三维欧氏空间中梯度、散度、旋度的计算公式

下面各式中, f, g 为 $\mathbf{R}^3$ 中的光滑函数, $\mathbf{v}$, $\mathbf{w}$ 为 $\mathbf{R}^3$ 中的光滑向量场.

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$
$$\nabla(f\mathbf{v}) = f\nabla\cdot\mathbf{v} + (\nabla f)\cdot\mathbf{v}$$
$$\nabla\times(f\mathbf{v}) = f\nabla\times\mathbf{v} + (\nabla f)\times\mathbf{v}$$
$$\nabla(\mathbf{v}\cdot\mathbf{w}) = (\mathbf{w}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{w} + \mathbf{w}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{w})$$
$$\nabla\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot\nabla\times\mathbf{v} - \mathbf{v}\cdot\nabla\times\mathbf{w}$$
$$\nabla\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{w}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{w} + (\nabla\cdot\mathbf{w})\mathbf{v} - (\nabla\cdot\mathbf{v})\mathbf{w}$$
$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\Big(\frac{1}{2}||\mathbf{v}||^2\Big) - \mathbf{v}\times(\nabla\times\mathbf{v})$$
$$\nabla\cdot(\nabla f) = \Delta f$$
$$\nabla\times(\nabla f) = 0$$
$$\nabla\cdot(\nabla\times\mathbf{v}) = 0$$
$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \Delta\mathbf{v}$$

在柱坐标 (r, θ, z) 下,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$
$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}$$
$$\nabla\times\mathbf{v} = \Big(\frac{1}{r}\frac{\partial v_z}{\partial\theta} - \frac{\partial v_\theta}{\partial z}\Big)\mathbf{e}_r + \Big(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\Big)\mathbf{e}_\theta + \Big(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\Big)\mathbf{e}_z$$
$$\triangle f = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial f}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$$

在球坐标 (r, θ, ϕ) 下,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi$$
$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$
$$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\Big(\frac{\partial}{\partial\theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial\phi}\Big)\mathbf{e}_r + \Big(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rv_\phi)\Big)\mathbf{e}_\theta + \Big(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\Big)\mathbf{e}_\phi$$
$$\triangle f = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial f}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial f}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$