习题 1.2.7(3) 中由递推关系

$$x_1 = 1, \quad x_{n+1} = 1 + \frac{x_n}{1+x_n} \quad (n = 1, 2, \cdots)$$

决定的数列的通项.

(1) 求"不动点". 设 x 满足 $x = 1 + \dfrac{x}{1+x}$ (即将递推关系中的 x_n 和 x_{n+1} 均用 x 代入), 即 $x^2 - x - 1 = 0$. 解得 $x = \dfrac{1 \pm \sqrt{5}}{2}$, 任取其中之一, 例如取 $x = \alpha \equiv \dfrac{1+\sqrt{5}}{2}$.

(2) 令 $x_n = \alpha + t_n$, 则利用 $\alpha^2 - \alpha - 1 = 0$ 得到 $\{t_n\}$ 满足的递推关系

$$t_1 = 1 - \alpha, \quad t_{n+1} = \frac{2-\alpha}{1+t_n+\alpha} = \frac{\dfrac{3-\sqrt{5}}{2}t_n}{1+t_n+\alpha} = \frac{\alpha^{-2}t_n}{t_n+\alpha^2}$$

(3) 取倒数. 令 $s_n = \dfrac{1}{t_n}$, 得到线性的递推关系

$$s_1 = \frac{1}{1-\alpha} = -\alpha, \quad s_{n+1} = \alpha^4 s_n + \alpha^2,$$

再设 s 满足 $s = \alpha^4 s + \alpha^2$, 解得 $s = \dfrac{\alpha^2}{1-\alpha^4} = -\dfrac{1}{\sqrt{5}}$.

(4) 令 $s_n = u_n - \dfrac{1}{\sqrt{5}}$, 则

$$u_1 = \frac{1}{\sqrt{5}} - \alpha = -\frac{1}{\sqrt{5}}\alpha^2, \quad u_{n+1} = \alpha^4 u_n,$$

它的解为 $u_n = -\dfrac{1}{\sqrt{5}}\alpha^{4n-2}$.

(5) 代回, 得

$$x_n = \alpha + \frac{1}{u_n - \dfrac{1}{\sqrt{5}}} = \frac{1}{\alpha}\frac{\alpha^{4n}-1}{\alpha^{4n-2}+1}.$$