beta_{4,\xi} &= \frac{1}{2}R_1^2(1+\pi_2-\sigma_2) + \frac{1}{2}R_2^2(1-\pi_2+\sigma_2) \\
&\quad - c_3\cos^2\theta_4(\lambda-\mu-(R_1^2-R_2^2)\sigma_2),
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
&\theta_{1,\eta} = 0, \\
&\alpha_{j,\eta} = \Gamma_j(a_1,a_2,a_3,\lambda), \quad \beta_{j,\eta} = \Gamma_j(a_1,a_2,a_3,\mu), \\
&(j = 1,2,3,4),
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
&\theta_{1,\tau} = 0, \\
&\alpha_{j,\tau} = \Gamma_j(b_1,b_2,b_3,\lambda), \quad \beta_{j,\tau} = \Gamma_j(b_1,b_2,b_3,\mu), \\
&(j = 1,2,3,4).
\end{aligned} \tag{63}
$$

Here

$$
\begin{aligned}
\Gamma_1(\kappa_1,\kappa_2,\kappa_3,\kappa_0) &= \gamma(\kappa_1,\kappa_2,\kappa_3)\frac{R_3^2}{R_4^2}\kappa_0 \\
&\quad + \gamma(\kappa_1,\kappa_2,\kappa_3)R_3^2 - c_2(\kappa_1-\kappa_2)K, \\
\Gamma_2(\kappa_1,\kappa_2,\kappa_3,\kappa_0) &= \gamma(\kappa_1,\kappa_2,\kappa_3)\frac{R_3^2}{R_4^2}\kappa_0 \\
&\quad + (\kappa_2-\kappa_3)R_3^2 - c_1(\kappa_1-\kappa_2)(\lambda+\mu+R_4^2-K), \\
\Gamma_3(\kappa_1,\kappa_2,\kappa_3,\kappa_0) &= \gamma(\kappa_1,\kappa_2,\kappa_3)\frac{R_3^2}{R_4^2}\kappa_0 \\
&\quad + \frac{\gamma(\kappa_1,\kappa_2,\kappa_3)}{2(R_3^2-R_4^2)}\Sigma + \frac{1}{2}(\kappa_1-\kappa_3)R_1^2 + \frac{1}{2}(\kappa_2-\kappa_3)R_2^2, \\
\Gamma_4(\kappa_1,\kappa_2,\kappa_3,\kappa_0) &= \gamma(\kappa_1,\kappa_2,\kappa_3)\frac{R_3^2}{R_4^2}(\kappa_0-\lambda-\mu) \\
&\quad - \frac{\gamma(\kappa_1,\kappa_2,\kappa_3)}{2(R_3^2-R_4^2)}\frac{R_3^2}{R_4^2}\Sigma - (k_1-k_2)c_1c_2(R_1^2-R_2^2) \\
&\quad + \gamma(\kappa_1,\kappa_2,\kappa_3)\frac{R_3^2}{2R_4^2}(R_1^2+R_2^2),
\end{aligned} \tag{64}
$$

with

$$\begin{aligned}\Sigma &= (2K - R_3^2 - R_4^2)(R_1^2 - R_2^2) \\ &\quad - 2\lambda(R_3^2 - R_4^2) - 2\mu(R_1^2 - R_2^2) - 2G(\lambda - \mu).\end{aligned} \tag{65}$$

$\Gamma_j(\kappa_1, \kappa_2, \kappa_3, \kappa_0)$ $(j = 1, 2, 3, 4)$ are all constants for constant κ_1, κ_2, κ_3, κ_0, and θ_4, δ are related with θ_1 by (50).

Let $\rho = \cos^2\theta_1$, then, with (50), the first equation of (61) becomes

$$\rho_\xi = \pm\frac{R_4^2}{R_3^2 - R_4^2}\sqrt{P(\rho)} \tag{66}$$

where

$$\begin{aligned}P(\rho) &= 4(R_1^2 - R_2^2)(\lambda - \mu)\rho^3 \\ &\quad + (4(\mu - \lambda)G + 4(R_1^2 - R_2^2)(\mu + R_3^2 - K) - r_1^2)\rho^2 \\ &\quad + (4G(K - \mu - R_3^2) + 2r_1r_2)\rho - r_2^2\end{aligned} \tag{67}$$

is a cubic polynomial of ρ,

$$\begin{aligned}r_1 &= \lambda - \mu + R_1^2 - R_2^2 + R_3^2 - R_4^2, \\ r_2 &= K + G - \mu - R_4^2.\end{aligned} \tag{68}$$

Suppose $R_1^2 > R_2^2$, $\lambda > \mu$ and P has three different real roots $\rho_1 < \rho_2 < \rho_3$. Moreover, suppose

$$\begin{aligned}&K + G - \mu - R_4^2 \neq 0, \\ &K - G - \lambda + R_1^2 - R_2^2 - R_3^2 \neq 0, \\ &\max\left(0, \frac{G + R_3^2 - R_4^2}{R_1^2 - R_2^2}\right) < \min\left(1, \frac{G}{R_1^2 - R_2^2}\right).\end{aligned} \tag{69}$$

