曲面积分的导数

定理: 设 Σ 是一个曲面, $\mathbf{f}$ 是一个向量场, 则

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\mathbf{f}(t,\mathbf{x})\cdot\mathrm{d}\mathbf{S}=\int_{\Sigma_t}\Big(\frac{\partial\mathbf{f}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{f}-(\mathbf{f}\cdot\nabla)\mathbf{v}+(\nabla\cdot\mathbf{v})\mathbf{f}\Big)\,\mathrm{d}\mathbf{S}$$
$$=\int_{\Sigma_t}\Big(\frac{\partial\mathbf{f}}{\partial t}+\nabla\times(\mathbf{f}\times\mathbf{v})+(\nabla\cdot\mathbf{f})\mathbf{v}\Big)\,\mathrm{d}\mathbf{S}.$$

证明: 在 $t=t_0$ 处求导, 对重积分的导数的证明中已得

$$\left.\frac{\partial y_i}{\partial x_j}\right|_{t=t_0}=\delta_{ij},\quad \left.\frac{\partial^2 y_i}{\partial x_j\partial t}\right|_{t=t_0}=\frac{\partial v_i}{\partial x_j}.$$

因此,

$$\int_{\Sigma_t}\mathbf{f}(t,\mathbf{x})\cdot\mathrm{d}\mathbf{S}=\frac{1}{2}\int_{\Sigma_t}\epsilon_{ijk}f_i(t,\mathbf{x})\,\mathrm{d}x_j\wedge\mathrm{d}x_k$$
$$=\frac{1}{2}\int_{\Sigma_{t_0}}\epsilon_{ijk}f_i(t,\mathbf{y}(t,\mathbf{x}))\frac{\partial y_j(t,\mathbf{x})}{\partial x_a}\frac{\partial y_k(t,\mathbf{x})}{\partial x_b}\,\mathrm{d}x_a\wedge\mathrm{d}x_b,$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\mathbf{f}(t,\mathbf{x})\cdot\mathrm{d}\mathbf{S}\right|_{t=t_0}$$
$$=\frac{1}{2}\int_{\Sigma_{t_0}}\epsilon_{ijk}\left(\Big(\frac{\partial f_i}{\partial t}+(\mathbf{v}\cdot\nabla)f_i\Big)\delta_{ja}\delta_{kb}+f_i\frac{\partial v_j}{\partial x_a}\delta_{kb}+f_i\delta_{ja}\frac{\partial v_k}{\partial x_b}\right)\mathrm{d}x_a\wedge\mathrm{d}x_b.$$

由于

$$\frac{1}{2}\epsilon_{abc}\epsilon_{ijk}\left(\Big(\frac{\partial f_i}{\partial t}+(\mathbf{v}\cdot\nabla)f_i\Big)\delta_{ja}\delta_{kb}+f_i\frac{\partial v_j}{\partial x_a}\delta_{kb}+f_i\delta_{ja}\frac{\partial v_k}{\partial x_b}\right)$$
$$=\frac{1}{2}\epsilon_{abc}\epsilon_{ijk}\Big(\frac{\partial f_i}{\partial t}+(\mathbf{v}\cdot\nabla)f_i\Big)\delta_{ja}\delta_{kb}+\frac{1}{2}\epsilon_{abc}\epsilon_{ijk}f_i\frac{\partial v_j}{\partial x_a}\delta_{kb}+\frac{1}{2}\epsilon_{abc}\epsilon_{ijk}f_i\delta_{ja}\frac{\partial v_k}{\partial x_b}$$
$$=\frac{1}{2}\epsilon_{abc}\epsilon_{iab}\Big(\frac{\partial f_i}{\partial t}+(\mathbf{v}\cdot\nabla)f_i\Big)+\frac{1}{2}\epsilon_{abc}\epsilon_{ijb}f_i\frac{\partial v_j}{\partial x_a}+\frac{1}{2}\epsilon_{abc}\epsilon_{iak}f_i\frac{\partial v_k}{\partial x_b}$$
$$=\frac{\partial f_c}{\partial t}+v_k\frac{\partial f_c}{\partial x_k}-f_k\frac{\partial v_c}{\partial x_k}+f_c\frac{\partial v_k}{\partial x_k},$$

所以

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}\mathbf{f}(t,\mathbf{x})\cdot\mathrm{d}\mathbf{S}\right|_{t=t_0}=\int_{\Sigma}\Big(\frac{\partial f_c}{\partial t}+v_k\frac{\partial f_c}{\partial x_k}-f_k\frac{\partial v_c}{\partial x_k}+f_c\frac{\partial v_k}{\partial x_k}\Big)n_c\,\mathrm{d}S$$
$$=\int_{\Sigma}\Big(\frac{\partial\mathbf{f}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{f}-(\mathbf{f}\cdot\nabla)\mathbf{v}+(\nabla\cdot\mathbf{v})\mathbf{f}\Big)\cdot\,\mathrm{d}\mathbf{S}.$$