Then the solution ρ can be expressed by an elliptic function of ξ. Let $\rho = \rho_1 + (\rho_2 - \rho_1)\omega^2$, (66) becomes

$$\omega_\xi = \pm p\sqrt{(1 - \omega^2)(1 - k^2\omega^2)} \tag{70}$$

where

$$\begin{aligned}k &= \sqrt{(\rho_2 - \rho_1)/(\rho_3 - \rho_1)}, \\ p &= \frac{R_4^2}{R_3^2 - R_4^2}\sqrt{(R_1^2 - R_2^2)(\lambda - \mu)(\rho_3 - \rho_1)}.\end{aligned} \tag{71}$$

Hence $\omega = \pm\,\mathrm{sn}(p(\xi + C_0))$ where the function *sn* has parameter k. Thus

$$\rho = \rho_1 + (\rho_2 - \rho_1)\,\mathrm{sn}^2(p(\xi + C_0)), \tag{72}$$

with an arbitrary constant C_0. ρ is a periodic function of ξ.

Remark 3. *Since ρ_j is a root of P, (67) leads to*

$$\begin{aligned}&4\rho_j(1 - \rho_j)((R_1^2 - R_2^2)\rho_j - G) \\ &\quad \cdot (R_3^2 - R_4^2 + G - (R_1^2 - R_2^2)\rho_j) \\ &= (K - \mu - R_4^2 - (\lambda - \mu + R_3^2 - R_4^2)\rho_j \\ &\quad + (2\rho_j - 1)((R_1^2 - R_2^2)\rho_j - G))^2 \geq 0.\end{aligned} \tag{73}$$

(This is equivalent to the second equation of (50) with $\delta = \pi/2$.) Hence,

$$\max\left(0, \frac{G + R_3^2 - R_4^2}{R_1^2 - R_2^2}\right) \leq \rho_1 < \rho_2 \leq \min\left(1, \frac{G}{R_1^2 - R_2^2}\right)$$

holds if there is a solution locally, since $0 \leq \cos^2\theta_1 \leq 1$ and $0 \leq \cos^2\theta_4 \leq 1$ should be satisfied. This also guarantees that the solution is global because $\rho_1 \leq \rho \leq \rho_2$. Moreover, under the assumptions (69), $P(0) \neq 0$, $P(1) \neq 0$. Hence $0 < \rho_1 < \rho_2 < 1$ and $0 < \rho < 1$.

According to (61), any one of α_j, β_j $(j = 1, 2, 3, 4)$ is a sum of a function of $\rho(\xi)$ and a linear function of ξ, η and τ. For example,

$$\begin{aligned}(\alpha_1 - \alpha_2)_\xi &= R_4^2\pi_1, \\ (\alpha_1 - \alpha_2)_\eta &= \Gamma_1(a_1, a_2, a_3, \lambda) - \Gamma_2(a_1, a_2, a_3, \lambda), \\ (\alpha_1 - \alpha_2)_\tau &= \Gamma_1(b_1, b_2, b_3, \lambda) - \Gamma_2(b_1, b_2, b_3, \lambda).\end{aligned}$$

By (50) and (57), π_1 is a rational function of $\rho(\xi)$. Write $Q(\rho(\xi)) = \pi_1(\xi)$. Suppose the minimal positive period of the function *sn* with parameter k is $T(k)$ and

$$\begin{aligned}A &= R_4^2\int_0^{T(k)/p} Q(\rho(\xi))\,\mathrm{d}\xi, \\ \Delta &= \Gamma_1(a_1, a_2, a_3, \lambda) - \Gamma_2(a_1, a_2, a_3, \lambda).\end{aligned} \tag{74}$$

Then, when (ξ, η) is changed to $(\xi + \frac{T(k)}{p}, \eta - \frac{A}{\Delta})$ or $(\xi, \eta + \frac{2\pi}{\Delta})$, u_{12} is invariant respectively. Hence

$$\begin{aligned}u_{12}\left(x + \frac{T(k)}{p} - \frac{\gamma_1 A}{\Delta}, y - \frac{A}{\Delta}, t\right) &= u_{12}(x, y, t), \\ u_{12}\left(x + \frac{2\pi\gamma_1}{\Delta}, y + \frac{2\pi}{\Delta}, t\right) &= u_{12}(x, y, t).\end{aligned} \tag{75}$$

This means that u_{12} is a double periodic function on the (x, y) plane, so are u_{13} and u_{23}. However, the periods of these three functions are different. Therefore, as a whole, (u_{12}, u_{13}, u_{23}) gives an almost-periodic solution of the 2+1 dimensional three-wave equation.

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